Can. J. Math., Vol. XXVII, No. 5, 1975, pp. 986–990

METRIZATION OF SYMMETRIC SPACES

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1. Introduction. A *distance function* d on a set X is a function $X \times X \rightarrow [0, \infty)$ satisfying (1) $d(x, y) = 0$ if and only if $x = y$, and (2) $d(x, y) = d(y, x)$. Such a function determines a topology T on X by agreeing that U is an open set if it contains an ϵ-sphere $N(p; \epsilon)(= \{x: d(p, x) < \epsilon\})$ about each of its points. Equivalently, F is closed if and only if $d(x, F) > 0$ for each $x \in X - F$. A topological space is *symmetrizable* via a distance function d if its topology is determined by d as above, and *semi-metrizable* via d if $x \in \bar{A}$ is equivalent to $d(x, A) = 0$. Although neither need be Hausdorff, and symmetrizable spaces are not generally first countable, a space that is semi-metrizable via d is first countable and symmetrizable via d. We also remark that there are distance functions which are semi-metrics for no topology. Denoting by G^*S the union of all members of G that intersect the set S, we say the sequence $G_1, G_2, \ldots$ of open covers for a space X is a *development* for X if

(1) G_{n+1} refines G_n, $n = 1, 2, 3, \ldots$, and

(2) $G_1{}^*x, G_2{}^*x, \ldots$ form a local base at x, whereupon X is *developable* via $G_1, G_2, \ldots$.

A T_0 space, developable via $G_1, G_2, \ldots$, is always semi-metrizable by setting $d(x, y) = 1/\min\{n: y \notin G_n{}^*x\}$.

F. B. Jones in [**3**] introduced and demonstrated the usefulness of the following metrization theorems, one due to R. L. Moore, the other to himself. R. E. Hodel also mentions these theorems in [**2**].

THEOREM 1 (Moore). *A regular, T_1 space X, developable via $G_1, G_2, \ldots$, is metrizable provided that whenever F is closed and $x \in X - F$, there is a positive integer n such that $G_n{}^*x \cap G_n{}^*F = \emptyset$.*

THEOREM 2 (Jones). *A regular, T_1 space X, developable via $G_1, G_2, \ldots$, is metrizable provided that whenever K and F are closed, with K compact and $K \cap F = \emptyset$, there is a positive integer n such that $G_n{}^*K \cap F = \emptyset$.*

Stating the hypotheses of Theorem 2 in terms of the associated semi-metric yields $d(K, F) > 0$ whenever K is closed and compact, F closed, and $K \cap F = \emptyset$. A. V. Arhangelskii [**1**] greatly strengthened Theorem 2 by showing that a Hausdorff space, symmetrizable via d satisfying the above, is metrizable. No assumption of first countability or regularity is made. Later H. W. Martin

Received January 30, 1974.

[4] was able to remove even the Hausdorff assumption to give the theorem below.

THEOREM 3 (Arhangelskii-Martin). *A topological space S, symmetrizable via d satisfying $d(K, F) > 0$ whenever K is compact, F closed, and $K \cap F = \emptyset$, is metrizable.*

The authors have been equally successful in stating and proving Moore's Theorem for symmetrizable spaces.

THEOREM 4. *Let the topological space X be symmetrizable via d. Suppose that for each closed set F and each $x \in X - F$ there exists $\epsilon > 0$ such that $N(x, \epsilon) \cap N(F, \epsilon) = \emptyset$. Then X is metrizable.*

This theorem will be proved in § 2 and several examples will be given in § 3 to show that this is the best possible result of this type.

2. Proofs. We begin with some preliminary lemmas.

LEMMA 1. *Let X be symmetrizable via d and K be a compact subset of X. Then for every sequence (x_n) in K, there is a point $x \in X$ and a subsequence (x_{n_i}) of (x_n) such that $d(x_{n_i}, x) \to 0$.*

Proof. Otherwise, consider $F_1 = \{x_1, x_2, \ldots\}$. If $x \in X - F_1$ we must have $d(x, F) > 0$, so that F_1 is closed. Similarly, $F_n = \{x_n, x_{n+1}, \ldots\}$ is closed and $X - F_1, X - F_2, \ldots$ cover K with no finite subcover.

LEMMA 2. *Let X be symmetrizable via d satisfying the condition that whenever $x \neq y$, there exists $\epsilon > 0$ such that $N(x, \epsilon) \cap N(y, \epsilon) = \emptyset$. Then compact subsets of X are closed.*

Proof. Assume that K is compact, but not closed. Then there is a point $x \in X - K$ with $d(x, K) = 0$. Choose a sequence (x_n) in K for which $d(x_n, x) \to 0$ and put $F = \{x_1, x_2, \ldots\} \cup \{x\}$. If F is not closed, there is a point $y \in X - F$ such that $d(y, F) = 0$. We may assume $d(x_n, y) \to 0$. But this contradicts our hypothesis, since $d(x_n, x) \to 0$, also. Now proceed as in Lemma 1.

LEMMA 3. *Let X be compact and satisfy the hypothesis of Theorem 4. Then x lies in the interior of $N(x, \epsilon)$, for $\epsilon > 0$.*

Proof. Put $S = X - N(x, \epsilon)$ and $L = \{x: d(x, S) = 0\}$. If L is closed, we are through. Otherwise, there is a point $x' \in X - L$ such that $d(x', L) = 0$. Hence there are one-to-one sequences (x_n) in $L - S$ and (y_n) in S such that (1) $d(x_n, x') \to 0$, and (2) $d(x_n, y_n) \to 0$. Since X is compact, by Lemma 1 we may assume that there is a point $y \notin \{x_1, x_2, \ldots\} \cup \{x'\}$ for which $d(y_n, y) \to 0$. Put $F = \{x_1, x_2, \ldots\} \cup \{x'\}$. Then F is compact ($d(x_n, x) \to 0$ implies $x_n \to n$), thus closed by Lemma 2. Hence there does not exist $\epsilon > 0$ for which $N(F, \epsilon)$ and $N(y, \epsilon)$ are disjoint, which is a contradiction.

https://doi.org/10.4153/CJM-1975-102-x Published online by Cambridge University Press

Proof of Theorem 4. Let K be compact, F closed, and $K \cap F = \emptyset$. By Lemma 2, K is closed. Although symmetrizability is not in general hereditary, since K is closed, $d|K \times K$ will induce the relative topology on K. Moreover, the hypothesis on points and closed sets is inherited by K. Thus by Lemma 3, for $x \in K$, x belongs to the interior of $N(x, \epsilon) \cap K$, where the interior is taken relative to K. Hence, for each $x \in K$, choose $\epsilon_x > 0$ such that $N(x, \epsilon_x) \cap N(F, \epsilon_x) = \emptyset$. It follows that there are points $x_1, x_2, \ldots, x_n \in K$ such that $N(x_1, \epsilon_{x_1}), \ldots, N(x_n, \epsilon_{x_n})$ cover K. Putting $\epsilon = \min\{\epsilon_{x_1}, \ldots, \epsilon_{x_n}\}$ we have $d(K, F) \geqq \epsilon > 0$. Thus X is metrizable by Theorem 3.

3. Examples and other conditions on d**.** Let R denote the set of real numbers and Z the integers.

Example 1. Let $X = R$ and d be defined below. $d(x, y) = |x - y|$, if neither x nor y is 0; $d(0, x) = d(x, 0) = 1$, if $x \in X - Z$; $d(0, \pm n) = d(\pm n, 0) = 1/n$, if $n \in Z$, $n > 0$. Then d is a distance function, thereby determining a topology on X. To describe the topology more fully, let F be closed and $0 \notin F$. Then $d(0, F) > 0$, so there exists a positive integer N such that for $n \geqq N$ and $n \in Z$, $\pm n \in X - F$. Thus for $n \geqq N$, there exists $\epsilon_n > 0$ such that the intervals $(\pm n - \epsilon_n, \pm n + \epsilon_n)$ do not meet F. Denote by U the union of these intervals together with 0. Then U is open and it follows that all sets of this form constitute a local base at 0. At $x \neq 0$, a local base consists of open intervals (chosen sufficiently small, depending on x).

From this description one can easily see that X is Hausdorff, regular, and Lindelof, thus paracompact. However, X is not first countable (To see this easily, show that for Hausdorff symmetrizable spaces $x_n \to x$ implies $d(x_n, x) \to 0$. Then observe that $0 \in \overline{X - Z}$, whereas $d(0, X - Z) = 1$ so that no sequence in $X - Z$ converges to 0) or even Fréchet but every point is a G_δ. Also, X is not locally compact.

Let K be a compact subset of X. Then $K - Z$ is bounded. Otherwise, there would be an open neighborhood U of Z for which $K - U$ is unbounded, implying that U together with the open intervals $(z, z + 1)$, $z \in Z$, cover K with no finite subcover. Hence, if K and L are disjoint compact subsets of X with $0 \in K$, then L is compact in the usual topology for the reals and K is contained in a set of the form $K_1 \cup \{0, \pm(n), \pm(n + 1), \ldots\}$ which does not intersect L, where K_1 is compact in the usual topology for the reals. From this, it follows that any two disjoint compact subsets of X have disjoint ϵ-spheres, but X is not metrizable.

Next after several lemmas, we give an example of a compact, non-Hausdorff symmetrizable space wherein distinct points have disjoint ϵ-spheres.

LEMMA 4. *Let X denote the space of Example* 4. *Then there is a positive valued function f on X satisfying*

(1) *$\inf f(K) > 0$, when K is compact, and*

(2) *$\inf f(F) = 0$, when F is closed but not compact.*

https://doi.org/10.4153/CJM-1975-102-x Published online by Cambridge University Press

Proof. Put $f(x) = 1$, if $x \in Z$; $f(x) = |x|$, if $-1 < x < 1$, $x \neq 0$; $f(x) = 1/|x|$, if $|x| > 1$ but $x \in X - Z$. Let K be compact. Then, as seen before, $K - Z$ is bounded. But $K - Z$ must also be bounded away from 0, so that $\inf f(K) > 0$. Suppose that F is closed but not compact. Then there is an infinite set (X is paracompact, F Closed) $\{x_1, x_2, \ldots\}$ in F with no limit point. We may assume $x_n \in X - Z$. If $\{x_1, x_2, \ldots\}$ is unbounded, we are through. Otherwise, it clusters with the usual topology at some point x. But x must be 0 or $\{x_1, x_2, \ldots\}$ would cluster at x with the given topology. Hence, $\inf f(F) = 0$.

A set is *sequentially closed* if it contains the limits of its convergent sequences. A space is *sequential* if sequentially closed sets are closed.

LEMMA 5. *Let X^* denote the one point compactification of X, obtained by adjoining ∞. Then X^* is sequential.*

Proof. Let F be sequentially closed in X^*. Assume $\infty \in F$. If $0 \in F$, F is closed, X^* being first countable at all points outside F. If $0 \in X - F$, since F is sequentially closed there exists a positive integer N with $\pm n \in X - F$, whenever $n \geqq N$. Hence there is a basic neighborhood of 0 which does not meet F, so that F is closed. If $\infty \in X^* - F$, F is a sequentially closed subset of X, thus closed (All symmetrizable spaces are sequential). If F is not compact, it is not countably compact, being closed in paracompact X. Hence, there is an infinite set $\{x_1, x_2, \ldots\}$ in F with no limit point. Clearly $x_1, x_2, x_3, \ldots \to \infty$, since no compact subset of X contains more than finitely many of the terms. Thus, F is not sequentially closed.

LEMMA 6. *X^* is symmetrizable by extending d as follows: $d(\infty, x) = d(x, \infty) = f(x)$, where f is defined in Lemma 4.*

Proof. Let F be d-closed. If $\infty \in X^* - F$, F is d-closed in X, thus closed in X. If F is not compact, we have $d(\infty, F) = \inf f(F) = 0$, which is a contradiction. If $\infty \in F$ and F is not closed, it is not sequentially closed, by Lemma 5. Hence, there is a sequence (x_n) in F converging to $x \in X^* - F$, x real. But this implies that $d(x_n, x) \to 0$, which is a contradiction, establishing that all d-closed sets are closed. Now suppose F is closed. If $\infty \in X^* - F$, then F is closed and compact in X. Thus $d(\infty, F) > 0$, since F is compact. If $\infty \in F$, $F \cap X$ is closed in X, thus d-closed in X, from which it follows that F is d-closed in X^*.

Example 2. X^* is a compact, non-Hausdorff, symmetrizable space in which distinct points have disjoint ϵ-spheres. To see this, note that $N(0, 1/2)$ and $N(\infty, 1/2)$ are disjoint.

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