REMARK ON THE DEFINING RELATION OF THE σ-SYMBOLS

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Summary. It is shown algebraically that the full defining relation for the $\sigma^{k\dot{\mu}\nu}$ is effectively contained in its symmetric part.

The development of the spinor analysis of Infeld and van der Waerden [1] [1] has as its starting point the possibility of establishing a linear one-to-one correspondence between real world vectors and hermitean spinors. Thus if a^k is a world vector and $\alpha_{\dot{\mu}\nu}$ the corresponding spinor,

(1) $$a^k = \sigma^{k\dot{\mu}\nu}\alpha_{\dot{\mu}\nu}.$$

If spin indices be raised and lowered by means of the skew-symmetric basic spinor $\gamma_{\mu\nu}$ and its inverse $\gamma^{\mu\nu}$, i.e.

(2) $$\zeta^\mu = \gamma^{\mu\nu}\zeta_\nu, \quad \zeta_\mu = \gamma_{\nu\mu}\zeta^\nu, \quad (\gamma^{\mu\lambda}\gamma_{\nu\lambda} = \delta^\mu_\nu),$$

one has the quadratic invariant $I(\alpha) = \alpha^{\dot{\mu}\nu}\alpha_{\dot{\mu}\nu}$ which has the same signature as the invariant $I(a) = a^k a_k$, and these two invariants are to be identified with each other. One infers the identity

(3) $$g_{kl}\sigma^{k\dot{\mu}\nu}\sigma^{l\dot{\alpha}\beta} = \gamma^{\dot{\mu}\dot{\alpha}}\gamma^{\nu\beta}.$$

It follows without much difficulty that

(4) $$\sigma^{k\dot{\lambda}\mu}\sigma^l{}_{\dot{\lambda}\nu} + \sigma^{l\dot{\lambda}\mu}\sigma^k{}_{\dot{\lambda}\nu} = g^{kl}\delta^\mu_\nu.$$

Eqs. (3) and (4) are identical with eqs. (11) and (15) of IW.

Now, for instance Bade and Jehle [2], and Corson [3] proceed in much the same way, but the latter remarks that all the properties of the σ-symbols can be obtained without reference to any particular representation by adopting the following defining relation for them:

(5) $$\sigma^{k\dot{\lambda}\mu}\sigma^l{}_{\dot{\lambda}\nu} = \tfrac{1}{2}g^{kl}\delta^\mu_\nu - \tfrac{1}{2}ie^{klst}\sigma_s{}^{\dot{\lambda}\mu}\sigma_{t\dot{\lambda}\nu},$$

granted that $\sigma^{k\dot{\lambda}\mu}$ is hermitean. ($e^{klst} = (-g)^{-\frac{1}{2}}\varepsilon^{klst}$, where ε^{klst} is the Levi-Civita alternating tensor density.) Harish-Chandra [4] also takes the

[1] This paper will be referred to as IW.

σ's to be defined by (5). On the other hand this relation nowhere appears in IW: (4) is only the symmetric part of (5). Yet one would expect the stronger relation (5) to be implicit in the linear correspondence (1), the hermiticity of $\sigma^{k\dot{\mu}\nu}$, and the identity of $I(a)$ with $I(\alpha)$, bearing in mind the group-theoretical basis of spinor calculus in flat space, i.e. the homomorphism between the Lorentz group and the group of linear two-dimensional unimodular complex transformations. It is the purpose of this note to show that (5) can indeed be inferred from (4) by means of a simple algebraic argument, an inevitable ambiguity of sign aside.

Consider the spin-invariant tensor of valence 4

$$\Omega^{klmn} = 2\sigma^k{}_{\dot\lambda\nu}\sigma^{l\dot\lambda\alpha}\sigma^m{}_{\dot\mu\alpha}\sigma^{n\dot\mu\nu} - g^{kl}g^{mn} - g^{kn}g^{lm} + g^{km}g^{ln}. \tag{6}$$

Interchanging k and l one has

$$\Omega^{lkmn} = 2(g^{kl}\delta^\alpha_\nu - \sigma^k{}_{\dot\lambda\nu}\sigma^{l\dot\lambda\alpha})\sigma^m{}_{\dot\mu\alpha}\sigma^{n\dot\mu\nu} - g^{kl}g^{mn} - g^{kn}g^{lm} + g^{km}g^{ln}, \tag{7}$$

where (4) has been used. In the first term

$$g^{kl}\delta^\alpha_\nu\sigma^m{}_{\dot\mu\alpha}\sigma^{n\dot\mu\nu} = g^{kl}g^{mn},$$

from the spin trace of (4). It follows by inspection that

$$\Omega^{(kl)mn} = 0. \tag{8}$$

Interchanging other pairs of indices one finds in exactly the same way that

$$\Omega^{k(lm)n} = \Omega^{kl(mn)} = 0. \tag{9}$$

In other words, Ω^{klmn} is an alterating tensor and therefore a scalar multiple of e^{klmn}, say

$$\Omega^{klmn} = \omega e^{klmn}. \tag{10}$$

Now transvect (7) with $\sigma_l^{\dot\rho\beta}\sigma_{n\dot\rho\gamma}$. One gets

$$2\sigma^k{}_{\dot\lambda\nu}\sigma^{l\dot\lambda\alpha}\sigma_l{}^{\dot\rho\beta}\sigma^m{}_{\dot\mu\alpha}\sigma^{n\dot\mu\nu}\sigma_{n\dot\rho\gamma} = \sigma^{k\dot\rho\beta}\sigma^m{}_{\dot\rho\gamma} + \sigma^{m\dot\rho\beta}\sigma^k{}_{\dot\rho\gamma} - g^{km}\sigma^{l\dot\rho\beta}\sigma_{l\dot\rho\gamma} \\ + \omega e^{klmn}\sigma_l{}^{\dot\rho\beta}\sigma_{n\dot\rho\gamma}.$$

On the left the second and third factors give $\gamma^{\dot\lambda\dot\rho}\gamma^{\alpha\beta}$, and the fifth and sixth $\delta^{\dot\mu}{}_{\dot\rho}\delta^\nu_\gamma$, each time because of (3). The left hand member thus becomes $-2\sigma^k{}_{\dot\lambda\gamma}\sigma^{m\dot\lambda\beta}$. In view of (4) the first two terms on the right combine to give $g^{km}\delta^\beta_\nu$, whilst the third term is $-2g^{km}\delta^\beta_\gamma$. (11) thus becomes after some relabelling

$$\sigma^{k\dot\lambda\mu}\sigma^l{}_{\dot\lambda\nu} = \tfrac{1}{2}g^{kl}\delta^\mu_\nu - \tfrac{1}{2}\omega e^{klst}\sigma_s^{\dot\lambda\mu}\sigma_{t\dot\lambda\nu}. \tag{12}$$

This may be written as

$$S^{kl\mu}{}_\nu = -\tfrac{1}{2}\omega e^{klst}S_{st}{}^\mu{}_\nu, \tag{13}$$

where

$$S^{kl\mu}{}_{\nu} = \tfrac{1}{2}(\sigma^{k\dot{\lambda}\mu}\sigma^{l}{}_{\dot{\lambda}\nu} - \sigma^{l\dot{\lambda}\mu}\sigma^{k}{}_{\dot{\lambda}\nu}).$$

Now transvect (13) with e_{klmn} and recall that

$$e^{klst}e_{klmn} = \delta^{klst}_{klmn} = -2\delta^{st}_{mn}.$$

One gets

$$e_{klmn}S^{kl\mu}{}_{\nu} = 2\omega S_{mn}{}^{\mu}{}_{\nu},$$

which may be used in (13) to give

$$(\omega^2+1)S^{kl\mu}{}_{\nu} = 0. \tag{14}$$

$S^{kl\mu}{}_{\nu}$ cannot be zero. (Note that $S^{kl\mu\nu}S_{kl\mu\nu} = 6$.) It therefore follows finally that

$$\omega = \pm i, \tag{15}$$

so that (12) agrees with (5) except for the ambiguity of sign of the second term on the right. However, this is as should be since the choice of the positive sign is purely conventional. With (12) and (15) one is now in a position to *verify* that the integrability conditions on the Lorentz group,

$$S^{kl\mu}{}_{\rho}S^{mn\rho}{}_{\nu} - S^{mn\mu}{}_{\rho}S^{kl\rho}{}_{\nu} = 4g^{[k[m}S^{n]l]\mu}{}_{\nu},$$

are satisfied.

References

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[2] Bade, W. L., and Jehle, H., An introduction to spinors, Revs. Mod. Phys., 25 (1953), 714–729.
[3] Corson, E. M., Tensors, spinors, and relativistic wave equations. Blackie, London (1953), Chap. II, §§ 6–13.
[4] Harish-Candra, A note on the σ-symbols, Proc. Ind. Acad. Sci., 23 (1946), 152–163.

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