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Gröbner Flags and Gorenstein Algebras

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Abstract. The goal of this paper is to study the Koszul property and the property of having a Gröbner basis of quadrics for classical varieties and algebras as canonical curves, finite sets of points and Artinian Gorenstein algebras with socle in low degree. Our approach is based on the notion of Gröbner flags and Koszul filtrations. The main results are the existence of a Gröbner basis of quadrics for the ideal of the canonical curve whenever it is defined by quadrics, the existence of a Gröbner basis of quadrics for the defining ideal of $s \leqslant 2n$ points in general linear position in $\mathbf{P}^n$, and the Koszul property of the 'generic' Artinian Gorenstein algebra of socle degree 3.

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Introduction

A standard graded K-algebra R is Koszul if K has a linear resolution as an R-module. The algebra R is said to be quadratic if its defining ideal I is generated by polynomials of degree 2 and it is said to be G-quadratic if I has a Gröbner basis of quadrics in some coordinate system and with respect to some term order. It is well known that G-quadratic algebras are Koszul, that Koszul algebras are quadratic, and that these implications are generally strict. These properties appear naturally in various contexts and their study have attracted the attention of many researchers in the last three decades. For instance, many classical varieties (Grassmannians, Schubert varieties, flag manifolds, etc.) are Koszul and even G-quadratic in their natural embedding and any projective variety can be embedded in such a way that it is G-quadratic, see [12, 18]. The study of the Koszulness and of the G-quadraticity of semigroup rings (i.e. toric varieties) and their relation with the underlying combinatorial objects gave rise to beautiful theories and results, see, for instance, [3, 16, 22, 28]. The relation between the Koszul property (and more generally the study of the resolution of the residue field) and the structure of the Yoneda Hopf algebra $\mathrm{Ext}^*_R(K, K)$ is another important aspect of the theory, see [20, 21, 26].

In this paper we are mainly concerned with the study of the G-quadraticity and of the Koszulness of three classes of algebras: coordinate rings of canonical curves, coordinate rings of finite sets of points and Artinian Gorenstein algebras with socle

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in degree 3. Our approach is based on the notion of Koszul filtrations and Gröbner flags. Koszul filtrations have been introduced in [7]. Roughly speaking, a Koszul filtration for an algebra R is a family of ideals of R generated by linear forms such that any ideal of the family can be filtered in such a way that all the successive colon ideals belong to the family. If an algebra has a Koszul filtration then it is necessarily Koszul. A Gröbner flag is just a Koszul filtration supported on a single complete flag of linear forms. In Section 2 it is proved that if an algebra has a Gröbner flag then it is G-quadratic.

Section 3 is devoted to points in projective space. Let X be a set of s points in general linear position in $\mathbf{P}^n$ with $s \leqslant 2n$ and let R be its coordinate ring. Kempf proved in [19] that R is Koszul and in [7] it is proved that R has a Koszul filtration. The Koszul filtration of R given in [7] is not supported on a flag, but we show how to modify the argument of [7] to get a Gröbner flag, hence proving that R is G-quadratic.

Section 4 contains a discussion of whether the above-mentioned results for points can be extended to a larger number of points in linear general position. It is proved in [7] that s points in $\mathbf{P}^n$ with generic coordinates are Koszul if and only if $s \leqslant 1 + n + (n^2/4)$. So one may ask whether Kempf's theorem can be extended to $2n + 1$ (or more) points in linear general position with a quadratic defining ideal. The answer is negative. We show that there exists a set of 9 points in $\mathbf{P}^4$ which are in general linear position and are quadratic but not Koszul. These 9 points have been obtained, with the help of the computer algebra system CoCoA [6], as a generic lifting of a quadratic Artinian algebra, described by Roos in [25], with Hilbert series $1 + 4z + 4z^2$ which is not Koszul. The points of the above set are in linear general position; even more, we have checked that the minimal free resolution over the polynomial ring of any subset of the 9 points is the generic one.

Butler [5] and Polishchuk [24] asked whether the homogeneous coordinate ring of a projectively normal, smooth, connected, complex curve is Koszul, provided it is quadratic. Sturmfels gave a negative answer to this question, see [29]. His example is a curve in $\mathbf{P}^5$ with a Hilbert series, say $P(z)$, such that $1/P(-z)$ has a negative coefficient and it is known that no algebra with this Hilbert series can be Koszul. We show that Roos's example can be lifted also to get a quadratic non-Koszul smooth curve in $\mathbf{P}^5$ such that $1/P(-z)$ has positive coefficients.

The goal of Section 5 is to show that the coordinate ring of a canonical curve is G-quadratic provided it is quadratic. Let $\mathcal{C}$ be a smooth algebraic curve of genus g over an algebraically closed field of characteristic zero. If $\mathcal{C}$ is not hyperelliptic, then the canonical sheaf on $\mathcal{C}$ gives a canonical embedding $\mathcal{C} \to \mathbf{P}^{g-1}$ and the coordinate ring $R_{\mathcal{C}}$ of $\mathcal{C}$ in this embedding is the canonical ring of $\mathcal{C}$. It is known that $R_{\mathcal{C}}$ is quadratic unless $\mathcal{C}$ is a trigonal curve of genus $g \geqslant 5$ or a plane quintic. Vishik and Finkelberg [30] proved that $R_{\mathcal{C}}$ is Koszul if it is quadratic (see also [24] and [23]). We prove that if $R_{\mathcal{C}}$ is quadratic then it has a Gröbner flag.

Artinian Gorenstein algebras with socle in degree 3 are in bijective correspondence with cubic forms via the so-called Macaulay inverse system. Explicitly, every cubic form $F \in K[y_1, \dots, y_n]$ corresponds to the Artinian Gorenstein algebra $R_F = K[x_1, \dots, x_n]/I_F$ which has the socle in degree 3. Here I_F is the ideal of the polynomials $G(x_1, \dots, x_n)$ such that the $G(\partial/\partial y_1, \dots, \partial/\partial y_n)(F) = 0$. The aim of Section 6 is to study the Koszulness and the G-quadraticity of R_F. To avoid trivial cases, we assume that $n > 2$ and that F is not a cone (i.e. F cannot be represented as a form with less than n variables). In general R_F need not be quadratic and we do not know a simple description of those F such that R_F is quadratic. It is not difficult to show that if F is smooth then R_F cannot be G-quadratic. Our main result is that R_F has a Koszul filtration if F is generic and that R_F has a Gröbner flag if F is singular and generic.

Subsection 6.1 is devoted to plane cubics (i.e. $n = 3$). It is easy to see that R_F is quadratic if and only if it is a complete intersection of quadrics, so that R_F is quadratic if and only if it is Koszul. We show that R_F is quadratic if and only if no polar quadric of F is a double line, i.e. the space of the first derivatives of F does not contain a rank 1 quadric. Further we show that R_F is G-quadratic if and only if F is singular and no polar quadric of F is a double line. Recall that a plane cubic F is said to be anharmonic if it is in the Zariski closure of the PGL_3-orbit of Fermat cubic $y_1^3 + y_2^3 + y_3^3$. Dolgachev and Kanev shown that F is anharmonic if and only if F has a polar quadric which is a double line, see [10]. Therefore we have that R_F is quadratic if and only if F is not anharmonic and that R_F is G-quadratic if and only if F is not anharmonic and singular. The first of this two statements has been observed also by Eisenbud and Stillman (unpublished), see [11].

Subsection 6.2 is devoted to space cubics (i.e. $n = 4$). We show that R_F is quadratic if and only if R_F is Koszul and this is equivalent to the fact that no polar quadric of F is a double line. Also we show that R_F is G-quadratic if and only if no polar quadric of F is a double line and F is singular.

We do not know whether in higher embedding dimension (i.e. $n > 4$) the quadraticity of R_F implies already its Koszulness and whether the quadraticity of R_F can be characterized in terms of ranks of polar quadrics. But for $n > 5$ the fact that no polar quadric is a double line does not suffice to have the quadraticity of R_F.

1. Notation and Generalities

Let K be a field. For the sake of simplicity we will always assume that K is algebraically closed of characteristic 0.

A graded commutative Noetherian K-algebra $R = \oplus_{i \in \mathbf{N}} R_i$ is said to be standard graded if $R_0 = K$ and R is generated (as a K-algebra) by elements of degree 1. We may present such an algebra R as a quotient of a polynomial ring $K[x_1, \dots, x_n]$ by a homogeneous ideal I. The presentation is minimal if $n = \dim R_1$ and in this case the ideal I is non-degenerate, i.e. $I \subseteq (x_1, \dots, x_n)^2$. The Hilbert function of R is defined as $H_R(t) = \dim_K(R_t)$ with $t \in \mathbf{N}$.

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The Hilbert series $P_R(z)$ of R is by definition $P_R(z) = \sum_{t \geqslant 0} H_R(t)z^t$.

A standard graded algebra R is said to be Koszul if K has a linear resolution as a graded R-module. This means that the graded R-module $\mathrm{Tor}_i^R(K, K)$ is zero in degree j for all $j \neq i$ and for all $i \geqslant 0$.

An ideal I of a polynomial ring is said to be quadratic if it is generated by quadrics, i.e. homogeneous polynomials of degree 2. A standard graded algebra R is said to be quadratic if it can be presented as a quotient $K[x_1, \dots, x_n]/I$ where I is quadratic. Since we want a polynomial ring to be a quadratic algebra, in the previous definitions I is allowed to be 0.

A standard graded algebra R is said to be G-quadratic if it can be presented as a quotient $K[x_1, \dots, x_n]/I$ where I is an ideal with the following property: there exists a set of coordinates $x'_1, \dots, x'_n$ (linear combinations of $x_1, \dots, x_n$) and a monomial order with respect to which I has a Gröbner basis of quadrics.

It is well known that a G-quadratic algebra is Koszul and that a Koszul algebra is quadratic. None of these implications can be reversed in general. For instance, it is known that certain generic complete intersections of quadrics are not G-quadratic [12, Corollary 20], although every complete intersection of quadrics is a Koszul algebra. For an updated survey and a rich bibliography on these topics, we refer the reader to the recent paper of Fröberg [15].

2. Koszul Filtrations and Gröbner Flags

Let us recall the definition of Koszul filtration:

DEFINITION 2.1. Let R be a standard graded K-algebra. A family $\mathcal{F}$ of ideals of R is said to be a Koszul filtration of R if:

(1) Every ideal $I \in \mathcal{F}$ is generated by linear forms.
(2) The ideal 0 and the maximal homogeneous ideal $\mathcal{M}$ of R belong to $\mathcal{F}$.
(3) For every $I \in \mathcal{F}$ different from 0, there exists $J \in \mathcal{F}$ such that $J \subset I$, I/J is cyclic and $J{:}I \in \mathcal{F}$.

This notion has been introduced in [7]. It was inspired by the work of Herzog, Hibi and Restuccia on strongly Koszul algebras [16]. It has been proved in [7], Prop.1.2, that if R has a Koszul filtration $\mathcal{F}$ then all the ideals of $\mathcal{F}$ have a linear R-free resolution. In particular R is Koszul.

We now present the notion of Gröbner flag.

DEFINITION 2.2. Let R be a standard graded K-algebra. A Gröbner flag of R is a complete flag of R_1, say $F{:}\ V_0 = 0 \subset V_1 \subset V_2 \subset \dots \subset V_{n-1} \subset V_n = R_1$, where V_i is a space of dimension i, such that the ideals (V_i) form a Koszul filtration of R, that is, for every $i = 1, \dots, n$, there exists j_i such tat $(V_{i-1}){:}(V_i) = (V_{j_i})$.

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This is equivalent to say that there exists an ordered system of generators $l_1, \ldots, l_n$ of R_1 (a basis of the flag) such that for every $i = 1, \ldots, n$ we have

$$(l_1, \ldots, l_{i-1})\colon l_i = (l_1, l_2, \ldots, l_{j_i}).$$

The sequence of numbers $j_1, j_2, \ldots, j_n$ will be denoted by $j(F)$.

We need the following auxiliary result.

LEMMA 2.3. *Let R be a standard graded algebra and assume that R has a Gröbner flag F. Then the Hilbert series of R depends only on j(F).*

Proof. For every $i = 1, \ldots, n$ we have short exact sequences

$$0 \to R/(V_{j_i})[-1] \to R/(V_{i-1}) \to R/(V_i) \to 0.$$

It is clear that $P_{R/(V_n)}(z) = 1$ and $n \geqslant j_i \geqslant i-1$ for every $i = 1, \ldots, n$. Hence

$$P_{R/(V_{n-1})}(z) = \begin{cases} 1+z & \text{if } j_n = n \\ \dfrac{1}{1-z} & \text{if } j_n = n-1 \end{cases}.$$

Similarly, one proves, by decreasing induction on i, that for every i the Hilbert series of $R/(V_i)$ depends only on $j(F)$. $\square$

The importance of Gröbner flags is explained by the following theorem.

THEOREM 2.4. *Let R be a standard graded algebra. If R has a Gröbner flag, then R is G-quadratic.*

Proof. Let $F\colon V_0 = 0 \subset V_1 \subset V_2 \subset \ldots \subset V_{n-1} \subset V_n = R_1$ be a Gröbner flag of R with $j(F) = j_1, \ldots, j_n$. Let $l_1, \ldots, l_n$ be a basis of the flag. Consider the presentation $K[x_1, \ldots, x_n]/I \simeq R$ of R obtained by sending x_i to l_i. For every $i = 1, \ldots, n$ by assumption we have $(l_1, \ldots, l_{i-1})\colon l_i = (l_1, l_2, \ldots, l_{j_i})$. Hence, for all k with $i \leqslant k \leqslant j_i$ we get a relation $l_k l_i = \sum_{h=1}^{i-1} l_h L_{k,i,h}$ where $L_{k,i,h} \in R_1$. Therefore for every $i = 1, \ldots, n$ and every k with $i \leqslant k \leqslant j_i$, in I there are quadrics $Q_{i,k} = x_i x_k - \sum_{h=1}^{i-1} x_h L_{k,i,h}$ where the $L_{k,i,h}$ are linear forms. Now consider a term order τ on $K[x_1, \ldots, x_n]$ such that $\mathrm{in}_\tau(Q_{i,k}) = x_i x_k$. For example one can take the degree reverse lexicographic order induced by the total order $x_n > x_{n-1} > \ldots > x_1$. We want to prove that the $Q_{i,k}$ are a Gröbner basis of I with respect to τ. Let $J = (x_i x_k\colon 1 \leqslant i \leqslant n \text{ and } i \leqslant k \leqslant j_i)$. It suffices to show that $K[x_1, \ldots, x_n]/J$ and R have the same Hilbert series. This follows from 2.3 if we show that $K[x_1, \ldots, x_n]/J$ has a Gröbner flag G with $j(G) = j(F)$. The Gröbner flag of $K[x_1, \ldots, x_n]/J$ is given by $G\colon 0 = W_0 \subset W_1 \subset W_2 \subset \ldots \subset W_n$ where W_i is the space generated by the residue classes of $x_1, \ldots, x_i$. To show that G is a Gröbner flag with $j(G) = j(F)$ one has to show that $(J + (x_1, \ldots, x_{i-1}))\colon x_i = J + (x_1, \ldots, x_{j_i})$ and this is a simple exercise on monomial ideals. $\square$

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The following proposition contains a characterization of the algebras which have a Gröbner flag, in terms of their initial ideals. This result has been obtained independently also by Blum in [5].

PROPOSITION 2.5. *Let R be a standard graded algebra. The following conditions are equivalent:*

(1) *R has a Gröbner flag,*
(2) *there exists a presentation of R, say $R \simeq K[x_1, \dots, x_n]/I$, such that if τ is the degree reverse lexicographic order induced by the total order $x_n > x_{n-1} > \dots > x_1$, then $\mathrm{in}_\tau(I)$ is generated by monomials of degree 2 and if $x_a x_b \in \mathrm{in}_\tau(I)$ with $a \leqslant b$ then $x_a x_c \in \mathrm{in}_\tau(I)$ for all $a \leqslant c \leqslant b$.*

Proof. That (1) implies (2) is a consequence of the proof of Theorem 2.4. In order to show that (2) implies (1) let us set:

$$A_i = \{b\colon i \leqslant b \leqslant n \quad \text{and} \quad x_i x_b \in \mathrm{in}_\tau(I)\}.$$

Then set $j_i = \max A_i$ if $A_i \neq \emptyset$ and $j_i = i-1$ otherwise. It is enough to prove that $(I + (x_1, \dots, x_{i-1}))\colon x_i = I + (x_1, \dots, x_{j_i})$.

We start with the inclusion $(I + (x_1, \dots, x_{i-1}))\colon x_i \supseteq I + (x_1, \dots, x_{j_i})$. If $j_i = i-1$ there is nothing to prove. So assume that $j_i \geqslant i$ and let k any number between i and j_i. Since $x_i x_k \in \mathrm{in}_\tau(I)$, there exists $f \in I$ such that $\mathrm{in}(f) = x_i x_k$ and we may take f to be reduced in the sense that among the terms of f only the initial one belongs to $\mathrm{in}_\tau(I)$. Then $f = \lambda x_i x_k + g$ with $\lambda \in K^*$ and $g \in (x_1, \dots, x_{i-1})$. It follows that $x_k \in (I + (x_1, \dots, x_{i-1}))\colon x_i$.

As for the other inclusion, let $f \in (I + (x_1, \dots, x_{i-1}))\colon x_i$ and assume by contradiction that $f \notin I + (x_1, \dots, x_{j_i})$. We may assume that no term of f belongs either to $\mathrm{in}_\tau(I)$ or to $(x_1, \dots, x_{j_i})$. Since $f x_i \in I + (x_1, \dots, x_{i-1})$, we have that $f x_i = g + h$ with $g \in I$ and $h \in (x_1, \dots, x_{i-1})$. Then by construction $\mathrm{in}(f) x_i = \mathrm{in}(g) \in \mathrm{in}_\tau(I)$. Since $\mathrm{in}(f) \notin \mathrm{in}_\tau(I)$, we have that x_i times one of the variables of $\mathrm{in}(f)$ belongs to $\mathrm{in}_\tau(I)$. Note that the variables of f and hence of $\mathrm{in}(f)$ have indices $> j_i$. This is a contradiction because of the definition of j_i. $\square$

We have seen that an algebra with a Gröbner flag is G-quadratic. The following example shows that the converse does not hold.

EXAMPLE 2.6. Let R be the algebra $K[x, y, z]/(x^2, xy, yz, z^2)$. Clearly R is G-quadratic. We claim that R does not have a Gröbner flag. Assume, by contradiction, that $0 = V_0 \subset V_1 \subset V_2 \subset V_3 = R_1$ is a Gröbner flag of R. Since the depth of R is 0, the ideal $0\colon (V_1)$ cannot be 0 and hence V_1 must be generated by an element whose square is 0. It follows that either $V_1 = \langle x \rangle$ or $V_1 = \langle z \rangle$ (where by abuse of notation we denote by $x, z, \dots$ the classes of $x, z, \dots$ in R). By symmetry we may assume that $V_1 = \langle x \rangle$. Then $0\colon (V_1) = (x, y)$. It follows that $V_2 = \langle x, y \rangle$. But $(V_1)\colon (V_2) = (x, z)$ and this is a contradiction since $V_2 \neq \langle x, z \rangle$.

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Let us mention also that to have a Koszul filtration is a stronger property than just being Koszul. A necessary condition for an Artinian algebra $R = A/I$ to have a Koszul filtration is that the ideal I contains a reducible quadric. Now if I is the ideal generated by 5 generic quadrics in 5 variables, then $R = A/I$ is Koszul because it is a complete intersection. On the other hand a dimensional argument shows that I does not contain reducible quadrics and hence R does not have a Koszul filtration. Explicitely, the ideal

$$I = (x_1^2 - x_3x_4, x_2^2 - x_1x_5, x_3^2 - x_2x_5, x_4^2 - x_1x_2, x_5^2 - 3x_1x_4)$$

in $A = K[x_1, x_2, x_3, x_4, x_5]$ is a complete intersection of quadrics, but one can check that every nonzero linear combination of the above quadratic forms is a quadric of rank at least three.

The first instance where Gröbner flags arise is described in the following lemma.

LEMMA 2.7. *Let R be a standard graded algebra. If there exists a nonzero element $l \in R_1$ such that $l^2 = 0$ and $lR_1 = R_2$, then R has a Gröbner flag.*

Proof. Since $0{:}\,l \supseteq (l) \supset R_2$, then $0{:}\,l$ does not have generators in degree > 1. Therefore we have $0{:}\,l = (l, l_2, \dots, l_r)$ for suitable independent linear forms $l_2, \dots, l_r$ in R_1. We can complete $l, l_2, \dots, l_r$ to a basis $\{l, l_2, \dots, l_r, l_{r+1}, \dots, l_n\}$ of R_1. We claim that the flag associated to this ordered basis is a Gröbner flag of R. To this end one has just to note that, by construction, we have $0{:}\,l = (l, l_2, \dots, l_r)$ and if $i \geqslant 2$, then $(l, l_2, \dots, l_{i-1}){:}\,l_i = \mathcal{M}$ because $R_2 \subset (l)$. □

Let R be a standard graded K-algebra. Given an element $l \in R_1$, the short exact sequence

$$0 \to R/(0{:}\,l)(-1) \to R \to R/lR \to 0$$

gives

$$P_R(z) = P_{R/lR}(z) + zP_{R/(0:l)}(z) \tag{1}$$

Let $\phi\colon R_1 \xrightarrow{l} R_2$ be the map given by multiplication by l. We let

$$\mathrm{rank}(l) = \dim_K(lR_1) = \dim_K \mathrm{Image}(\phi).$$

One has

$$H_R(1) = \mathrm{rank}(l) + \dim_K(0{:}\,l)_1. \tag{2}$$

In order to determine the existence of Koszul filtrations and of Gröbner flags we need to detect linear forms with small rank or/and linear forms whose square is 0. One has:

LEMMA 2.8. *Let R be a standard graded algebra with $H_R(1) = n$ and $H_R(2) = m$.*

(1) *Let V be a subspace of R_1. Set $a = \dim V$, $b = \dim VR_1$, $c = \dim(0{:}_{R_1} V)$.*

If $a+b+c-n-2 \geqslant 0$, *then there exists a nonzero* $l \in V$ *such that* $\operatorname{rank}(l) < b$. *In particular:*

(1.1) *If* $m \geqslant 2$ *then there exists* $l \in R_1$ *with* $\operatorname{rank}(l) < m$.
(1.2) *Let* $l \in R_1$, $l \neq 0$ *and* $V = 0:_{R_1} l$. *If* $\operatorname{rank}(l) < \dim VR_1$ *then there exists a nonzero* $l_1 \in V$ *such that* $\operatorname{rank}(l_1) < \dim VR_1$.

(2) *If* $m < n$, *then there exists a nonzero* $l \in R_1$ *such that* $l^2 = 0$.

Proof. (1) Set $W = 0:_{R_1} V$ and let $x \in V$. The multiplication by x from R_1 to R_2 factors through R_1/W and its image is contained in VR_1. Therefore the rank of x is the rank of the map from R_1/W to VR_1 given by multiplication by x. Let $x_1, \ldots, x_a$ be a basis of V, $z_1, \ldots, z_{n-c}$ be a basis of R_1/W and $y_1, \ldots, y_b$ be a basis of VR_1; we have relations

$$z_i x_j = \sum_k \lambda_{ij}^{(k)} y_k,$$

with $\lambda_{ij}^{(k)} \in K$. Let us consider the projective space $\mathbf{P}(V) = \mathbf{P}^{a-1}$, and use coordinates $t_1, \ldots, t_a$ for it. Hence we write $x = \sum_i t_i x_i$ for an element $x \in V$.

If $n-c < b$ then all the linear forms of V have rank $< b$. Hence we may assume that $n-c \geqslant b$. We have

$$z_i x = \sum_j z_i t_j x_j = \sum_k \left(\sum_j \lambda_{ij}^{(k)} t_j \right) y_k.$$

The matrix associated to the map $\phi: R_1/W \xrightarrow{x} VR_1$ with respect to the given basis is the $b \times (n-c)$ matrix M whose (j,k) entry is $\sum_j \lambda_{ij}^{(k)} t_j$. It follows that the set of elements $x \in R_1$ such that $\operatorname{rank}(x) < b$ is the variety X in $\mathbf{P}(V)$ defined by the $b \times b$ minors of M. This determinantal variety has codimension at most $(n-c)-b+1$ so that

$$\dim X \geqslant a-1-((n-c)-b+1) = a+b+c-n-2 \geqslant 0.$$

This proves (1). To prove (1.1) one just takes $V = R_1$. To prove (1.2), one just notes that $\dim V = n - \operatorname{rank}(l)$ and $l \in \{x \in R_1 : xV = 0\}$, so that the corresponding c is positive.

(2) Let $x_1, \ldots, x_n$ be a basis R_1, and $y_1, \ldots, y_m$ be a basis of R_2. We have relations $x_i x_j = \sum_k \lambda_{ij}^{(k)} y_k$. Let $x = \sum_i t_i x_i$ be an element in R_1; then

$$x^2 = \sum_{1 \leqslant i,j \leqslant n} t_i t_j x_i x_j = \sum_k \left(\sum_{1 \leqslant i,j \leqslant n} \lambda_{ij}^{(k)} t_i t_j \right) y_k.$$

Hence the elements $l \in \mathbf{P}(R_1)$ with $l^2 = 0$ form a variety X in $\mathbf{P}(R_1)$ defined by the quadrics $Q_k = \sum_{1 \leqslant i,j \leqslant n} \lambda_{ij}^{(k)} t_i t_j$, with $k = 1, \ldots, m$. It follows that the codimension of X in $\mathbf{P}^{n-1}$ is at most m, so that $\dim X \geqslant n-1-m \geqslant 0$. This proves (2). □

The Hilbert series of elements of small rank is easily related to that of R. An element l of rank 0 is a linear form which annihilates R_1. Thus, if $\operatorname{rank}(l) = 0$, then $P_{R/(0:l)}(z) = 1$ so that by (1)

$$P_R(z) = P_{R/lR}(z) + z.$$

Under the assumption that R is quadratic and Artinian, we have similar results for the elements of rank one.

LEMMA 2.9. *Let R be a quadratic and Artinian graded algebra and let l be a linear form of R. If* $\operatorname{rank}(l) = 1$, *then*

$$P_R(z) = P_{R/lR}(z) + z + z^2, \qquad P_{R/(0:l)}(z) = 1 + z.$$

Proof. Let $V = (0: l)_1 = \{x \in R_1 : xl = 0\}$. By assumption V is a space of codimension 1 in R_1. We have $(V) \subseteq 0: l$. Since R is quadratic and Artinian we have that $R/(V)$ is also quadratic and Artinian. Further the embedding dimension of $R/(V)$ is 1. It follows that

$$R/(V) \simeq K[x]/(x^2).$$

This implies that $P_{R/(V)}(z) = 1 + z$. Since $P_{R/(V)}(z) \geqslant P_{R/(0:l)}(z)$ and the ideals $(0: l)$ and (V) coincide in degree 1, we get $(0: l) = (V)$. The conclusion follows by (1).□

We recall that if $R = \oplus_{i=0}^{s} R_i$ is an Artinian graded algebra with $R_s \neq 0$, then the ideal $0: R_1$ of R is called the socle of R. The dimension of $0: R_1$ as a vector space is denoted by $\tau(R)$ and is called the Cohen–Macaulay type of R. Since $R_s \subseteq 0: R_1$, the integer s is called the socle degree of R.

LEMMA 2.10. *If R is a quadratic and Artinian graded algebra and $H_R(2) = 1$, then R has socle degree* 2.

Proof. Since $R_2 \neq 0$ there is an element $l \in R_1$ such that $\operatorname{rank}(l) = 1$. By Lemma 2.9

$$P_R(z) = P_{R/lR}(z) + z + z^2$$

The conclusion follows since $P_{R/lR}(z) = 1 + (n-1)z$. □

LEMMA 2.11. *Let R be a standard graded algebra and let l be a linear form of R. If one of the following conditions holds:*

(a) $(0: l) = 0$ *i.e. l is regular in R,*
(b) $(0: l) = (l)$,
(c) $(0: l) = \mathcal{M}$ *i.e.* $\operatorname{rank}(l) = 0$,

then every Gröbner flag of R/lR can be lifted to a Gröbner flag of R.

Proof. Let $\overline{l_2}, \dots, \overline{l_n}$ be a basis of a Gröbner flag of R/lR. Then one easily shows that $l, l_2, \dots, l_n$ is a basis of a Gröbner flag of R. □

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Using these results we can establish the following useful criteria:

PROPOSITION 2.12. *Let R be a standard graded algebra with $P_R(z) = 1 + nz + z^2$, $n \geqslant 2$. The following conditions are equivalent:*

(a) *R has a Gröbner flag,*
(b) *R is quadratic,*
(c) $\tau(R) < n$.

Proof. If R has a Gröbner flag, then R is G-quadratic, and hence quadratic. Now we prove that (b) implies (c). If, by contradiction, $\tau(R) = n$, then $R_1 = \langle x_1, \dots, x_{n-1}, x_n\rangle$ where $x_1, \dots, x_{n-1}$ are in the socle. Then the Hilbert series of $R/(x_1, \dots, x_{n-1})$ would be $1 + z + z^2$ which is impossible for a quadratic algebra. Finally we prove that (c) implies (a). By Lemma 2.8, (2), there exists an element $l \in R_1$ such that $l^2 = 0$. If $\operatorname{rank}(l) = 1$, then R has a Gröbner flag by Lemma 2.7. If $\operatorname{rank}(l) = 0$ then $P_{R/lR}(z) = 1 + (n-1)z + z^2$ and $\tau(R/lR) = \tau(R) - 1 < n - 1$. Hence we get the conclusion by induction and Lemma 2.11, (c), after remarking that, if $n = 2$, then R is Gorenstein so that l cannot be in the socle and thus has rank 1. □

PROPOSITION 2.13. *Let R be a standard graded algebra with $P_R(z) = 1 + nz + nz^2 + z^3$ and $n \geqslant 3$. Assume that R is either quadratic or Gorenstein. Then we have:*

(a) *If there exists $l \in R_1$ such that $l^2 = 0$ and $\operatorname{rank}(l) = n - 1$, then R has a Gröbner flag.*
(b) *If there exist $l, m \in R_1$ such that $lm = 0$ and $\operatorname{rank}(l) = \operatorname{rank}(m) = n - 1$, then R has a Koszul filtration.*

Proof. Let l be a linear form with $\operatorname{rank}(l) = n - 1$. Then it is clear that $H_{R/lR}(0) = 1$, $H_{R/lR}(1) = n - 1$, $H_{R/lR}(2) = 1$. If R is quadratic, then R/lR is quadratic and by Lemma 2.10 its socle degree is 2. If R is Gorenstein, then $lR_2 = R_3$ (since otherwise l would be in the socle) and this implies that R/lR has socle degree equal to 2. Hence, in both cases, we have $P_{R/lR}(z) = 1 + (n-1)z + z^2$, and, by (1), $P_{R/(0:l)}(z) = 1 + (n-1)z + z^2$. Further it is well known (and easy to see) that if R is Artinian Gorenstein then $R/(0\colon l)$ is also Gorenstein for all nonzero linear form l.

Let us prove (a). If $l^2 = 0$, then $(0\colon l) \supseteq (l)$ so that $(0\colon l) = (l)$ because they have the same Hilbert function. If R is Gorenstein then $R/lR = R/(0\colon l)$ is also Gorenstein, while if R is quadratic then R/lR is quadratic. By Proposition 2.12 R/lR has a Gröbner flag in both cases. By Lemma 2.11, (b), this Gröbner flag can be lifted to a Gröbner flag of R.

We prove now (b). We have $(l) \subseteq (0\colon m)$ and $(m) \subseteq (0\colon l)$ so that $(l) = (0\colon m)$ and $(m) = (0\colon l)$ since they have the same Hilbert function. This implies, as above, that R/lR and R/mR have a Gröbner flag. Let $\{\overline{x_1}, \dots, \overline{x_{n-1}}\}$ be a basis of a Gröbner flag of R/lR, and let $\{\overline{y_1}, \dots, \overline{y_{n-1}}\}$ be a basis of a Gröbner flag of R/mR. Then

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it is easy to see that

$$\mathcal{F} = \{(0), (l), (m), (l, x_1), (m, y_1), (l, x_1, x_2), (m, y_1, y_2), \dots, \mathcal{M}\}$$

is a Koszul filtration of R. □

3. Quadratic Gröbner Bases for Ideals of Points

In this section we prove:

THEOREM 3.1. *Let X be a set of s distinct points in general linear position in $\mathbf{P}^n$. If $s \leqslant 2n$ then the coordinate ring R of X has a Gröbner flag.*

Under the same assumptions Kempf [19] proved that R is Koszul and later Conca, Trung and Valla [7] proved that R has a Koszul filtration. The Koszul filtration of R given in [7] is not supported on a flag, but we show how to modify the argument to get a Gröbner flag.

Given a set X of distinct points $P_1, \dots, P_s$ in $\mathbf{P}^n$, we denote by I its defining ideal in $A = K[x_0, \dots, x_n]$. We can write $I = \wp_1 \cap \wp_2 \cap \dots \cap \wp_s$ where $\wp_i$ is the prime ideal corresponding to P_i. The homogeneous coordinate ring of X is the standard graded algebra $R = A/I$.

One says that the points are in general linear position if any subset of X that lies on a d-dimensional linear subspace has cardinality $\leqslant d+1$.

Proof. [of 3.1] If $s \leqslant n+1$, the general linear position property implies $P_R(z) = (1+sz)/(1-z)$. Hence any Artinian reduction of R has a Gröbner flag. It follows from Lemma 2.11 that such a flag can be lifted to a flag of R.

Let $s \geqslant n+1$; we have $I = \wp_1 \cap \wp_2 \cap \dots \cap \wp_s$ and we let $J = \wp_1 \cap \wp_2 \cap \dots \cap \wp_n$. By the general linear position assumption there exists a linear form $L \in A$ such that the hyperplane $L = 0$ contains the points $P_1, \dots, P_n$ and avoids the points $P_{n+1}, \dots, P_s$. Similarly there exists a linear form $M \in A$ such that the hyperplane $M = 0$ contains the points $P_{n+1}, \dots, P_s$ and avoids the points $P_1, \dots, P_n$. Then $LM \in I$ and $L + M$ is regular on $R = A/I$ and also on A/J.

As in [7], Lemma 2.2, one proves that $I + (L) = J$ so that

$$P_{A/(I+(L,M))}(z) = P_{A/(J+(L+M))}(z) = (1-z)P_{A/J}(z) = 1 + (n-1)z. \tag{3}$$

Denote by S the ring $A/I + (L+M)$ and by l and m the classes of L and M in S. We have $0 = (l+m)l = l^2$. Since $S/(l) = A/I + (L, M)$, by (3) we have $lS_1 = S_2$. Then, by Lemma 2.7, S has a Gröbner flag. By virtue of 2.11, (a), this flag can be lifted to a Gröbner flag of R since $L + M$ is a regular element on R. □

4. Non-Koszul Points and Curves

One may ask whether the above result for ideals of points can be improved. In [7] it was proved that if X is a set of s points with 'generic coordinates' in $\mathbf{P}^n$, then R is Koszul if and only if $s \leqslant 1 + n + (n^2/4)$. Here having generic coordinates means that the coordinates of the points are algebraically independent over $\mathbf{Q}$. So, for example, 9 points with generic coordinates in $\mathbf{P}^4$ are Koszul.

The goal of this section is to present a set X of 9 points in $\mathbf{P}^4$ which are in linear general position, are intersection of quadrics but not Koszul. We will also see that any subset of X has the expected generic graded Betti numbers over the polynomial ring.

This set of points arises in the following way. J. E. Roos made an extensive study of quadratic algebras of embedding dimension 4. He published in [25] the list of all the possible homological behaviours for the algebras of this class. According to Roos's list, there are four homological behaviours for a quadratic algebra with Hilbert series $1 + 4z + 4z^2$: they are number (54), (55), (56) and (57) in the list. For each class he gave a specific example. The algebras of the class (54) are Koszul. The typical example of an algebra of the class (55) is the algebra $K[x, y, z, t]/J$ defined by

$$J = (f_1 = x^2 + xy,\quad f_2 = y^2,\quad f_3 = xz + yt,\quad f_4 = z^2,\quad f_5 = xt,\quad f_6 = zt + t^2)$$

Our goal is to lift J to an ideal of points, (i.e. a radical ideal of dimension 1). There is a standard way to lift monomial ideals to points. On the other hand, for a non-monomial ideal as J there is no reason (a priori) to believe that such an ideal can be lifted. Nevertheless we have been able, by using CoCoA [6], to lift J to a radical ideal I in $K[x, y, z, t, w]$. Let us sketch the argument. First deform each f_i by adding a linear form times the new variable w: $F_i = f_i + w(a_{i1}x + a_{i2}y + a_{i3}z + a_{i4}t + a_{i5}w)$ and set $I = (F_1, \ldots, F_6)$. Let Lift (J) be the subset of the affine space $\mathbf{A}(a_{ij})$ whose points are those collections of (a_{ij}) such that w is a regular element modulo I. Let τ be any term order on $K[x, y, z, t]$ and let τ' be the degree reverse lexicographic product of τ with the term order on $K[w]$. This means that τ' is the term order of $K[x, y, z, t, w]$ defined as follows: if m_1 and m_2 are monomials in the variables x, y, z, t, then $m_1w^i > m_2w^j$ with respect to τ' if and only if $\deg(m_1w^i) > \deg(m_2w^j)$ or $\deg(m_1w^i) = \deg(m_2w^j)$ and $i < j$ or $\deg(m_1w^i) = \deg(m_2w^j)$, $i = j$ and $m_1 > m_2$ with respect to τ. Then it is not difficult to prove that w is a regular element modulo I if and only if $\mathrm{in}_\tau(J) = \mathrm{in}_{\tau'}(I)$. This implies that Lift (J) is an affine variety and its defining equations can be computed by imposing that the Buchberger algorithm applied to the F_i produces only initial terms which are in $\mathrm{in}_\tau(J)$. After this is done, one just takes a random point on Lift (J) and hopes to get a reduced ideal. In this way we have found that the ideal I generated by following quadrics:

$$\begin{aligned}
&F_1 = f_1 + w(-x + y - z - t - w), \quad F_2 = f_2 + w(x - 2y + 2z + t - 2w),\\
&F_3 = f_3 + w(2x + 2y + z + t + w), \quad F_4 = f_4 + w(-x - 3y - z + t - 2w),\\
&F_5 = f_5 + w(2x + t + 4w), \quad F_6 = f_6 + w(2x + 2y + 2z + t - 6w),
\end{aligned}$$

defines a set of 9 distinct points in $\mathbf{P}^4$ (over the rationals) which are not Koszul. One can prove that I does not contain quadrics of rank 2, thus proving that the corresponding points are in general linear position. Just one of these 9 points has rational coordinates, it is $(x = 0, y = 0, z = 3, t = -4, w = 1)$. One can look for a prime number p such that the 9 points given by I have coordinates in $\mathbf{Z}/(p)$. This boils down to finding a p such that the univariate polynomial $x^8 + 6x^7 + 16x^6 + 6x^5 - 57x^4 - 83x^3 + 7x^2 + 43x - 3$ (obtained by eliminating y, z, t from $I + (w-1)$ and then getting rid of the trivial root $x = 0$) has 8 roots in $\mathbf{Z}/(p)$. This can be done by using the factorization package written by J. Abbott and available in CoCoA. We have found that $p = 30341$ is such a prime, so that over the finite field $\mathbf{Z}/(30341)$ one can explicitly determine the coordinates of the points. We have checked that every subset of this set of points has the generic Betti numbers over the polynomial ring.

Summing up: 9 points in $\mathbf{P}^4$ with 'generic coordinates' are Koszul, while the above set of very 'general' 9 points are not Koszul. This example suggests that it is hard to guess a generalization of Kempf's theorem, describing the Koszul locus in geometric terms, for sets of $2n+1$ points in $\mathbf{P}^n$.

We have tried to lift also the examples (56) and (57) in Roos's list. Lifting the example (57) and taking, as before, a random point of the lifting variety, we got 9 points in $\mathbf{P}^4$ in general linear position which are not Koszul and have a different homological behaviour than that of the previous example.

On the contrary, lifting the example (56) we always got 9 points in $\mathbf{P}^4$ with this configuration: 5 points on a $\mathbf{P}^3$ and 4 points on a $\mathbf{P}^2$. This may of course depend on the specific example and we cannot exclude that there exist 9 points in $\mathbf{P}^4$ in linear general position and whose homological behaviour corresponds to number (56) in Roos' list.

Butler [5], Problem 5, and Polishchuk [24], p. 123, asked whether the homogeneous coordinate algebra of a projectively normal, smooth, connected, complex curve is Koszul, provided it is quadratic. Sturmfels [29], Thm. 3.1, gave a negative answer to this question. He considered the prime ideal $\wp \subset K[a, b, c, d, e, f]$ generated by

$$
\begin{aligned}
&a(c+d) - bf,\, b(b+f) - ce,\, c(a+c+f) - d(c+d),\, df - e(b+f),\\
&e^2 - a(a+c+f).
\end{aligned}
$$

This ideal defines a smooth, projectively normal curve in $\mathbf{P}^5$ of genus 7, degree 11, which is quadratic but not Koszul. The Hilbert series of $K[a, b, c, d, e, f]/\wp$ is:

$$
\frac{1 + 4z + 5z^2 + z^3}{(1-z)^2}.
$$

It is well-known (and easy to prove) that for a Koszul algebra S the Poincaré series of the Tor's

$$
T_S(z) = \sum_{i \geqslant 0} \dim_K \operatorname{Tor}_i^S(K, K) z^i
$$

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and the Hilbert series $P_S(z)$ satisfy the following relation:

$$T_S(z) = 1/P_S(-z)$$

Since the coefficient of z^9 in the series $(1+z)^2/(1-4z+5z^2-z^3)$ is negative (it is equal to -1220), no algebra with Hilbert series $(1+4z+5z^2+z^3)/(1-z)^2$ can be Koszul.

At this point one can ask whether the question of Butler and Polishchuk has a positive answer under the additional assumption that all the coefficients of the series $1/P_R(-z)$ are positive. The answer is negative. We have lifted the above radical ideal I to a prime ideal $\wp$ in $K[x, y, z, t, w, u]$ by using the same methods as before. We got an arithmetically Cohen–Macaulay, smooth, reduced, irreducible curve in $\mathbf{P}^5$ which has degree 9, genus 4, it is intersection of quadrics but is not Koszul. The Hilbert series is $P(z) = (1+4z+4z^2)/(1-z)^2$ and all the coefficients of $P(-z)^{-1}$ are positive. Note that this curve has maximal genus with respect to the classical Castelnuovo bound. Explicitly, over $\mathbf{Q}$ the generators of $\wp$ are:

$$\begin{aligned}
G_1 &= F_1 + u(-2x+4y+8z+8t-20w-40u),\\
G_2 &= F_2 + u(+2x-11y+8t-18w-6u),\\
G_3 &= F_3 + u(+2x+4y+4z-t+6w+52u),\\
G_4 &= F_4 + u(-2x+2y-4z+t+4w-8u),\\
G_5 &= F_5 + u(+4x+4t-8w+48u),\\
G_6 &= F_6 + u(-4x-4y+4z+2t-12w+16u).
\end{aligned}$$

5. A Gröbner Bases of Quadrics for the Ideal of the Canonical Curve

In this section we prove the following:

THEOREM 5.1. *Let $\mathcal{C}$ be a smooth, non-hyperelliptic, non-trigonal curve of genus $g \geqslant 5$ which is not a plane quintic and let $R_{\mathcal{C}}$ be its canonical ring. Then $R_{\mathcal{C}}$ has a Gröbner flag.*

Under the additional assumption that $\mathcal{C}$ is not bielliptic, Vishik and Finkelberg in [30] proved that $R_{\mathcal{C}}$ is a Koszul algebra. Later Polishchuk in [24] and Pareschi and Purnaprajna in [23], gave different proofs of the result of Vishik and Finkelberg, which work also in the bielliptic case.

Let $\mathcal{C}$ be a smooth algebraic curve of genus g over an algebraically closed field of characteristic zero. If $\mathcal{C}$ is not hyperelliptic, then the canonical sheaf on $\mathcal{C}$ gives a canonical embedding $\mathcal{C} \to \mathbf{P}^{g-1}$. The image of this embedding is a non degenerate smooth algebraic curve of degree $2g-2$ whose extrinsic geometry reflects intrinsic properties of the abstract curve $\mathcal{C}$.

The homogeneous coordinate ring $R_{\mathcal{C}}$ of this embedding is called the canonical ring of the curve $\mathcal{C}$. For a description of the properties of $R_{\mathcal{C}}$ in algebraic terms we refer the reader to the paper of Eisenbud [11]. We just recall that $R_{\mathcal{C}}$ is a Gorenstein domain of dimension 2 with Hilbert series:

$$\frac{1+(g-2)z+(g-2)z^2+z^3}{(1-z)^2}.$$

According to Petri's theorem (see [1], pg. 131), $R_{\mathcal{C}}$ is quadratic if and only if $\mathcal{C}$ is non-hyperelliptic, non-trigonal and not a plane quintic. Hence 5.1 says that $R_{\mathcal{C}}$ is G-quadratic provided it is quadratic.

We will need the following:

LEMMA 5.2. *Let $\mathcal{C}$ be a non-hyperelliptic, non-trigonal curve which is not a plane quintic and let K be its canonical class. Then there exist two effective divisors D_1 and D_2 of degree $g-1$ on $\mathcal{C}$ such that $D_1+D_2=K$ and $|D_1|$ and $|D_2|$ are base-point-free linear systems of dimension* 1.

Under the additional assumption that $\mathcal{C}$ is non bielliptic, Lemma 5.2 has been proved in [30], Lemma 1.1, as a consequence of theorems of Mumford and Martens. However, as remarked in [23], the arguments used in [30] also work for bielliptic curves.

As noticed in [30], a corollary of 5.2 is that there exists a hyperplane section of $\mathcal{C}$, say $Z=\mathcal{C}\cap H$, such that $Z=X\cup Y\subset \mathbf{P}^{g-2}$ where X and Y are sets of points of cardinality $g-1$ with the properties:

(i) The points of X are contained on a unique hyperplane H_1 of $\mathbf{P}^{g-2}$. Furthermore H_1 does not contain points of Y.
(ii) The points of Y are contained on a unique hyperplane H_2 of $\mathbf{P}^{g-2}$. Furthermore H_2 does not contain points of X.

By virtue of 2.11, (a), to prove 5.1 it suffices to show that the coordinate ring of the set of points Z has a Gröbner flag. This follows from:

THEOREM 5.3. *Let X be a set of $2n+2$ distinct points in $\mathbf{P}^n$, $n\geqslant 3$. Denote by R the coordinate ring of X and assume that*

(a) $P_R(z)=(1+nz+nz^2+z^3)/(1-z)$,
(b) *R is quadratic (or Gorenstein),*
(c) *X has a decomposition as $X=X_1\cup X_2$ where, for $i=1,2$, X_i is a set of $n+1$ points such that there exists an unique hyperplane H_i containing X_i. Further H_i does not contain points of X_j if $j\neq i$.*

Then R has a Gröbner flag.

Proof. Let $A=K[x_0,\dots,x_n]$. Let I_i be the defining ideal of X_i and let I be that of X, so that $I=I_1\cap I_2$. Let L_i be the linear form defining H_i. Then $L_i\in I_i$ and it

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is regular on $A/I_j, j \neq i$. Further $L_1 + L_2$ is regular on R. Since the points in X_i lie on a unique hyperplane, we have

$$P_{A/I_1}(z) = P_{A/I_2}(z) = \frac{1 + (n-1)z + z^2}{1 - z}.$$

We have

$$I: L_1 = (I_1 \cap I_2): L_1 = (I_1: L_1) \cap (I_2: L_1) = I_2: L_1 = I_2.$$

From (1) of Section 2 we get

$$P_{A/(I+(L_1))}(z) = P_{A/I}(z) - zP_{A/I_2}(z) = \frac{1 + (n-1)z + z^2}{1 - z}.$$

Therefore, since $I + (L_1) \subseteq I_1$ and they have the same Hilbert function, we get $I + (L_1) = I_1$. Since L_2 is regular on A/I_1, we have

$$P_{A/(I+(L_1,L_2))}(z) = 1 + (n-1)z + z^2.$$

Let $S = A/I + (L_1 + L_2)$ and denote by l_i the class of L_i in S. Since $L_1L_2 \in I$, we get that $0 = (l_1 + l_2)l_1 = l_1^2$. The Hilbert series of S is $1 + nz + nz^2 + z^3$ and we have seen that

$$P_{S/l_1S}(z) = P_{A/(I+(L_1,L_2))}(z) = 1 + (n-1)z + z^2$$

which in turn says that l_1 has rank $n - 1$ in S.

Summing up, S is a quadratic (or Gorenstein) algebra with Hilbert series $1 + nz + nz^2 + z^3$ and l_1 is a linear form of S with $l_1^2 = 0$ and $\operatorname{rank}(l_1) = n - 1$. It follows from 2.13, (a), that S has a Gröbner flag. Then by Lemma 2.11, (a), also R has a Gröbner flag. □

6. Gorenstein Algebras of Socle Degree 3

The aim of this section is the study of the Koszul property and the existence of quadratic Gröbner basis for Artinian graded algebras which are Gorenstein of socle degree 3.

As we have seen in the preceding section, these algebras arise as Artinian reduction of the canonical ring of a curve. They arise also as Artinian reduction of the homogeneous coordinate ring of a set of self associated points (see [9] and [14]).

Every Artinian Gorenstein graded algebra $R = K[x_1, \dots, x_n]/I$ of socle degree j, corresponds, up to scalars, to a form F of degree j in another set of variables as follows. Let $A = K[x_1, \dots, x_n]$ and $B = K[y_1, \dots, y_n]$. Regard B as a A-module via the action $x_i \circ F = \partial F/\partial y_i$; then every element $G \in A$ acts as a differential operator on the elements of B:

$$G \circ F = G(\partial/\partial y_1, \dots, \partial/\partial y_n)(F).$$

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Given a form $F \in B$ of degree j, we denote by I_F the ideal of the elements of A which annihilate F:

$$I_F = \{G \in A \mid G \circ F = 0\}.$$

Set $R_F = A/I_F$. It is easy to see that R_F is a standard graded Artinian Gorenstein algebra of socle degree j. Moreover, every ideal $I \subset A$ which defines a standard graded Artinian Gorenstein algebra of socle degree j arises in this way.

This construction gives a bijective correspondence between the set of Artinian graded Gorenstein quotients of A of socle degree j and forms (up to scalars) of degree j in B. This correspondence is compatible with the action of $\mathrm{GL}_n(K)$. Furthermore cones (i.e. forms that can be represented with less than n variables) correspond to degenerate ideals (i.e. ideals containing a linear form).

This correspondence is sometime called the 'inverse systems' of Macaulay. The study of the geometric objects arising from this correspondence between forms and ideals is a classical theme in algebraic geometry and commutative algebra. For a modern treatment and an updated list of references, the reader can consult the recent book of Iarrobino and Kanev [17].

Our goal is the study of the Koszulness and G-quadraticity of the Artinian graded Gorenstein algebras of socle degree 3. Hence, from now on, F will be a cubic form of B and we will always consider the non-degenerate situation. We may hence identify the degree 1 part of R_F with A_1. In the case $n = 2$ the algebra R_F is never quadratic, hence we will always assume that $n \geqslant 3$.

Every linear form $l \in A_1$, say $l = a_1x_1 + a_2x_2 + \ldots + a_nx_n$, can be considered as a point $P = (a_1, a_2, \ldots, a_n) \in \mathbf{P}(A_1) = \mathbf{P}^{n-1}$. The polar quadric of F with respect to P, denoted by Q_P, is the quadric of B:

$$Q_P = l \circ F = \sum_{i=1}^{n} a_i \frac{\partial F}{\partial y_i}.$$

The unique symmetric matrix associated to Q_P will be denoted by M_P; note that $2M_P$ is just the Hessian matrix

$$He(F) = \left(\frac{\partial^2 F}{\partial y_i \partial y_j}\right)$$

of F evaluated at P.

The following lemma contains well-known facts. They can be found for instance in [10]. We include them for ease of reference.

LEMMA 6.1. *Let* $l = \sum a_i x_i$ *and* $m = \sum b_i x_i$ *be linear forms and let* P *and* P' *be the corresponding points of* $\mathbf{P}^{n-1}$. *Then we have*

(1) $lm \in I_F \Longleftrightarrow P' \in \mathrm{Sing}(Q_P)$.
(2) $l^2 \in I_F \Leftrightarrow P \in \mathrm{Sing}(Q_P) \Leftrightarrow P \in \mathrm{Sing}(F)$.

(3) $\operatorname{rank}(l) = \operatorname{rank}(M_P) = \operatorname{rank}(He(F)_P)$, *where* $\operatorname{rank}(l)$ *is the rank of* l *in* R_F. *In particular, the variety of the linear forms of rank* $< t$ *in* R_F *is defined by the ideal of the* $t \times t$*-minors of the Hessian matrix* $He(F)$.

As a consequence of these simple facts, we have:

PROPOSITION 6.2. *If* F *is a smooth cubic, then* R_F *is not G-quadratic.*

Proof. Let us assume by contradiction that R_F is G-quadratic. This means that in a specific system of coordinates, say $x_1, \dots, x_n$, and with respect to a suitable term order, the ideal I_F has a Gröbner basis of quadrics. We may assume that $x_1 > x_2 > \cdots > x_n$ so that x_n^2 is the smallest monomial of degree 2. Since R_F is Artinian and $\operatorname{in}(I_F)$ is generated in degree 2, we must have $x_i^2 \in \operatorname{in}(I_F)$ for all i, and in particular $x_n^2 \in \operatorname{in}(I_F)$. Since x_n^2 is the smallest monomial of degree 2, it follows that $x_n^2 \in I_F$. Then, by the Lemma 6.1(2), the point $(0, 0, \dots, 0, 1)$ is singular for F, a contradiction. □

Since we are assuming that F is not a cone, the Hilbert series of R_F is $1 + nz + nz^2 + z^3$. Hence the rank of a linear form is bounded above by n. The variety of the linear forms of rank $< n$ is defined by the determinant $\det He(F)$ of the Hessian matrix of F.

If we take a point on the Hessian hypersurface $\det He(F) = 0$ of F, say P, then the corresponding form l has rank $< n$ in R_F and so it must annihilate another linear form, say m, which also has $\operatorname{rank}(m) < n$. If we can find such an l and m so that their rank is exactly $n - 1$, then by 2.13, (b), we may conclude that R_F has a Koszul filtration. This is what happens for a generic F.

THEOREM 6.3. *Let* F *be a generic cubic. Then* R_F *has a Koszul filtration.*

Moreover, if F is singular then any singular point P corresponds to a linear form l such that $l^2 = 0$ and hence $\operatorname{rank}(l) < n$. As above, if we can find such an l so that $\operatorname{rank}(l) = n - 1$, then we may conclude by virtue of 2.13, (a), that R_F has a Gröbner flag. This is what happens for a generic singular F.

THEOREM 6.4. *Let* F *be a generic singular cubic. Then* R_F *has a Gröbner flag.*

Before embarking in the proof of 6.3 and 6.4, let us introduce some notation. The space of cubics of A is a projective space $\mathbf{P}(A_3)$ and the set of the singular cubics $\mathcal{SC}$ is a Zariski closed subset of $\mathbf{P}(A_3)$. Let F be a cubic. Denote by

$$X(F) = \{l \in \mathbf{P}(A_1)\colon \text{ the rank of } l \text{ in } R_F \text{ is } < n - 1\}.$$

Note that $X(F)$ is the determinantal locus given by the $(n-1)$-minors of the Hessian matrix of F. Further set

$$Y(F) = \{(l, m) \in \mathbf{P}(A_1) \times \mathbf{P}(A_1)\colon lm = 0 \text{ in } R_F\}$$

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and

$$Z(F) = \{(l, m) \in \mathbf{P}(A_1) \times \mathbf{P}(A_1) : l \in X(F) \text{ or } m \in X(F)\}.$$

By the above discussion we have that the Theorems 6.3 and 6.4 follow if we show that:

CLAIM 6.5. *There is a nonempty Zariski open subset U of $\mathbf{P}(A_3)$ such that for all $F \in U$ one has $Y(F) \not\subseteq Z(F)$.*

and

CLAIM 6.6. *There is a nonempty Zariski open subset U of $\mathcal{SC}$ such that for all $F \in U$ one has* $\operatorname{Sing}(F) \not\subseteq X(F)$.

In order to prove the two claims one argues like this: first one shows that the property under consideration is open and then one presents a concrete example to ensure that the relevant open set is not empty. The first part of the argument is standard since the properties under discussion are described in terms of inclusions between varieties and we leave the details to the reader. We just present the two examples. Consider first the following cubic:

$$G = (y_1 + y_2)(y_3^2 + y_4^2 + \ldots + y_n^2) + y_1^3 + y_2^3 + y_3^3 + y_4^3 + \ldots + y_n^3.$$

It is clear that $\partial G/\partial y_1$ and $\partial G/\partial y_2$ are quadrics of rank $n-1$ and that $x_1x_2 \in I_G$ and hence that $Y(G) \not\subseteq Z(G)$. Also, in order to be sure that the example is not too special one has to check that $Y(G)$ has the smallest possible dimension (which is $n-2$). But this follows easily from the fact that G is non-singular. This proves Claim 6.5. As for 6.6, consider the following cubic:

$$H = y_1(y_2^2 + y_3^2 + y_4^2 \ldots + y_n^2) - 2/3(y_2^3 + y_3^3 \ldots + y_n^3).$$

Note that $\partial H/\partial y_1$ is a quadric of rank $n-1$ and that $x_1^2 \in I_H$. This shows that $\operatorname{Sing}(H) \not\subseteq X(H)$. This time to be sure that the example is general enough one has to check that H has just one singular point and that the Hilbert function of the ring defined by its jacobian ideal is equal to 1 in large degrees. The reader can check that this is indeed the case. This concludes the proof of 6.6.

In the study of the correspondence between Artinian Gorenstein algebras with socle in degree 3 and cubics, a natural problem is to find geometric properties of the cubic hypersurface $F = 0$ which imply (or are equivalent) to the quadraticity of R_F. It follows from 6.3 that for a generic F the ring R_F is quadratic. Note that the smoothness of F does not imply that R_F is a quadratic algebra. Take for example the Fermat cubic $F = y_1^3 + y_2^3 + \ldots + y_n^3$. Then F is smooth and not quadratic since the ideal I_F is minimally generated by x_ix_j with $i < j$ and $x_i^3 - x_n^3$ with $i < n$.

The next lemma gives two necessary conditions for R_F to be quadratic.

LEMMA 6.7. *Let F be a cubic in n variables. We have:*

(a) *If R_F is quadratic then for every nonzero linear form $l \in R_F$ one has* $\operatorname{rank}(l) \geqslant 2$, *i.e. the codimension of the ideal of the 2×2 minors of the Hessian matrix $He(F)$ of F is n.*
(b) *Assume that, in some coordinate system, F has a decomposition $F = G + H$ where G and H are cubics on disjoint sets of variables. Then R_F is not quadratic.*

Proof. (a) Since R_F is Gorenstein, the socle of R_F is concentrated in degree 3. Hence $\operatorname{rank}(l) \geqslant 1$ for every nonzero linear form $l \in R_F$. Since R_F is quadratic, if for some linear form $l \in R_F$ we have $\operatorname{rank}(l) = 1$, then by Lemma 2.9 the Hilbert series of R_F/lR_F is $1 + (n-1)z + (n-1)z^2 + z^3$. This implies $lR_2 = 0$, a contradiction.

(b) By assumption we have that F splits as $G + H$ where G is a cubic in $y_1, \ldots, y_j$ and H is a cubic in $y_{j+1}, \ldots, y_n$ with $1 \leqslant j < n$. Assume, by contradiction, that R_F is quadratic. We may assume that F is not a cone so that G and H are not cones in their embedding. Let I_G and I_H be the ideals of $K[x_1, \ldots, x_j]$ and $K[x_{j+1}, \ldots, x_n]$ which correspond to G and H respectively and let R_G and R_H be the corresponding quotients. Let J be the ideal of $A = K[x_1, \ldots, x_n]$ defined by:

$$J = (x_1, \ldots, x_j)(x_{j+1}, \ldots, x_n) + I_G A + I_H A.$$

By construction $J \subseteq I_F$. Note that A/J in any positive degree i is isomorphic to $(R_G)_i \oplus (R_H)_i$ (i.e. A/J is the fiber product of R_G and R_H). It follows that the dimension of A/J in degree 2 is $j + (n-j) = n$, which is also the dimension of R_F is degree 2. Then J and I_F coincide in degree 2. Since we are assuming that R_F is quadratic, we may conclude that $I_F = J$. But the dimension of A/J in degree 3 is given by $1 + 1 = 2$ and this is a contradiction since the dimension of R_F in degree 3 is 1. □

Remark 6.8. With the notation of the proof of 6.7 part (b), the ideal I_F equals to $J + (G' - H')$ where G' is a cubic of $K[x_1, \ldots, x_j]$ with $G' \circ G = 1$ and H' is a cubic of $K[x_{j+1}, \ldots, x_n]$ with $H' \circ H = 1$.

Remark 6.9. The two conditions of 6.7 are independent in general. For instance, if $F = y_1 y_2^2 - y_2 y_3^2$, then in R_F there is an element of rank 1 (namely x_1) but F does not split since its Hessian determinant is $8y_2^3$. On the other hand if $F = y_1y_2y_3 + y_4y_5y_6$ then in R_F there is no linear form of rank 1 since this property clearly holds for the cubic $y_1y_2y_3$.

We do not know an example of a cubic F such that R_F is quadratic and not Koszul. So we ask the following:

QUESTION 6.10. *Let F be a cubic such that R_F is quadratic. Is R_F a Koszul algebra?*

We will see that if $n = 3$ or $n = 4$ the answer to this question is positive and we will also prove that, in these cases, R_F is quadratic if and only if every nonzero linear form in R_F has rank at least two.

We also remark that one could try to attack the above question by using Proposition 2.13. The first step in this approach would be to prove the existence of a linear form in R_F of rank $n-1$. Unfortunately, there are cubics F such that no linear form of R_F has rank $n-1$, see for instance [27], p. 173. Hence this approach breaks down completely.

6.1. PLANE CUBICS

Let F be a cubic form of $K[y_1, y_2, y_3]$. We assume as usual that F is non-degenerate, so that its zero locus is a plane cubic. For simplicity we will say that F itself is a plane cubic. The ideal I_F contains three independent quadrics and it has codimension 3. Hence the following conditions are equivalent:

(1) R_F is a complete intersection,
(2) R_F is Koszul,
(3) R_F is quadratic.

If R_F is quadratic, then by Lemma 6.7, (a), $\operatorname{rank}(l) \geqslant 2$ for every nonzero linear form $l \in R_F$. The converse holds:

PROPOSITION 6.11. *Let F be a plane cubic. Then:*

(1) *R_F is quadratic (and hence a complete intersection) if and only if* $\operatorname{rank}(x) \geqslant 2$ *for every nonzero linear form $x \in R_F$, if and only if the ideal of the 2×2 minors of the Hessian matrix $He(F)$ has codimension three.*
(2) *R_F is G-quadratic if and only if R_F has a Gröbner flag, if and only if F is singular and* $\operatorname{rank}(x) \geqslant 2$ *for every nonzero linear form $x \in R_F$, if and only if F is singular and the ideal of the 2×2 minors of the Hessian matrix $He(F)$ has codimension three.*

Proof. Since by 6.1(3) the condition that $\operatorname{rank}(x) \geqslant 2$ for every nonzero linear form $x \in R_F$ is equivalent to the condition that the ideal of the 2×2 minors of the Hessian matrix $He(F)$ has codimension three, in order to prove the first assertion we need only to prove that if $\operatorname{rank}(x) \geqslant 2$ for every nonzero linear form $x \in R_F$ then R_F is quadratic. We have already seen that there exist nonzero linear forms $x, y \in R_F$ such that $xy = 0$ and $\operatorname{rank}(x), \operatorname{rank}(y) \leqslant n-1 = 2$. Hence $\operatorname{rank}(x) = \operatorname{rank}(y) = 2$ and by Proposition 2.13, (b), the algebra R_F has a Koszul filtration. In particular R_F is quadratic.

Also for the second assertion, we need only to prove that if F is singular and $\operatorname{rank}(x) \geqslant 2$ for every nonzero linear form $x \in R_F$ then R_F has a Gröbner flag. But we know from Lemma 6.1(2) that every singular point of F corresponds to a linear form x such that $x^2 = 0$ and $\operatorname{rank}(x) \leqslant n-1 = 2$. Hence, by assumption, $\operatorname{rank}(x) = 2$, so that by Proposition 2.13, (a), the algebra R_F has a Gröbner flag. □

A plane cubic F is said to be anharmonic if it is in the closure of the orbit of the Fermat cubic $x^3+y^3+z^3$ under the action of $PGL_3(K)$. In the projective space of the plane cubics, the set of anharmonic cubics is the hypersurface defined by a polynomial of degree 4, the Aronhold invariant $\mathbf{I}_4$ of degree 4 of plane cubics. If F is given in the form

$$\begin{aligned} F = {} & ax^3+by^3+cz^3+3dx^2y+3ex^2z+3fxy^2+3gy^2z \\ & +3hxz^2+3iyz^2+6jxyz, \end{aligned}$$

then the Aronhold invariant is

$$\begin{aligned} \mathbf{I}_4 = {} & abcj-(bcde+cafg+abhi)-j(agi+bhe+cdf)+ \\ & +(afi^2+ahg^2+bdh^2+bie^2+cgd^2+cef^2)-j^4+ \\ & +2j^2(fh+id+eg)-3j(dgh+efi)-(f^2h^2+i^2d^2+e^2g^2)+ \\ & +(ideg+egfh+fhid). \end{aligned}$$

It is known that a plane cubic F is anharmonic if and only if F has a polar conic which is a double line, see [10], Prop. 5.13.2. But by 6.1(3), this is equivalent to the fact that $\operatorname{rank}(l) < 2$ for some nonzero linear form $l \in R_F$.

Summing up, Proposition 6.11 together with the above mentioned result of Dolgachev and Kanev implies that:

COROLLARY 6.12. *Let F be a plane cubic. Then R_F is quadratic if and only if F is not anharmonic. Furthermore R_F is G-quadratic if and only if F is not anharmonic and singular.*

The first of these two assertions has been observed also by Eisenbud and Stillman (unpublished), see [11], Section IV.

EXAMPLE 6.13. Let $F = x^3+y^3+z^3+6jxyz$ with $j \in K$. The Aronhold invariant of F is j^4-j. Furthermore it is known (and easy to see) that F is singular if and only if $8j^3+1=0$. It follows that R_F is quadratic if and only if $j^4 \neq j$ and R_F is G-quadratic if and only if $8j^3=-1$. For instance, the form $F = x^3+y^3+z^3+6xyz$ corresponds to a non-quadratic algebra, the form $F = x^3+y^3+z^3+xyz$ corresponds to a quadratic but not G-quadratic algebra and the form $F = x^3+y^3+z^3-3xyz$ corresponds to a G-quadratic algebra.

Note that Proposition 6.11 gives, in the case of plane cubics, the answer to the problem of finding conditions on F such that R_F is a complete intersection. This question was mentioned by Iarrobino and Kanev in [17].

6.2. SPACE CUBICS

The goal of this section is to characterize the Koszulness, the quadraticity, and the G-quadraticity of R_F for space cubics (i.e. cubics in 4 variables). We need the following:

LEMMA 6.14. *Let R be a standard graded Gorenstein algebra with Hilbert series* $1+4z+4z^2+z^3$. *Assume that* $\operatorname{rank}(x)\geqslant 2$ *for every nonzero linear form x in R. Then one has:*

(1) *Let V be a space of linear forms of R Then* $\dim(0:_{R_1} V)+\dim VR_1=4$. *Further if* $\dim V>1$ *then there exists $x\in V$ such that* $\operatorname{rank}(x)<\dim VR_1$.
(2) *Let V be a space of linear forms of R. If* $\dim V=2$ *then* $\dim VR_1\geqslant 3$ *and if* $\dim V=3$ *then* $VR_1=R_2$.
(3) *Let $x\in R_1$ with* $\operatorname{rank}(x)=2$. *Then there exists a space V of linear forms of R such that $x\in V$,* $\dim V=2$ *and* $\dim VR_1=3$.
(4) *Let V be a space of linear forms of R such that* $\dim V=2$ *and* $\dim VR_1=3$. *Then there exists $x\in V$ such that* $\operatorname{rank}(x)=2$.
(5) *Let $x\in R_1$ with* $\operatorname{rank}(x)=2$. *Then the ideal $0{:}\,x$ is generated by a 2-dimensional space V of linear forms such that* $\dim VR_1=3$.

Proof. (1) The first assertion is a consequence of the standard duality which holds in a Gorenstein Artinian ring. The second follows from 2.8, (1).

(2) Assume first that $\dim V=2$. If, by contradiction, $\dim VR_1<3$ then by (1) there exists a nonzero $x\in V$ such that $\operatorname{rank}(x)<\dim VR_1\leqslant 2$. This contradicts the assumption. Assume now that $\dim V=3$. If, by contradiction, $\dim VR_1<4$ then by (1) the space $0:_{R_1} V$ is nonzero and hence there exists in R an element of rank 1, a contradiction.

(3) Let $x\in R_1$ with $\operatorname{rank}(x)=2$. The Hilbert series of R/x is $1+3z+2z^2$. Then by 2.8, 1.1), there exists a nonzero linear form $\bar{y}\in R/xR$ with $\operatorname{rank}(\bar{y})\leqslant 1$. If $\operatorname{rank}(\bar{y})=0$, then $\dim\langle x,y\rangle R_1=2$ and this contradicts 2). Hence $\operatorname{rank}(\bar{y})=1$, which means $\dim\langle x,y\rangle R_1=3$.

(4) Let V be a space of linear forms of R such that $\dim V=2$ and $\dim VR_1=3$. By 1) there exists $x\in V$ such that $\operatorname{rank}(x)<3$ and hence $\operatorname{rank}(x)=2$.

(5) Set $V=(0{:}\,x)_1$. Then $\dim V=2$ and $VR_1\subseteq(0{:}\,x)_2$. But $\dim(0{:}\,x)_2=3$ and hence by (2) we may conclude that $VR_1=(0{:}\,x)_2$ and this proves the assertion. □

We are ready to prove:

THEOREM 6.15. *Let F be a space cubic. The following conditions are equivalent:*

(1) *R_F is Koszul,*
(2) *R_F is quadratic,*
(3) $\operatorname{rank}(x)\geqslant 2$ *for every nonzero linear form x in R_F,*
(4) *The ideal of the 2×2 minors of the Hessian matrix $He(F)$ has codimension four.*

Proof. For simplicity of notation set $R=R_F$. The implications $(1)\Rightarrow(2)\Rightarrow(3)$ hold, no matter what n is, and (3) is equivalent to (4) by 6.1 (3). We have to show that $(3)\Rightarrow(1)$. To this end, assume that every nonzero linear form of R has rank at least two. Consider the following family of ideals $\mathcal{F} = \{(0)\} \cup \mathcal{F}_1 \cup \mathcal{F}_2$, where

$$\mathcal{F}_1 = \{(x) \mid x \in R_1,\ \operatorname{rank}(x) = 2\},$$

$$\begin{aligned}\mathcal{F}_2 = &\{I \mid I \text{ is generated by linear forms and}\\ &\exists\ V \subseteq I_1 \text{ such that } \dim V = 2 \text{ and } \dim VR_1 = 3\}.\end{aligned}$$

It is enough to prove that $\mathcal{F}$ is a Koszul filtration of R. First of all we must prove that $\mathcal{M} \in \mathcal{F}$, that is $\mathcal{M} \in \mathcal{F}_2$.

Since the Hessian matrix $He(F)$ is a 4×4 symmetric matrix, the ideal generated by its 3×3 minors has codimension not greater than 3. Therefore there are always linear forms of rank $\leqslant 2$ (and hence 2 by assumption). Let x be a linear form of rank 2. By Lemma 6.14, (3), there exists a space V of linear forms such that $x \in V$, $\dim V = 2$ and $\dim VR_1 = 3$. Since $\mathcal{M}$ contains V we may conclude that $\mathcal{M} \in \mathcal{F}_2$.

We have now to prove that condition (3) in the Definition 2.1 of Koszul filtration holds for $\mathcal{F}$. Let us start with an ideal $I \in \mathcal{F}_1$, that is $I = (x)$ with $\operatorname{rank}(x) = 2$. We have to prove that $0\colon x \in \mathcal{F}$. This is a consequence of Lemma 6.14, (5), which implies that $0\colon x \in \mathcal{F}_2$. Let now $I \in \mathcal{F}_2$. By assumption there exists a subspace $V \subseteq I$ of linear forms such that $\dim V = 2$ and $\dim VR_1 = 3$. Let us first consider the case $I = (V)$. By 6.14, (4), we know that there exists $x \in V$ such that $\operatorname{rank}(x) = 2$. Choose an element y so that x, y is a basis of V. Since $(x) \in \mathcal{F}_1$, we need only to prove that $(x)\colon(y) \in \mathcal{F}_2$. Denote by W the degree 1 part of $(x)\colon(y)$. By comparing dimensions one has that $\dim W = 3$ and hence by 6.14, (2), one has $(x)\colon(y) = (W)$. It remains to show that W contains a 2-dimensional subspace W_1 with $\dim W_1 R_1 = 3$. Set $S = R/xR$ and denote by $\overline{y}$ the class of y in S. It is enough to show that $0{:}_S\,\overline{y}$ contains a linear form $\overline{w}$ whose rank in S is 1. Since $\dim VR_1 = 3$ the rank of $\overline{y}$ in S is 1. By Lemma 2.8, (1.2), we know that there exists in $0{:}_S\,\overline{y}$ a linear form $\overline{w}$ such that $\operatorname{rank}(\overline{w}) \leqslant 1$. If $\operatorname{rank}(\overline{w}) = 0$ in S, then $\dim\langle x, w\rangle R_1 = 2$ which contradicts 6.14, (2). Then $\operatorname{rank}(\overline{w}) = 1$ and we are done.

It remains to consider the case in which (V) is a proper subideal of I. If $I = \mathcal{M}$ then let W be any 3-dimensional space containing V. Then $(W) \in \mathcal{F}_2$ by construction and $(W)\colon\mathcal{M} = \mathcal{M}$ since by 6.14, (2), $WR_1 = R_2$. Finally we have to consider the case in which I is generated by a 3-dimensional space of linear forms, that is $I = (V) + (z)$ where z is a linear form not in V. We need only to prove that $(V)\colon z \in \mathcal{F}$. By dimension considerations one has that $(V)\colon z$ contains a 3-dimensional space W of linear forms. By 6.14, (2), we have that W generates $(V)\colon z$. Since clearly $V \subset W$ we may conclude that $(V)\colon z \in \mathcal{F}_2$. □

Note that Theorem 6.15 is in accordance with what Roos predicts in [25] for an algebra of this type. As far as G-quadraticity is concerned, we have:

THEOREM 6.16. *Let F be a space cubic. The following conditions are equivalent:*

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(1) *R_F is G-quadratic,*
(2) *F is singular and* $\mathrm{rank}(x) \geqslant 2$ *for every nonzero linear form x in R_F.*
(3) *F is singular and the ideal of the 2×2 minors of the Hessian matrix $He(F)$ has codimension four.*

Proof. We have seen already that (2) is equivalent to (3) and (1) $\Rightarrow$ (2) holds for any n. To prove that (2) $\Rightarrow$ (1) we argue as follows: let P be a singular point for F and let x be the corresponding linear form in R_F. Then either $\mathrm{rank}(x) = 3$ or $\mathrm{rank}(x) = 2$. If $\mathrm{rank}(x) = 3$, then we have seen already that R_F has a Gröbner flag (it is the argument that works for the generic singular cubic). We are left with the case in which $\mathrm{rank}(x) = 2$. Set $S = R_F/(x)$. The Hilbert series of S is $1 + 3z + 2z^2$. By 6.15, we know that R_F is quadratic and hence S is quadratic. It has been shown in [8] that any quadratic Artinian algebra A with $\dim A_2 = 2$ admits a linear form y such that $y^2 = 0$ and $yA_1 = A_2$. Therefore there exists a linear form $\bar{y} \in S$ such that $\bar{y}^2 = 0$ and $\bar{y}S_1 = S_2$. We extend x, y to a basis of the vector space of linear forms in R_F with, say, z, t and consider the presentation $K[x_1, x_2, x_3, x_4] \to R_F$ of R_F obtained by sending x_1 to x, x_2 to y, x_3 to z and x_4 to t. By the above discussion we know that the defining ideal I_F of R_F contains polynomials of the form:

$$\begin{array}{lll} x_1^2, & x_1 L, & x_2^2 + x_1 L_1, \\ x_3^2 + x_1 L_2 + x_2 L_3, & x_3 x_4 + x_1 L_4 + x_2 L_5, & x_4^2 + x_1 L_6 + x_2 L_7 \end{array}$$

where L is a nonzero linear form in x_2, x_3, x_4 and L_i is a linear form in x_1, x_2, x_3, x_4 for every $i = 1, \dots, 7$. Consider the reverse lexicographic order induced by $x_4 > x_3 > x_2 > x_1$. We have that $\mathrm{in}(I_F)$ contains $x_1^2, x_2^2, x_3^2, x_3x_4, x_4^2$ and $x_1\mathrm{in}(L)$, where $\mathrm{in}(L)$ is either x_2 or x_3 or x_4. But $\mathrm{in}(L)$ cannot be x_2, otherwise R_F would be 0 in degree 3. Hence $\mathrm{in}(L)$ is either x_3 or x_4 and this implies that the given equations form a Gröbner basis of I_F, because the Hilbert function of their initial terms is the right one. □

Remark 6.17. We have seen that for a planar cubic F the ring R_F has a Gröbner flag as soon as it is G-quadratic. This is not the case for space cubics. To see this, consider a space cubic F with exactly one singular point P. Let x be the corresponding linear form and assume that x has rank 2 in R_F. Should F have a Gröbner flag, say $0 \subset V_1 \subset V_2 \subset V_3 \subset V_4$, then V_1 would be equal to $\langle x \rangle$. Since x has rank 2, the ideal $0{:}\,x$ must be equal to (V_2). This would imply that $(0{:}\,x)^2 \subseteq (x)$ since $V_2 \subset (V_1){:}\,(V_2)$. The inclusion $(0{:}\,x)^2 \subseteq (x)$ does not always hold. For example, let

$$F = 6xyz + 3y^2t - 3z^2t + t^3 \in \mathbf{Q}[x, y, z, t].$$

Then

$$I_F = (t^2 - y^2, tz + yx, z^2 + y^2, ty - zx, tx, x^2)$$

and it is easy to see that $(1, 0, 0, 0)$ is the only singular point of F and that the

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corresponding linear form x has rank 2 in R_F. In R_F one has $0: x = (x, t)$ and $t^2 \notin (x)$. Hence R_F does not have a Gröbner flag.

Note that according to 6.16 the algebra R_F is G-quadratic. The proof of 6.16 indicates how to get an explicit Gröbner basis of quadrics. One has to detect a linear form, say L, such that $L^2 = 0$ in $R_F/(x)$ and L has rank 2 in $R_F/(x)$. One easily checks that $L = z + t$ is such a form. Then in the coordinate system $t_1 = y, z_1 = z$, $y_1 = t + z, x_1 = x$ and with respect to the rev.lex. order induced by $t_1 > z_1 > y_1 > x_1$, the ideal I_F has a Gröbner basis of quadric. Namely, in the new coordinates, the ideal I_F is generated by

$$\begin{array}{lll} -t_1^2 + z_1^2 - 2z_1y_1 + y_1^2, & -z_1^2 + z_1y_1 + t_1x_1, & t_1^2 + z_1^2, \\ -t_1z_1 + t_1y_1 - z_1x_1, & -z_1x_1 + y_1x_1, & x_1^2 \end{array}$$

and the reduced Gröbner basis of I_F is, as predicted by 6.16, the following:

$$\begin{array}{lll} x_1^2, & z_1x_1 - y_1x_1, & y_1^2 + 2t_1x_1, \\ z_1^2 - z_1y_1 - t_1x_1, & t_1z_1 - t_1y_1 + y_1x_1, & t_1^2 + z_1y_1 + t_1x_1 \end{array}$$

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