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FILTERS AND OVERRINGS

H. H. BRUNGS

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Let R be an integral domain. It is well known (see Lambek (1971), Stenström (1971)), that idempotent filters of right ideals, torsion radicals and torsion theories are in one-to-one correspondence, but that different idempotent filters $\mathscr{F}$ of right ideals may lead to the same rings of quotiens $R_{\mathscr{F}}$. We have always $R \subset R_{\mathscr{F}} \subset Q_{\max}(R)$. Given this situation one can ask a number of questions. For example: Describe all different idempotent filters for a given ring. Determine all different rings of quotients. When do different filters lead to the same ring of quotients? When are all rings between R and $Q_{\max}(R)$ of the form $R_{\mathscr{F}}$? When is every $R_{\mathscr{F}}$ of the form RS^{-1}, where S is an Ore system?

Some of the problems mentioned above are easier to handle if it is possible to use a localizing procedure. This is described in section 1, and one can apply it for example to all noetherian commutative domains (where one knows all different filters). We then consider noncommutative Krull domains. These rings are a generalization of Krull domains, and it is possible to determine all their rings of quotients.

It is known (see for example Gilmer and Ohm (1964)) that every ring between a commutative noetherian integral domain R and its field of quotients is always a ring of quotients with respect to some multiplicative system if and only if R is a Dedekind domain with torsion class group. We will give a similar condition for the semigroup of divisorial ideals of a non commutative Krull domain to insure that every ring of quotients with respect to some torsion theory is a ring of quotients with respect to some Ore system.

We add some related results about principal ideal domains and Bezont domains in the final section.

We will use the word filter instead of idempotent filter and we will use the definition in Stenström (1971), page 12: A *filter* $\mathscr{F}$ is a non empty family of right deals of R satisfying:

F1. If $I \in \mathscr{F}$ and r in R then $r^{-1}I = \{a \in R; ra \in I\} \in \mathscr{F}$.

F2. If $J \in \mathscr{F}$ and $r^{-1}I \in \mathscr{F}$ for all r in J and a right ideal I in R, then $I \in \mathscr{F}$.

Similarly we will use the term Krull domain instead of non commutative Krull domain.

1

The results in this section show that in certain cases the rings of quotients can be computed locally. We will show further that this is not always possible.

LEMMA 1. *Let R be an integral domain, S a right Ore system. If $\mathscr{F}$ is a filter of right ideals in R such that the right ideal $\{r \in R;\ \exists t \in S \ni rt = sa$ for some $a \in A\}$ is again in $\mathscr{F}$ for every A in $\mathscr{F}$ and s in S, then $\mathscr{F}' = \{ARS^{-1}, A \in \mathscr{F}\}$ is a filter in $RS^{-1} = R'$.*

PROOF. We have to check the conditions F1 and F2 mentioned at the end of the introduction. Let A' be in $\mathscr{F}'$, $\alpha = as^{-1}$ in R' with a in $R, s \in S$. Then

$$(as^{-1})^{-1}A' = \{r' \in R';\ s^{-1}r' \in a^{-1}A'\}.$$

Since $a^{-1}A'$ contains $a^{-1}(A' \cap R) \in \mathscr{F}$, we have to prove only that with A' in $\mathscr{F}'$ and s in S the right ideal sA' is an element of $\mathscr{F}'$. But this will be the case if and only if $sA' \cap R = \{r \in R;\ r = sat^{-1}\}$ is contained in $\mathscr{F}$ with $a \in A' \cap R \in \mathscr{F}$, $t \in S$. This is exactly the above condition. To check F2. let A' be in $\mathscr{F}'$ and B' be a right ideal in R' such that $\alpha^{-1}B'$ is in $\mathscr{F}'$ for all α in A'. But $A' \cap R = A$ is in $\mathscr{F}$ and therefore $a^{-1}(B' \cap R)$ is in $\mathscr{F}$, which implies B' in $\mathscr{F}'$ and proves the Lemma.

The next lemma gives conditions under which the ring $R_{\mathscr{F}}$ is the intersection of the rings $R'_{\mathscr{F}'}$:

LEMMA 2. *Let R be a right Ore domain, S_i be right Ore systems in R and denote by R_i the rings of quotients RS_i^{-1}. Assume that for the filter $\mathscr{F}$ in R the set $\mathscr{F}_i = \{AR_i, A \in \mathscr{F}\}$ is a filter in R_i for all i. Assume further that $R = \cap R_i$ and that for every right ideal A in R the set $\{AR_i \cap R\}$ is finite. Then $R_{\mathscr{F}} = \cap (R_i)_{\mathscr{F}_i}$.*

For a proof let α be an element in $R_{\mathscr{F}}$. Then $\alpha A \subset R$ for some A in $\mathscr{F}$ and $\alpha AR_i \subset R_i$ which proves $R_{\mathscr{F}} \subseteq \cap (R_i)_{\mathscr{F}_i}$. If on the other hand α is in $\cap (R_i)_{\mathscr{F}_i}$, we know that $A_i = \{r \in R_i \ni \alpha r \in R_i\}$ is in $\mathscr{F}_i$.

Consider $C = \{r \in R \ni \alpha r \in R\}$. It follows that $CR_i \subset A_i$.

For $as^{-1} \in A_i$, $a \in R$, $s \in S_i$, elements b in R and t in S_i exist such that $\alpha as^{-1} = bt^{-1}$, and further $\alpha at' = bs'$ with elements t', s' in R, t' in S_i and $st' = ts'$.

This means that at' is an element of C and as^{-1} is contained in CR_i, proving that $A_i = CR_i$ for all i.

We conclude from $\alpha A_i \subset R_i$ that $\alpha \cap (A_i \cap R) \subseteq \cap R_i = R$. Using the assumption in the lemma we see that $\cap (A_i \cap R)$ is a finite intersection of members of $\mathscr{F}$ and therefore in $\mathscr{F}$. This proves that α is contained in R and the lemma follows.

Compare Arnold and Brewer (1971) for related results in the commutative case. We observe that the condition $|\{AR_i \cap R, \in I\}| < \aleph_0$ is satisfied for any commutative noetherian domain R and any set of overrinas $R_i = RS_i^{-1}$. Consider a finite primary decomposition $A = \cap Q_j$, where Q_j is P_j-primary, $j \in \Lambda$. We obtain $AR_i = \bigcap_j (Q_j R_i)$ and $AR_i \cap R = \bigcap_j (Q_j R_i \cap R)$. But $Q_j R_i \cap R$ is either equal to R or Q_j and this shows that $AR_i \cap R = \bigcap_{j \in \Lambda_i} Q_j$ for a subset Λ_i of Λ. To illustrate that $R = \cap RS_i^{-1}$ does not necessarily imply that $R_{\mathscr{F}} = \cap (RS_i^{-1})_{\mathscr{F}_i}$, consider the rina R of all entire functions with the field K of meromorphic functions as field of quotients. Consider the maximal fixed ideals (see Henriksen (1952)). $M_a = \{f(z) \in R;\, f(a) = 0\}$ for every complex number a. It follows that $R_a = R_{M_a}$ are discrete rank 1 valuation rings and that $R = \cap R_a$. Define $\mathscr{F} = \{A \subset R$ with $A \supset M_{a_1} \cdots M_{a_n}$, some $a_i\}$. $\mathscr{F}$ is a filter in R, $\mathscr{F}_a = \{AR_a\,;\, A \in \mathscr{F}\}$ are filters in R_a, but since $(R_a)_{\mathscr{F}_a} = K$, we obtain $\cap (R_a)_{\mathscr{F}_a} = K \underset{+}{\supseteq} R_{\mathscr{F}}$ the set of those meromorphic functions with finitely many poles only.

2

We will apply the result of section 1 to Krull domains. These rings were defined in Brungs (1973). An integral domain in which all right ideals are inversely well ordered by inclusion is called a *generalized discrete valuation ring* or a G.D.V. domain. These rings can be defined equivalently as follows: There exists a mapping v from $R/0$ onto the semigroup $\Gamma = \{\alpha, \alpha < \omega^l\}$ of ordinals less than ω^l with addition as operation such that

$$v(ab) = v(a) + v(b)$$

$$v(a + b) \leqq \max\{v(a), v(b)\}$$

and

$$v(a) = v(b) \text{ if and only if } a = b\varepsilon \text{ for a unit } \varepsilon \text{ in } R.$$

We say in this case that R is of type $\omega^l + 1$.

A *Krull domain* R is defined by the following properties:

1) $R = \cap V_i$, where the V_i are G.D.V. domains s.t. $Q(V_i) = D$ is a common skewfield of fractions.

2) $V_i = R_{P_i}$, where $\{P_i\}$ is a set of strong prime ideals in R such that $S_i = R \backslash P_i$ is an Ore system and $V_i = R_{P_i} = RS_i^{-1}$. Further $P_i \cap P_j$ contains no non zero prime ideal for $i \neq j$.

3) $a \neq 0$ in R implies $aV_i = V_i$ for almost all i.

We say a right ideal A in R is *divisorial* if $A = \cap AV_i$ and say that two right ideals A and B in R are *quasi equal* if $\cap AV_i = \cap BV_i$. The equivalence classes of quasi equal ideals form a semigroup isomorphic to $\Gamma = \oplus \Gamma_i$ where $\Gamma_i = \{\alpha, \alpha < \omega^{l_i}\}$ with $\omega^{l_i} + 1$ as type of V_i. The mapping w with $w(A) = \{v_i(a)\} \in \Gamma$ defines a homomorphism from the semigroup of all right ideals $\neq 0$ of R onto Γ.

THEOREM 1. *Let R be a Krull domain with the associated semigroup Γ. Let $H = \oplus \Lambda_i$ with $\Lambda_i = \{\alpha, \alpha < \omega^{j_i}; j_i \leqq l_i\} \subset \Gamma_i$. Then H is a subsemigroup of Γ and $\mathscr{F}_H = \{A \subset R, w(A) \in H\}$ is a filter of right ideals of R. Different subgroups H and H' of the above type lead to different filters and different rings of quotients. Every ring of quotients with respect to any torsion theory of R is a ring of quotients with respect to some $\mathscr{F}_H$.*

PROOF. The condition F.1 for a filter is satisfied since right ideals in R are two-sided. To prove F.2 let A be in $\mathscr{F}_H$ and $a^{-1}B$ be in $\mathscr{F}_H$ for all a in A and a right ideal B of R. Assume $B \notin \mathscr{F}_H$, say $v_i(B) \geqq \omega^{j_i}$ for a certain i. There exists a in A with $v_i(a) < \omega^{j_i}$ and r in $a^{-1}B$ with $v_i(r) < \omega^{j_i}$. From this $v_i(ar) < \omega^{j_i}$ follows and gives a contradiction, since ar is in B. To prove the second part of the theorem we consider first the filters of a G.D.V. domain V of type $\omega^l + 1$. It is clear that the sets of right ideals $\mathscr{F}_j = \{A \subset V, v(A) < \omega^j, j \leqq l\}$ form filters and that these are the only filters of V. Further consider $S_j = V \backslash P_j$, where P_j is the prime ideal in V with $v(P_j) = \omega^j$. S_j is an Ore system and we obtain $V_{\mathscr{F}_j} = VS_j^{-1}$.

We can now apply Lemmas 1 and 2.

For any filter $\mathscr{F}$ in the Krull domain R we obtain

$$R_{\mathscr{F}} = \cap (V_i)_{\mathscr{F}_i} = \cap V_i S_{j_i}^{-1} \text{ for certain } j_i.$$

Since further $R_{\mathscr{F}_H} = \cap V_i S_{j_i}^{-1}$ if H and j_i are chosen as in the formulation of the theorem, we see that every $R_{\mathscr{F}}$ is equal to some $R_{\mathscr{F}_H}$ for a suitable H. To prove that one obtains different rings of quotients for different $\mathscr{F}_H$ we show first that every $R_{\mathscr{F}}$ itself is a Krull domain. It is clear that every $V_i S_{j_i}^{-1} = V_{i,j_i}$ is a G.D.V. domain, and that the defining conditions 1 and 3 are satisfied for $R_{\mathscr{F}} = R'$. If $\hat{P}_{i,j_i}$ is the maximal ideal of V_{i,j_i} then $P'_{i,j_i} = \hat{P}_{i,j_i} \cap R'$ and $P_{i,j_i} = \hat{P}_{i,j_i} \cap R$ are prime ideals in R' and R respectively. Since

$$V_{i,j_i} \supset R'_{Pi,j_i} \supset R_{P_{i,j_i}} \supset V_{i,j_i}, \text{ we obtain } V_{i,j_i} = R'_{P'_{i},q_i}$$

and the second half of condition 2 for a Krull domain is inherited from the same property for the P_i's in R.

Now let H, H' be two different subsemigroups of Γ of the form described in the theorem, and $\mathscr{F} = \mathscr{F}_H$, $\mathscr{F}' = \mathscr{F}_{H'}$ the corresponding filters. Then there exists an i with $j_i > j'_i$ say. In R exists an ideal A with $\omega^{j'_i} < v_i(A) < \omega^{j_i}$, $v_k(A) = 0$ for $k \neq i$. It follows that $AR_{\mathscr{F}'} \neq R_{\mathscr{F}'}$ and $AR_{\mathscr{F}} = R_{\mathscr{F}}$ which implies $R_{\mathscr{F}} \neq R_{\mathscr{F}'}$ and proves the theorem.

We actually proved the following Corollary also:

COROLLARY. *Let $R = \cap V_i$ be a Krull domain, R' a ring of quotients of R with respect to some filter. Then R' is a Krull domain, $R' = \cap V_{i,j_i}$ where $V_{i,j_i} = V_i S_{j_i}^{-1}$ for Ore-systems S_{i_j} of V_i.*

Lambek (1971), page 41 gives an example where two different filters lead to the same ring of quotients. The above considerations show that there is an abundance of such incidents. For example let R be a commutative noetherian unique factorization domain R with Krull dimension > 1 and infinitely many maximal ideals M_i with weight $(M_i) \geqq 2$. Then every filter $\mathscr{F}_i = \{A \subset R, A \supset M_i^n, \text{some } n\}$ will lead to the same ring of quotients, namely R, even though all these $\mathscr{F}_i$ are different filters

Contrary to the commutative case one does not obtain all rings between R and its field of quotients as rings of quotients if R is a (non-commutative) Dedekind domain, i.e. a Krull domain where the set $\{P_i\}$ in condition 2 of the definition is the set of the maximal ideals.

This is not even true for a G.D.V. of type $\geqq \omega^2 + 1$. For a proof of the last statement consider a G.D.V. domain of type $\omega^2 + 1$ with $R \supset xR \supset yR \supset 0$ as its prime ideals and $xy = y\alpha, \alpha$ a certain unit in R. Every right ideal in R has a unique form $y^n x^m R$. We claim that $T = yRy^{-1}$ is a subring of $Q(R)$ which cannot be realized as a ring of quotients with respect to some torsion theory. We use the criterion given in Lambek (1971) p. 39, prop. 2.6. Let $yxy^{-1} = s \in T$. One sees that $yxy^{-1}x^n = yx\alpha^n y^{-1} = y^k x^l \varepsilon \in R$ is not possible, since it leads to $yx\alpha^n = y^{k+1}\alpha\varepsilon'$. We obtain $(yxy^{-1})^{-1}R = yR$ but y^{-1} is not contained in T. But it follows from our results that in the Dedekind case the filters described in Theorem 1 are all the different filters, and different filters lead to different rings of quotients.

THEOREM 2. *Let R be a Krull domain. Every ring of quotients with respect to some torsion theory is a ring of quotients with respect to some right Ore system in R if and only if for every i and every $j < l_i$ there exists an ordinal $\alpha_{j,i}$ with $\omega^j \leqq \alpha_{j,i} < \omega^{j+1}$ and a principal ideal $A_{j,i}$ such that $v_k(A_{j,i}) = 0$ for $k \neq i$ and $v_i(A_{j,i}) = \alpha_{j,i}$.*

PROOF. Let $u \in R$ be an element in a ring of quotients with respect to some torsion with corresponding filter $\mathscr{F}$. It follows that $u^{-1}R = A$ is a divisorial right ideal in R. Let $w(A) = (\beta_1, \cdots, \beta_i, \cdots) \in \Gamma$.

We consider the ith component: $\beta_i = \omega^j n_j + \omega^{j-1} n_{j-1} + \cdots + n_0$, $n_j \neq 0$ where the n_i are nonnegative integers. Write $\alpha_{j,i} = \omega^j k_j + \cdots + k_0$ in similar form with $k_j \neq 0$ by assumption. Since a sufficiently large power of A is contained in $A_{j,i}$, it follows that $A_{j,i}$ is contained in $\mathscr{F}$ and there exists an integer m_i such that $v_i(A_{j,i}^{m_i}) \geqq v_i(A)$. We repeat this procedure for the finitely many i with $v_i(A) = \beta_i \neq 0$ and form the product C of the coresesponding $A_{j,i}^{m_i}$.

C is still a principal right ideal, it is contained in A and a member of $\mathscr{F}$. If we write $C = cR$ we obtain $uc = r \in R$; $u = rc^{-1}$ follows and $c^{-1} \in R_{\mathscr{F}}$. This proves that $R_{\mathscr{F}}$ is a ring of quotients with respect to a right Ore system.

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To prove the converse let $\mathscr{F}$ be a filter of R with ring of quotient $R_{\mathscr{F}} = \bigcap_{k \neq i} V_k \cap V_{i,j+1}$. We may assume that $\mathscr{F}$ is the filter corresponding to $H = \Lambda_i = \{\alpha, \alpha < \omega^{j+1} \leqq \omega^{l_i}\}$.

If $R_{\mathscr{F}}$ is a ring of quotients with respect to some right Ore system there exists $c^{-1} \in R_{\mathscr{F}}$ and $c^{-1}A \subset R$ for a right ideal A in $\mathscr{F}$. We conclude that cR is contained in $\mathscr{F}$ and therefore $v_k(cR) = 0$ for $k \neq i$, $\omega^j \leqq v_i(c) < \omega^{j+1}$ for some c with c^{-1} in R. The assumption $v_i(c) < \omega^j$ for every c^{-1} in R would lead to the contradiction

$$R_{\mathscr{F}} \subseteq R_{\mathscr{F}'} = \bigcap_{k \neq i} V_k \cap V_{i,j} \underset{+}{\subset} R_{\mathscr{F}}.$$

3

Let R be a right and left Ore domain with weak global dimension $R \leqq 1$. The classical ring of quotients $Q(R)$ with respect to all non zero elements of R is the maximal ring of quotients in the sense of Lambek-Utumi and it is flat as a right and left R module. R is therefore right and left semihereditary (Turnidge (1970)). It follows further that every ring S between R and $Q(R)$ is flat as a right and left R module and the injection R into S is an epimorphism in the category of rings. S is therefore a ring of quotients with respect to some filter which contains a cofinal family of finitely generated ideals. (Papescu and Spirzu (1970), Stenström (1971)).

If R is a Bezout domain i.e. if every finitely generated right or left ideal is principal, one concludes that every overring of R contained in $Q(R)$ is a ring of quotients with respect to some Ore system in R (Beauregard (1973), Stenström (1971)).

If we go one step further and consider principal right and left ideal domains, we can describe these Ore systems which correspond uniquely to all filters and uniquely to all overrings of R in the following way (Brungs (1971)): Let p be an irreducible element in R. We say p' is similar to p if R/pR and $R/p'R$ are isomorphic as R right modules. For $a \neq 0$ set $h_p(a)$ = number of irreducible factors similar to p in any irreducible factorization of a. Finally let π be any set of non-similar irreducible elements in R.

Set

$$S_\pi = \{a \neq 0 \text{ in } R; h_p(a) = 0 \text{ for all } p \in \pi\} \text{ and}$$

$$\mathscr{F}_\pi = \{aR, a \in S_\pi\}.$$

The sets S_π are right Ore systems in R, the sets $\mathscr{F}_\pi$ are all the different filters of right ideals in R and they correspond uniquely to the different overrings of R. If R is a principal right ideal domain with maximal condition on left principal ideals, one can define the sets S_π and the filters $\mathscr{F}_\pi$ as above. These are again all the filters of right ideals of R, corresponding to different rings of quotients between R and $Q(R)$, but not every ring between $Q(R)$ is necessarily of this form.

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Dept. Math.
University of Alberta
Edmonton, Alberta, Canada