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BUZANO'S INEQUALITY HOLDS FOR ANY PROJECTION

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Abstract

We show that, in an inner product space H, the inequality

$$\tfrac{1}{2}[\|x\|\,\|y\| + |\langle x, y\rangle|] \geq |\langle Px, y\rangle|$$

is true for any vectors x, y and a projection $P : H \to H$. Applications to norm and numerical radius inequalities of two bounded operators are given.

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1. Introduction

Let $(H, \langle\cdot,\cdot\rangle)$ be an inner product space over the real or complex number field $\mathbb{K}$. The following inequality is well known in the literature as the *Schwarz inequality*

$$\|x\|\,\|y\| \geq |\langle x, y\rangle| \quad \text{for any } x, y \in H. \tag{1.1}$$

Equality holds in (1.1) if and only if there exists a constant $\lambda \in \mathbb{K}$ such that $x = \lambda y$.

In 1985, the author [2] (see also [5, page 38]) established the following refinement of (1.1):

$$\|x\|\,\|y\| \geq |\langle x, y\rangle - \langle x, e\rangle\langle e, y\rangle| + |\langle x, e\rangle\langle e, y\rangle| \geq |\langle x, y\rangle| \tag{1.2}$$

for any $x, y, e \in H$ with $\|e\| = 1$.

Using the triangle inequality for the modulus, (1.2) yields

$$\|x\|\,\|y\| \geq |\langle x, y\rangle - \langle x, e\rangle\langle e, y\rangle| + |\langle x, e\rangle\langle e, y\rangle| \geq 2|\langle x, e\rangle\langle e, y\rangle| - |\langle x, y\rangle|,$$

which implies the *Buzano inequality* [1]

$$\tfrac{1}{2}[\|x\|\,\|y\| + |\langle x, y\rangle|] \geq |\langle x, e\rangle\langle e, y\rangle|, \tag{1.3}$$

which holds for any $x, y, e \in H$ with $\|e\| = 1$.

For other Schwarz and Buzano related inequalities in inner product spaces, see the monographs [3, 5, 7].

2. Buzano's inequality for projection

Assume that $P : H \to H$ is an *orthogonal projection* on H, namely, it satisfies the condition $P^2 = P = P^*$. We obviously have in the operator order of $\mathcal{B}(H)$, the Banach algebra of all linear bounded operators on H, that $0 \le P \le 1_H$.

A family $\{e_j\}_{j\in J}$ of vectors in H is called *orthonormal* if

$$e_j \perp e_k \quad \text{for any } j, k \in J \text{ with } j \neq k \text{ and } \|e_j\| = 1 \text{ for any } j \in J.$$

If the *linear span* of the family $\{e_j\}_{j\in J}$ is *dense* in H, it is an *orthonormal basis* in H.

For an orthonormal family $\mathcal{E} = \{e_j\}_{j\in J}$, we define the operator $P_{\mathcal{E}} : H \to H$ by

$$P_{\mathcal{E}} x := \sum_{j\in J} \langle x, e_j\rangle e_j, \quad x \in H.$$

Then $P_{\mathcal{E}}$ is an *orthogonal projection* and

$$\langle P_{\mathcal{E}} x, y\rangle = \sum_{j\in J} \langle x, e_j\rangle\langle e_j, y\rangle, \quad x, y \in H \quad \text{and} \quad \langle P_{\mathcal{E}} x, x\rangle = \sum_{j\in J} |\langle x, e_j\rangle|^2, \quad x \in H.$$

The particular case when the family reduces to one vector, namely, $\mathcal{E} = \{e\}$, $\|e\| = 1$, is of interest since, in this case, $P_e x := \langle x, e\rangle e$, $x \in H$,

$$\langle P_e x, y\rangle = \langle x, e\rangle\langle e, y\rangle, \quad x, y \in H$$

and Buzano's inequality can be written as

$$\tfrac{1}{2}[\|x\|\,\|y\| + |\langle x, y\rangle|] \ge |\langle P_e x, y\rangle|, \quad x, y, e \in H \text{ with } \|e\| = 1.$$

The following result holds.

THEOREM 2.1. *Let $P : H \to H$ be an orthogonal projection on H. Then, for any $x, y \in H$,*

$$\tfrac{1}{2}[\|x\|\,\|y\| + |\langle x, y\rangle|] \ge |\langle Px, y\rangle|. \tag{2.1}$$

PROOF. From the properties of projection,

$$\begin{aligned}\langle x - Px, y - Py\rangle &= \langle x, y\rangle - \langle Px, y\rangle - \langle x, Py\rangle + \langle Px, Py\rangle \\ &= \langle x, y\rangle - 2\langle Px, y\rangle + \langle P^2 x, y\rangle = \langle x, y\rangle - \langle Px, y\rangle \end{aligned} \tag{2.2}$$

for any $x, y \in H$. By the Schwarz inequality,

$$\|x - Px\|^2\,\|y - Py\|^2 \ge |\langle x - Px, y - Py\rangle|^2 \tag{2.3}$$

for any $x, y \in H$.

Since, by (2.2), $\|x - Px\|^2 = \|x\|^2 - \langle Px, x\rangle$ and $\|y - Py\|^2 = \|y\|^2 - \langle Py, y\rangle$, then, by (2.3), for any $x, y \in H$,

$$(\|x\|^2 - \langle Px, x\rangle)(\|y\|^2 - \langle Py, y\rangle) \ge |\langle x, y\rangle - \langle Px, y\rangle|^2. \tag{2.4}$$

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By the elementary inequality $(ac - bd)^2 \geq (a^2 - b^2)(c^2 - d^2)$, which holds for any real numbers a, b, c, d,

$$(\|x\|\,\|y\| - \langle Px, x\rangle^{1/2}\langle Py, y\rangle^{1/2})^2 \geq (\|x\|^2 - \langle Px, x\rangle)(\|y\|^2 - \langle Py, y\rangle) \tag{2.5}$$

for any $x, y \in H$. Since $\|x\| \geq \langle Px, x\rangle^{1/2}$ and $\|y\| \geq \langle Py, y\rangle^{1/2}$, then

$$\|x\|\,\|y\| - \langle Px, x\rangle^{1/2}\langle Py, y\rangle^{1/2} \geq 0,$$

for any $x, y \in H$. Now, by (2.4) and (2.5),

$$(\|x\|\,\|y\| - \langle Px, x\rangle^{1/2}\langle Py, y\rangle^{1/2})^2 \geq |\langle x, y\rangle - \langle Px, y\rangle|^2$$

for any $x, y \in H$, which, by taking the square root, is equivalent to

$$\|x\|\,\|y\| \geq \langle Px, x\rangle^{1/2}\langle Py, y\rangle^{1/2} + |\langle x, y\rangle - \langle Px, y\rangle| \tag{2.6}$$

for any $x, y \in H$. By the Schwarz inequality for nonnegative operators,

$$\langle Px, x\rangle^{1/2}\langle Py, y\rangle^{1/2} \geq |\langle Px, y\rangle| \tag{2.7}$$

for any $x, y \in H$. On making use of (2.6), (2.7) and the triangle inequality for the modulus,

$$\begin{aligned}\|x\|\,\|y\| &\geq \langle Px, x\rangle^{1/2}\langle Py, y\rangle^{1/2} + |\langle x, y\rangle - \langle Px, y\rangle| \\ &\geq |\langle Px, y\rangle| + |\langle x, y\rangle - \langle Px, y\rangle| \geq |\langle Px, y\rangle| + |\langle Px, y\rangle| - |\langle x, y\rangle|,\end{aligned}$$

which is equivalent to the desired result (2.1). □

Let $\mathcal{E} = \{e_j\}_{j\in J}$ be an orthonormal family in H. From Theorem 2.1, for any $x, y \in H$,

$$\frac{1}{2}[\|x\|\,\|y\| + |\langle x, y\rangle|] \geq \Big|\sum_{j\in J}\langle x, e_j\rangle\langle e_j, y\rangle\Big|. \tag{2.8}$$

The inequality (2.8) provides a generalisation of Buzano's inequality for orthonormal families $\mathcal{E} = \{e_j\}_{j\in J}$.

3. Inequalities for the norm and numerical radius

Let $(H; \langle\cdot,\cdot\rangle)$ be a complex Hilbert space. The *numerical range* of an operator T is the subset of the complex numbers $\mathbb{C}$ given by [8, page 1]

$$W(T) = \{\langle Tx, x\rangle, x \in H, \|x\| = 1\}.$$

The *numerical radius* $w(T)$ of an operator T on H is defined by [8, page 8]

$$w(T) = \sup\{|\lambda|, \lambda \in W(T)\} = \sup\{|\langle Tx, x\rangle|, \|x\| = 1\}.$$

It is well known that $w(\cdot)$ is a norm on the Banach algebra $B(H)$ and therefore

$$w(T) \leq \|T\| \leq 2w(T) \quad \text{for any } T \in B(H).$$

Utilising Buzano's inequality (1.3), we obtained the following inequality for the numerical radius (see [4] or [6]).

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THEOREM 3.1. *Let $(H; \langle\cdot,\cdot\rangle)$ be a Hilbert space and $T : H \to H$ a bounded linear operator on H. Then*

$$w^2(T) \le \tfrac{1}{2}[w(T^2) + \|T\|^2]. \tag{3.1}$$

The constant $\frac{1}{2}$ is the best possible in (3.1).

The following theorem gives a general result for the product of two operators [8, page 37].

THEOREM 3.2. *If A, B are two bounded linear operators on the Hilbert space $(H, \langle\cdot,\cdot\rangle)$, then $w(AB) \le 4w(A)w(B)$. In the case in which $AB = BA$, then $w(AB) \le 2w(A)w(B)$. The constant two is the best possible here.*

The following results are also well known [8, page 38].

THEOREM 3.3. *If A is a unitary operator that commutes with another operator B, then*

$$w(AB) \le w(B). \tag{3.2}$$

If A is an isometry and $AB = BA$, then (3.2) also holds true.

We say that A and B *double commute* if $AB = BA$ and $AB^* = B^*A$. The following result holds [8, page 38].

THEOREM 3.4. *If the operators A and B double commute, then*

$$w(AB) \le w(B)\|A\|.$$

As a consequence of the above, we have the following corollary [8, page 39].

COROLLARY 3.5. *Let A be a normal operator commuting with B. Then*

$$w(AB) \le w(A)w(B).$$

For other inequalities for the numerical radius, see the recent monograph [7] and the references therein.

THEOREM 3.6. *Let $P : H \to H$ be an orthogonal projection on the Hilbert space $(H, \langle\cdot,\cdot\rangle)$. If A, B are two bounded linear operators on H, then*

$$|\langle BPAx, x\rangle| \le \tfrac{1}{2}[\|Ax\|\,\|B^*x\| + |\langle BAx, x\rangle|] \tag{3.3}$$

and

$$\|BPAx\| \le \tfrac{1}{2}[\|Ax\|\,\|B\| + \|BAx\|] \tag{3.4}$$

for any $x \in H$. Moreover,

$$w(BPA) \le \tfrac{1}{2}[\|A\|\,\|B\| + w(BA)] \tag{3.5}$$

and

$$\|BPA\| \le \tfrac{1}{2}[\|A\|\,\|B\| + \|BA\|]. \tag{3.6}$$

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PROOF. From the inequality (2.1),

$$|\langle PAx, B^*y\rangle| \le \tfrac{1}{2}[\|Ax\|\,\|B^*y\| + |\langle Ax, B^*y\rangle|].$$

This is equivalent to

$$|\langle BPAx, y\rangle| \le \tfrac{1}{2}[\|Ax\|\,\|B^*y\| + |\langle BAx, y\rangle|] \tag{3.7}$$

for any $x, y \in H$. If we take $y = x$ in (3.7), then we get (3.3).

Taking the supremum over $y \in H$ with $\|y\| = 1$ in (3.7) yields

$$\begin{aligned}\|BPAx\| &= \sup_{\|y\|=1} |\langle BPAx, y\rangle| \le \frac{1}{2} \sup_{\|y\|=1} [\|Ax\|\,\|B^*y\| + |\langle BAx, y\rangle|]\\ &\le \frac{1}{2}\Big[\|Ax\| \sup_{\|y\|=1} \|B^*y\| + \sup_{\|y\|=1} |\langle BAx, y\rangle|\Big] = \frac{1}{2}[\|Ax\|\,\|B\| + \|BAx\|]\end{aligned}$$

for any $x \in H$. The inequalities (3.5) and (3.6) follow from (3.3) and (3.4) by taking the supremum over $x \in H$ with $\|x\| = 1$. □

COROLLARY 3.7. *Let $P : H \to H$ be an orthogonal projection on the Hilbert space $(H, \langle\cdot,\cdot\rangle)$. If A, B are two bounded linear operators on H, then*

$$|\langle APAx, x\rangle| \le \tfrac{1}{2}[\|Ax\|\,\|A^*x\| + |\langle A^2x, x\rangle|]$$

and

$$\|APAx\| \le \tfrac{1}{2}[\|Ax\|\,\|A\| + \|A^2x\|]$$

for any $x \in H$. Moreover,

$$w(APA) \le \tfrac{1}{2}[\|A\|^2 + w(A^2)]$$

and

$$\|APA\| \le \tfrac{1}{2}[\|A\|^2 + \|A^2\|].$$

Let $e \in H$ with $\|e\| = 1$. If we write the inequalities (3.3) and (3.4) for the projection P_e defined by $P_e x = \langle x, e\rangle e$, $x \in H$, then

$$|\langle Ax, e\rangle|\,|\langle Be, x\rangle| \le \tfrac{1}{2}[\|Ax\|\,\|B^*x\| + |\langle BAx, x\rangle|] \tag{3.8}$$

and

$$|\langle Ax, e\rangle|\,\|Be\| \le \tfrac{1}{2}[\|Ax\|\,\|B\| + \|BAx\|] \tag{3.9}$$

for any $x \in H$. Taking the supremum over $x \in H$, $\|x\| = 1$ in (3.9) yields

$$\|A^*e\|\,\|Be\| \le \tfrac{1}{2}[\|A\|\,\|B\| + \|BA\|] \tag{3.10}$$

for any $e \in H$, $\|e\| = 1$. If, in (3.10), we take $B = A$, then

$$\|A^*e\|\,\|Ae\| \le \tfrac{1}{2}[\|A\|^2 + \|A^2\|]$$

for any $e \in H$, $\|e\| = 1$. If, in (3.8), we take $B = A$, then

$$|\langle Ax, e\rangle|\,|\langle e, A^*x\rangle| \le \tfrac{1}{2}[\|Ax\|\,\|A^*x\| + |\langle A^2x, x\rangle|]$$

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for any $x \in H$ and $e \in H$ with $\|e\| = 1$ and, in particular,

$$|\langle Ae, e\rangle|^2 \le \tfrac{1}{2}[\|Ae\|\,\|A^*e\| + |\langle A^2e, e\rangle|] \tag{3.11}$$

for any $e \in H$, $\|e\| = 1$. Taking the supremum over $e \in H$, $\|e\| = 1$ in (3.11), we recapture the result in Theorem 3.1.

For a given operator T we consider the modulus of T defined as $|T| := (T^*T)^{1/2}$.

Corollary 3.8. *Let $P : H \to H$ be an orthogonal projection on the Hilbert space $(H, \langle\cdot,\cdot\rangle)$. If A, B are two bounded linear operators on H, then*

$$w(BPA) \le \tfrac{1}{2}w(BA) + \tfrac{1}{4}\|\,|A|^2 + |B^*|^2\|. \tag{3.12}$$

In particular,

$$w(APA) \le \tfrac{1}{2}w(A^2) + \tfrac{1}{4}\|\,|A|^2 + |A^*|^2\|.$$

Proof. From the inequality (3.3),

$$\begin{aligned}|\langle BPAx, x\rangle| &\le \tfrac{1}{2}[\|Ax\|\,\|B^*x\| + |\langle BAx, x\rangle|]\\ &\le \tfrac{1}{2}|\langle BAx, x\rangle| + \tfrac{1}{4}[\|Ax\|^2 + \|B^*x\|^2]\end{aligned} \tag{3.13}$$

for any $x \in H$, where, for the second inequality, we used the elementary inequality

$$ab \le \tfrac{1}{2}(a^2 + b^2), \quad a, b \in \mathbb{R}.$$

Since

$$\begin{aligned}\|Ax\|^2 + \|B^*x\|^2 &= \langle Ax, Ax\rangle + \langle B^*x, B^*x\rangle = \langle A^*Ax, x\rangle + \langle BB^*x, x\rangle\\ &= \langle(|A|^2 + |B^*|^2)x, x\rangle\end{aligned}$$

for any $x \in H$, then, from (3.13),

$$|\langle BPAx, x\rangle| \le \tfrac{1}{2}|\langle BAx, x\rangle| + \tfrac{1}{4}\langle(|A|^2 + |B^*|^2)x, x\rangle \tag{3.14}$$

for any $x \in H$. Taking the supremum over $x \in H$, $\|x\| = 1$ in (3.14) gives the desired result (3.12). □

We observe, by (3.11), that

$$\begin{aligned}|\langle Ae, e\rangle|^2 &\le \tfrac{1}{2}[\|Ae\|\,\|A^*e\| + |\langle A^2e, e\rangle|]\\ &\le \tfrac{1}{2}|\langle A^2e, e\rangle| + \tfrac{1}{4}[\|Ae\|^2 + \|A^*e\|^2]\\ &= \tfrac{1}{2}|\langle A^2e, e\rangle| + \tfrac{1}{4}\langle(|A|^2 + |A^*|^2)e, e\rangle\end{aligned} \tag{3.15}$$

for any $e \in H$ with $\|e\| = 1$. Taking the supremum over $e \in H$, $\|e\| = 1$ in (3.15) gives

$$w^2(A) \le \tfrac{1}{2}w(A^2) + \tfrac{1}{4}\||A|^2 + |A^*|^2\|, \tag{3.16}$$

for any bounded linear operator A. Since

$$\|\,|A|^2 + |A^*|^2\| \le \|\,|A|^2\| + \|\,|A^*|^2\| = 2\|A\|^2,$$

the inequality (3.16) is better than the inequality in Theorem 3.1.

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References

[1] M. L. Buzano, 'Generalizzazione della diseguaglianza di Cauchy–Schwarz', *Rend. Sem. Mat. Univ. Politech. Torino* **31** (1971/73), 405–409; (1974) (in Italian).

[2] S. S. Dragomir, 'Some refinements of Schwartz inequality', *Simpozionul de Matematici şi Aplicaţii, Timişoara, Romania* **1–2** (1985), 13–16.

[3] S. S. Dragomir, *Advances in Inequalities of the Schwarz, Grüss and Bessel Type in Inner Product Spaces* (Nova Science Publishers, Inc., Hauppauge, NY, 2005), viii+249.

[4] S. S. Dragomir, 'Inequalities for the norm and the numerical radius of linear operators in Hilbert spaces', *Demonstratio Math.* **40**(2) (2007), 411–417.

[5] S. S. Dragomir, *Advances in Inequalities of the Schwarz, Triangle and Heisenberg Type in Inner Product Spaces* (Nova Science Publishers, Inc., New York, 2007), xii+243.

[6] S. S. Dragomir, 'Some inequalities for the norm and the numerical radius of linear operators in Hilbert spaces', *Tamkang J. Math.* **39**(1) (2008), 1–7.

[7] S. S. Dragomir, *Inequalities for the Numerical Radius of Linear Operators in Hilbert Spaces*, Springer Briefs in Mathematics (Springer, Cham, Switzerland, 2013), x+120.

[8] K. E. Gustafson and D. K. M. Rao, *Numerical Range* (Springer, New York, 1997).

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