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SINGULARITIES OF EVOLUTES OF HYPERSURFACES IN HYPERBOLIC SPACE

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Abstract We study the differential geometry of hypersurfaces in hyperbolic space. As an application of the theory of Lagrangian singularities, we investigate the contact of hypersurfaces with families of hyperspheres or equidistant hyperplanes.

Keywords: hypersphere; equidistant hyperplane; hyperbolic (de Sitter) evolute; hyperbolic timelike (spacelike) height function; Lagrangian singularities

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1. Introduction

In this paper we study the differential geometry of hypersurfaces in hyperbolic space from a contact viewpoint as an application of singularity theory. There are several articles [**9**,**12**–**15**] concerning the contact of submanifolds in Euclidean space with hyperplanes or hyperspheres. Such hypersurfaces are known as totally umbilic hypersurfaces in Euclidean space. A singular point of the Gauss map of a hypersurface (i.e. a parabolic point) is a point at which the tangent hyperplane has degenerate contact with the hypersurface (cf. [**2**, **3**, **9**]). Therefore, we might say that the theory of singularities for Gauss maps describes the contact of hypersurfaces with hyperplanes. For the contact of hypersurfaces with hyperspheres, the evolute of a hypersurface plays a role similar to that of a Gauss map.

On the other hand, the basic notions and tools for the study of the differential geometry of hypersurfaces in hyperbolic space has recently been established in [**6**–**8**]. The hyperbolic Gauss indicatrix of a hypersurface in hyperbolic space has been explicitly described and the contact of hypersurfaces with hyperhorospheres has been systematically studied as an application of singularity theory to the hyperbolic Gauss indicatrix. In hyperbolic space there are four kinds of totally umbilic hypersurfaces (cf. § 2). The hyperhorosphere is one of the totally umbilic hypersurfaces in hyperbolic space. We have already studied the contact of hypersurfaces with hyperhorospheres in [**6**]. Therefore, we study the contact of hypersurfaces with totally umbilic hypersurfaces other than hyperhorospheres in this paper. In § 2 we review the basic notions and concepts in hyperbolic differential geometry on hypersurfaces. We adopt the model of hyperbolic space in Minkowski space, which is quite natural for the study of hypersurfaces from the contact viewpoint. We introduce the notion of hyperbolic (respectively, de Sitter) evolutes of hypersurfaces whose singularities describe the contact of hypersurfaces with hyperspheres (respectively, equidistant hyperplanes) in § 3. We also introduce the notion of timelike ridge points (respectively, spacelike ridge points) at which the hypersurface has $A_{k\geqslant 3}$-type contact with a hypersphere (respectively, an equidistant hyperplane). The ridges of surfaces in Euclidean 3-space were originally introduced by Porteous [**15**] as the sets of points at which the surface has a higher-order contact with some of their focal spheres. It is deeply related to the singularities of the distance-squared function on the surface. We define the analogous notion of hypersurfaces in hyperbolic space. For the study of their geometric meanings, we investigate hyperbolic timelike (respectively, spacelike) height functions on hypersurfaces. In § 4 we show that the hyperbolic (respectively, de Sitter) evolute of a hypersurface is a caustic of a certain Lagrangian submanifold in the cotangent bundle of the hyperbolic n-space whose generating family is the hyperbolic timelike (respectively, spacelike) height function. In § 5 we apply the theory of Lagrangian singularities and interpret a singularity of a hyperbolic (respectively, de Sitter) evolute as describing not only the contact of the hypersurface with a hypersphere (respectively, equidistant hyperplane) but also the contact of the hypersurface with a family of hypersurfaces (respectively, equidistant hyperplanes). This study leads us to the osculating spherical (respectively, equidistant planar) foliations. In § 6 we study generic properties, and we give a classification for $n = 3$ in § 7.

We shall assume throughout the paper that all the maps and manifolds are C^∞ unless the contrary is explicitly stated.

2. Basic concepts and notions

In this section we review basic notions and concepts on the differential geometry of hypersurfaces in hyperbolic space. We adopt the model of hyperbolic space in Minkowski space.

Let $\mathbb{R}^{n+1} = \{(x_0, x_1, \dots, x_n) \mid x_i \in \mathbb{R}\ (i = 0, 1, \dots, n)\}$ be an $(n+1)$-dimensional vector space. For any vectors $\boldsymbol{x} = (x_0, x_1, \dots, x_n)$, $\boldsymbol{y} = (y_0, y_1, \dots, y_n)$ in $\mathbb{R}^{n+1}$, the *pseudo-scalar product* of $\boldsymbol{x}$ and $\boldsymbol{y}$ is defined to be $\langle \boldsymbol{x}, \boldsymbol{y}\rangle = -x_0y_0 + \sum_{i=1}^n x_iy_i$. We call $(\mathbb{R}^{n+1}, \langle\cdot,\cdot\rangle)$ the $(n+1)$-dimensional *Minkowski space*. We use $\mathbb{R}^{n+1}_1$ instead of $(\mathbb{R}^{n+1}, \langle\cdot,\cdot\rangle)$.

We say that a vector $\boldsymbol{x} \in \mathbb{R}^{n+1}_1 \setminus \{\boldsymbol{0}\}$ is *spacelike*, *lightlike* or *timelike* if $\langle \boldsymbol{x}, \boldsymbol{x}\rangle > 0$, $= 0$ or < 0, respectively. The norm of the vector $\boldsymbol{x} \in \mathbb{R}^{n+1}$ is defined by $\|\boldsymbol{x}\| = \sqrt{|\langle \boldsymbol{x}, \boldsymbol{x}\rangle|}$. For any vector $\boldsymbol{v} \in \mathbb{R}^{n+1}$ and a real number c, we define the hyperplane with pseudo-normal

$\boldsymbol{v}$ by

$$\mathrm{HP}(\boldsymbol{v}, c) = \{\boldsymbol{x} \in \mathbb{R}^{n+1}_1 \mid \langle \boldsymbol{x}, \boldsymbol{v} \rangle = c\}.$$

We call $\mathrm{HP}(\boldsymbol{v}, c)$ a *spacelike hyperplane*, a *timelike hyperplane* or a *lightlike hyperplane* when $\boldsymbol{v}$ is timelike, spacelike or lightlike, respectively.

We now define the *hyperbolic n-space* by

$$H^n_+(-1) = \{\boldsymbol{x} \in \mathbb{R}^{n+1}_1 \mid \langle \boldsymbol{x}, \boldsymbol{x} \rangle = -1,\ x_0 > 0\}$$

and the *de Sitter n-space* by

$$S^n_1 = \{\boldsymbol{x} \in \mathbb{R}^{n+1}_1 \mid \langle \boldsymbol{x}, \boldsymbol{x} \rangle = 1\}.$$

We also define $H^n_-(-1) = \{\boldsymbol{x} \in \mathbb{R}^{n+1}_1 \mid \langle \boldsymbol{x}, \boldsymbol{x} \rangle = -1,\ x_0 < 0\}$.

For any $\boldsymbol{a}_1, \boldsymbol{a}_2, \dots, \boldsymbol{a}_n \in \mathbb{R}^{n+1}_1$, we define a vector $\boldsymbol{a}_1 \wedge \boldsymbol{a}_2 \wedge \cdots \wedge \boldsymbol{a}_n$ by

$$\boldsymbol{a}_1 \wedge \boldsymbol{a}_2 \wedge \cdots \wedge \boldsymbol{a}_n = \begin{vmatrix} -\boldsymbol{e}_1 & \boldsymbol{e}_2 & \cdots & \boldsymbol{e}_{n+1} \\ a^1_0 & a^1_1 & \cdots & a^1_n \\ a^2_0 & a^2_1 & \cdots & a^2_n \\ \vdots & \vdots & \cdots & \vdots \\ a^n_0 & a^n_1 & \cdots & a^n_n \end{vmatrix},$$

where $\boldsymbol{e}_1, \boldsymbol{e}_2, \dots, \boldsymbol{e}_{n+1}$ is the canonical basis of $\mathbb{R}^{n+1}_1$ and $\boldsymbol{a}_i = (a^i_0, a^i_1, \dots, a^i_n)$. We can easily check that

$$\langle \boldsymbol{a}, \boldsymbol{a}_1 \wedge \boldsymbol{a}_2 \wedge \cdots \wedge \boldsymbol{a}_n \rangle = \det(\boldsymbol{a}, \boldsymbol{a}_1, \dots, \boldsymbol{a}_n),$$

so that $\boldsymbol{a}_1 \wedge \boldsymbol{a}_2 \wedge \cdots \wedge \boldsymbol{a}_n$ is pseudo-orthogonal to any $\boldsymbol{a}_i$ $(i = 1, \dots, n)$.

We also define a set $\mathrm{LC}_c = \{\boldsymbol{x} \in \mathbb{R}^{n+1}_1 \mid \langle \boldsymbol{x} - \boldsymbol{c}, \boldsymbol{x} - \boldsymbol{c} \rangle = 0\}$, which is called *a closed lightcone* with the vertex $\boldsymbol{c}$. We define

$$\mathrm{LC}^*_+ = \{\boldsymbol{x} = (x_0, x_1 \dots x_n) \in \mathrm{LC}_0 \mid x_0 > 0\}$$

and we call it *the future lightcone* at the origin. If $\boldsymbol{x} = (x_0, x_1, \dots, x_n)$ is a non-zero lightlike vector, then $x_0 \neq 0$. Therefore, we have

$$\tilde{\boldsymbol{x}} = \left(1, \frac{x_1}{x_0}, \dots, \frac{x_n}{x_0}\right) \in S^{n-1}_+ = \{\boldsymbol{x} = (x_0, x_1, \dots, x_n) \mid \langle \boldsymbol{x}, \boldsymbol{x} \rangle = 0,\ x_0 = 1\}.$$

Here, we call S^{n-1}_+ *the lightcone $(n-1)$-sphere.*

Let $\boldsymbol{x} : U \to H^n_+(-1)$ be an embedding, where $U \subset \mathbb{R}^{n-1}$ is an open subset. We define $M = \boldsymbol{x}(U)$ and identify M and U by the embedding $\boldsymbol{x}$.

For any $p = \boldsymbol{x}(u) \in M \subset H^n_+(-1)$, we have $\langle \boldsymbol{x}(u), \boldsymbol{x}(u) \rangle = -1$. It follows that

$$\langle \boldsymbol{x}_{u_i}(u), \boldsymbol{x}(u) \rangle = 0 \quad (i = 1, 2, \dots, n-1),$$

where $u = (u_1, u_2, \dots, u_{n-1})$ and $\boldsymbol{x}_{u_i}(u) = \partial \boldsymbol{x}/\partial u_i(u) = (x_{0u_i}(u), x_{1u_i}(u), \dots, x_{nu_i}(u))$. Hence the tangent space of M at p is

$$T_pM = \langle \boldsymbol{x}_{u_1}(u), \boldsymbol{x}_{u_2}(u), \dots, \boldsymbol{x}_{u_{n-1}}(u) \rangle_{\mathbb{R}}.$$

Let N_pM be the normal space of M at $p = \boldsymbol{x}(u)$ in $\mathbb{R}^{n+1}_1$, then N_pM is a Lorentz plane. Define a spacelike unit vector

$$\boldsymbol{e}(u) = \frac{\boldsymbol{x}(u) \wedge \boldsymbol{x}_{u_1}(u) \wedge \cdots \wedge \boldsymbol{x}_{u_{n-1}}(u)}{\|\boldsymbol{x}(u) \wedge \boldsymbol{x}_{u_1}(u) \wedge \cdots \wedge \boldsymbol{x}_{u_{n-1}}(u)\|} \in S^1(N_pM \cap T_pH^n_+(-1)).$$

Since $\boldsymbol{x}(u) \in N_pM$, we have

$$N_pM = \langle \boldsymbol{x}(u), \boldsymbol{e}(u)\rangle_{\mathbb{R}}.$$

A map $\mathbb{E} : U \to S^n_1$ defined by $\mathbb{E}(u) = \boldsymbol{e}(u)$ is called the *de Sitter Gauss indicatrix* of $\boldsymbol{x}(U) = M$. We construct an extrinsic differential geometry on $\boldsymbol{x}$ by using the unit normal $\boldsymbol{e}$ as the unit normal of a hypersurface in Euclidean space. In this case, the de Sitter Gauss indicatrix of a hypersurface plays a role similar to that of the Gauss map for a hypersurface in Euclidean space. We can easily show that $D_v\boldsymbol{e} \in T_pM$ for any $p = \boldsymbol{x}(u_0) \in M$ and $\boldsymbol{v} \in T_pM$. Here D_v denotes the covariant derivative with respect to the tangent vector $\boldsymbol{v}$.

We call the linear transformation $A_p = -d\mathbb{E} : T_pM \to T_pM$ *the* (*de Sitter*) *shape operator* of $M = \boldsymbol{x}(U)$ at $p = \boldsymbol{x}(u_0)$. We denote the eigenvalue of A_p by κ_p, which we call a (*de Sitter*) *principal curvature*. We call the eigenvector of A_p the (*de Sitter*) *principal direction*. By definition, κ_p is a (de Sitter) principal curvature if and only if $\det(A_p - \kappa_p I) = 0$. We now define the notion of (de Sitter) Gauss–Kronecker curvatures as follows. *The* (*de Sitter*) *Gauss–Kronecker curvature* of $M = \boldsymbol{x}(U)$ at $p = \boldsymbol{x}(u_0)$ is defined to be $K_d(u_0) = \det A_p$.

We say that a point $p = \boldsymbol{x}(u_0) \in M$ is an *umbilic* point if $A_p = k_p\,\mathrm{id}_{T_pM}$. We also say that M is *totally umbilic* if all points of M are umbilic. A hypersurface given by the intersection of $H^n_+(-1)$ and a spacelike hyperplane, a timelike hyperplane or a lightlike hyperplane is, respectively, called a *hypersphere*, an *equidistant hyperplane* or a *hyperhorosphere*. Moreover, if the hypersurface is given by the intersection of $H^n_+(-1)$ and a timelike hyperplane through the origin of $\mathbb{R}^{n+1}_1$, the equidistant hyperplane is simply called a *hyperplane*. Then the following proposition is a well-known result.

Proposition 2.1. *Suppose that $M = \boldsymbol{x}(U)$ is totally umbilic, then $\kappa(p)$ is constant κ. Under this condition, we have the following classification.*

(1) *Suppose that $\kappa^2 \neq 1$.*

  (a) *If $\kappa \neq 0$ and $\kappa^2 < 1$, then M is part of an equidistant hyperplane.*

  (b) *If $\kappa \neq 0$ and $\kappa^2 > 1$, then M is part of a hypersphere.*

  (c) *If $\kappa = 0$, then M is part of a hyperplane.*

(2) *If $\kappa^2 = 1$, then M is part of a hyperhorosphere.*

Since $\boldsymbol{x}_{u_i}$ $(i = 1, \dots, n-1)$ are spacelike vectors, we induce the Riemannian metric (*first fundamental form*) $ds^2 = \sum_{i=1}^{n-1} g_{ij}du_idu_j$ on $M = \boldsymbol{x}(U)$, where $g_{ij}(u) = \langle \boldsymbol{x}_{u_i}(u), \boldsymbol{x}_{u_j}(u)\rangle$ for any $u \in U$. We define the *de Sitter second fundamental invariant* by $h_{ij}(u) = \langle -\mathbb{E}_{u_i}(u), \boldsymbol{x}_{u_j}(u)\rangle$ for any $u \in U$. By arguments similar to those of differential geometry on hypersurfaces in Euclidean space, we can show that the de Sitter Gauss–Kronecker curvature is given by

$$K_d = \frac{\det(h_{ij})}{\det(g_{\alpha\beta})}.$$

For a hypersurface $\boldsymbol{x} : U \to H^n_+(-1)$, we say that a point $u_0 \in U$ or $p = \boldsymbol{x}(u_0)$ is a (*de Sitter*) *flat point* if $h_{ij}(u_0) = 0$ for all i, j. Therefore, $p = \boldsymbol{x}(u_0)$ is a (de Sitter) flat point if and only if p is an umbilic point with the vanishing (de Sitter) principal curvature.

We now introduce the notion of evolutes of hypersurfaces in hyperbolic space. We say that a point $p = \boldsymbol{x}(u_0) \in M$ is a *horoparabolic point* if one of the de Sitter principal curvatures satisfies the condition that $\kappa^2(u_0) = 1$. For a hypersurface $\boldsymbol{x} : U \to H^n_+(-1)$, we define the *total evolute* of $\boldsymbol{x}(U) = M$ by

$$\mathrm{TE}^{\pm}_M = \left\{ \pm\frac{\kappa(u)}{\sqrt{|\kappa^2(u)-1|}}\left(\boldsymbol{x}(u) + \frac{1}{\kappa(u)}\boldsymbol{e}(u)\right) \,\middle|\, \kappa(u) \text{ is a de Sitter principal curvature at } p = \boldsymbol{x}(u),\ u \in U \right\}.$$

For a hypersurface as above, we have the following decomposition of the total evolute:

$$\mathrm{TE}^{\pm}_M(u) = \mathrm{HE}^{\pm}_M \cup \mathrm{SE}^{\pm}_M,$$

where

$$\mathrm{HE}^{\pm}_M = \left\{ \pm\frac{\kappa(u)}{\sqrt{|\kappa^2(u)-1|}}\left(\boldsymbol{x}(u) + \frac{1}{\kappa(u)}\boldsymbol{e}(u)\right) \,\middle|\, \kappa(u) \text{ is a de Sitter principal curvature with } \kappa^2(u) > 1 \text{ at } p = \boldsymbol{x}(u),\ u \in U \right\},$$

and

$$\mathrm{SE}^{\pm}_M = \left\{ \pm\frac{\kappa(u)}{\sqrt{|\kappa^2(u)-1|}}\left(\boldsymbol{x}(u) + \frac{1}{\kappa(u)}\boldsymbol{e}(u)\right) \,\middle|\, \kappa(u) \text{ is a de Sitter principal curvature with } \kappa^2(u) < 1 \text{ at } p = \boldsymbol{x}(u),\ u \in U \right\}.$$

We can show that $\mathrm{HE}^{\pm}_M \subset H^n_+(-1) \cup H^n_-(-1)$ and $\mathrm{SE}^{\pm}_M \subset S^n_1$. If $\boldsymbol{x} \in H^n_-(-1)$, then $-\boldsymbol{x} \in H^n_+(-1)$. It follows that $\mathrm{HE}^{+}_M \subset H^n_+(-1)$ or $\mathrm{HE}^{-}_M \subset H^n_+(-1)$. We consider the component of $\mathrm{HE}^{\pm}_M$ located on $H^n_+(-1)$. Therefore, we call $\mathrm{HE}^{\pm}_M$ (respectively, $\mathrm{SE}^{\pm}_M$) the *hyperbolic evolute* (respectively, *de Sitter evolute*) of $\boldsymbol{x}(U) = M$.

We define a smooth mapping $\mathrm{HE}^{\pm}_\kappa : U \to H^n_+(-1)$ by

$$\mathrm{HE}^{\pm}_\kappa(u) = \pm\frac{\kappa(u)}{\sqrt{|\kappa^2(u)-1|}}\left(\boldsymbol{x}(u) + \frac{1}{\kappa(u)}\boldsymbol{e}(u)\right),$$

where we fix a de Sitter principal curvature $\kappa(u)$ on U at u with $\kappa^2(u) > 1$. We can also define a smooth mapping $\mathrm{SE}^{\pm}_\kappa : U \to S^n_1$ in a similar way with $\kappa^2(u) < 1$. We have the following proposition.

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Proposition 2.2. *Let $M = \boldsymbol{x}(U)$ be a hypersurface in $H^n_+(-1)$ without horoparabolic points or de Sitter flat points.*

(A) *The following are equivalent.*

(1) *M is totally umbilic with $\kappa^2 > 1$.*

(2) *$\mathrm{HE}^{\pm}_M$ is a point in $H^n_+(-1)$.*

(3) *M is part of a hypersphere.*

(B) *The following are equivalent.*

(1) *M is totally umbilic with $0 < \kappa^2 < 1$.*

(2) *$\mathrm{SE}^{\pm}_M$ is a point in S^n_1.*

(3) *M is part of an equidistant hyperplane.*

Proof. (A) We assume that condition (1) holds. Then the (de Sitter) principal curvature $\kappa(u) = \kappa$ is constant and $\kappa^2 > 1$. Therefore, we have

$$\frac{\partial \mathrm{HE}^{\pm}_{\kappa}}{\partial u_i}(u) = \pm\frac{\kappa}{\sqrt{\kappa^2 - 1}}\left(\boldsymbol{x}_{u_i}(u) + \frac{1}{\kappa}\boldsymbol{e}_{u_i}(u)\right)$$

for any $u \in U$. By the definition of the (de Sitter) principal curvature, $-\boldsymbol{e}_{u_i} = \kappa \boldsymbol{x}_{u_i}$ for $i = 1, \dots, n-1$. It follows that $\partial\, \mathrm{HE}^{\pm}_{\kappa} / \partial u_i(u) = 0$ for $i = 1, \dots, n-1$. We conclude that $\mathrm{HE}^{\pm}_{\kappa}(u)$ is a point. Conversely, for any $u \in U$ and a de Sitter principal curvature $\kappa(u)$, we assume that

$$\mathrm{HE}^{\pm}_{\kappa}(u) = \pm\frac{\kappa(u)}{\sqrt{\kappa^2(u) - 1}}\left(\boldsymbol{x}(u) + \frac{1}{\kappa(u)}\boldsymbol{e}(u)\right)$$

is a point. We calculate that

$$\begin{aligned}\frac{\partial \mathrm{HE}^{\pm}_{\kappa}}{\partial u_i}(u) = {} & \frac{\mp\kappa_{u_i}(u)}{(\kappa^2(u) - 1)^{3/2}}\left(\boldsymbol{x}(u) + \frac{1}{\kappa(u)}\boldsymbol{e}(u)\right) \\ & \pm\frac{\kappa(u)}{\sqrt{\kappa^2(u) - 1}}\left(\boldsymbol{x}_{u_i}(u) - \frac{\kappa_{u_i}(u)}{\kappa^2(u)}\boldsymbol{e}(u) + \frac{1}{\kappa(u)}\boldsymbol{e}_{u_i}(u)\right).\end{aligned}$$

Since

$$\boldsymbol{e}_{u_i} \in \langle \boldsymbol{x}_{u_1}, \dots, \boldsymbol{x}_{u_{n-1}} \rangle_{\mathbb{R}} \text{ and } \{\boldsymbol{x}, \boldsymbol{x}_{u_1}, \dots, \boldsymbol{x}_{u_{n-1}}, \boldsymbol{e}\}$$

are linearly independent,

$$\partial\, \mathrm{HE}^{\pm}_{\kappa} / \partial u_i(u) = 0 \text{ if and only if } \kappa_i(u) = 0 \text{ for } i = 1, \dots, n-1.$$

Therefore, $\kappa(u) = \kappa$ is constant and $\kappa^2 > 1$. Moreover, we assume that there exists another (de Sitter) principal curvature $\bar{\kappa}$. Since $\mathrm{HE}^{\pm}_{\kappa}(u) = \mathrm{HE}^{\pm}_{\bar{\kappa}}(u)$ is a point, we have $\kappa = \bar{\kappa}$. This means that M is totally umbilic.

Since a hypersphere is totally umbilic, conditions (1) and (3) are equivalent by Proposition 2.1. This completes the proof of (A).

The proof of (B) is also given by straightforward calculations like those for the proof of (A). $\square$

3. Height functions

In this section we consider two kinds of families of height functions on a hypersurface in hyperbolic space in order to describe the hyperbolic and the de Sitter evolute of a hypersurface.

For this purpose we need some concepts and results in the theory of unfoldings of function germs. We give a brief review of the theory in the appendix.

We now define two families of functions

$$H^{\mathrm{T}} : U \times (H^n_+(-1) \setminus M) \to \mathbb{R}$$

by $H^{\mathrm{T}}(u, v) = \langle \boldsymbol{x}(u), \boldsymbol{v}\rangle$ and

$$H^{\mathrm{S}} : U \times S^n_1 \to \mathbb{R}$$

by $H^{\mathrm{S}}(u, v) = \langle \boldsymbol{x}(u), \boldsymbol{v}\rangle$. We call H^{T} (respectively, H^{S}) a *hyperbolic timelike height function* (respectively, *hyperbolic spacelike height function*) on $\boldsymbol{x} : U \to H^n_+(-1)$. We define $h^{\mathrm{T}}_v(u) = H^{\mathrm{T}}(u, \boldsymbol{v})$ (respectively, $h^{\mathrm{S}}_v(u) = H^{\mathrm{S}}(u, \boldsymbol{v})$). The following proposition is a standard result.

Proposition 3.1. *Let $\boldsymbol{x} : U \to H^n_+(-1)$ be a hypersurface. Then*

(1) $(\partial h^{\mathrm{T}}_v/\partial u_i)(u) = 0$ $(i = 1, \dots, n-1)$ *if and only if there exist real numbers λ, μ such that $\boldsymbol{v} = \lambda\boldsymbol{x}(u) + \mu\boldsymbol{e}(u)$, $\lambda^2 - \mu^2 = 1$; and*

(2) $(\partial h^{\mathrm{S}}_v/\partial u_i)(u) = 0$ $(i = 1, \dots, n-1)$ *if and only if there exist real numbers λ, μ such that $\boldsymbol{v} = \lambda\boldsymbol{x}(u) + \mu\boldsymbol{e}(u)$, $\lambda^2 - \mu^2 = -1$.*

Since $\boldsymbol{v} \notin M$, we have that $\mu \neq 0$ in case (1). By Proposition 3.1, we can detect both of the catastrophe sets (cf. the appendix) of H^{T} and H^{S} as follows:

$$C(H^{\mathrm{T}}) = \{(u, \boldsymbol{v}) \in U \times (H^n_+(-1) \setminus M) \mid \boldsymbol{v} = \lambda\boldsymbol{x}(u) + \mu\boldsymbol{e}(u)\},$$
$$C(H^{\mathrm{S}}) = \{(u, \boldsymbol{v}) \in U \times S^n_1 \mid \boldsymbol{v} = \lambda\boldsymbol{x}(u) + \mu\boldsymbol{e}(u)\}.$$

We also calculate that

$$\frac{\partial^2 H^{\mathrm{T}}}{\partial u_i \partial u_j}(u, \boldsymbol{v}) = \langle \boldsymbol{x}_{u_i u_j}(u), \boldsymbol{v}\rangle = -\lambda g_{ij} + \mu h_{ij}$$

on $C(H^{\mathrm{T}})$ and

$$\frac{\partial^2 H^{\mathrm{S}}}{\partial u_i \partial u_j}(u, \boldsymbol{v}) = \langle \boldsymbol{x}_{u_i u_j}(u), \boldsymbol{v}\rangle = -\lambda g_{ij} + \mu h_{ij}$$

on $C(H^{\mathrm{S}})$.

Therefore,

$$\det(\mathcal{H}(h^{\mathrm{T}}_v)(u)) = \det(\partial^2 H^{\mathrm{T}}/\partial u_i \partial u_j)(u, \boldsymbol{v}) = 0$$

(respectively, $\det(\mathcal{H}(h^{\mathrm{S}}_v)(u)) = 0$) if and only if $\kappa(u) = \lambda/\mu$ is a de Sitter principal curvature. Since $\boldsymbol{v} \in H^n_+(-1)$ (respectively, $\boldsymbol{v} \in S^n_1$) and $\kappa(u) = \lambda/\mu$ is a de Sitter principal curvature with $\kappa^2(u) > 1$ (respectively, $\kappa^2(u) < 1$), we have

$$\mathcal{B}_{H^{\mathrm{T}}} = \mathrm{HE}^+_M \cup \mathrm{HE}^-_M \quad (\text{respectively, } \mathcal{B}_{H^{\mathrm{S}}} = \mathrm{SE}^+_M \cup \mathrm{SE}^-_M).$$

Proposition 3.2. *We assume that $p = \boldsymbol{x}(u_0)$ is not a de Sitter flat point of $\boldsymbol{x}(U) = M$, then we have the following assertions.*

(1) *p is an umbilic point with $\kappa^2(p) > 1$ if and only if there exists $\boldsymbol{v}_0 \in H^n_+(-1) \setminus M$ such that u_0 is a singular point of $h^{\mathrm{T}}_{v_0}$ and $\operatorname{rank}\mathcal{H}(h^{\mathrm{T}}_{v_0})(u_0) = 0$.*

(2) *p is an umbilic point with $0 < \kappa^2(p) < 1$ if and only if there exists $\boldsymbol{v}_0 \in S^n_1$ such that u_0 is a singular point of $h^{\mathrm{S}}_{v_0}$ and $\operatorname{rank}\mathcal{H}(h^{\mathrm{S}}_{v_0})(u_0) = 0$.*

Proof. (1) Since p is an umbilic point, $A_p = \kappa_p \operatorname{id}_{T_pM}$. There exists an orthogonal matrix Q such that ${}^{\mathrm{t}}Q((h)^j_i)Q = \kappa_p I$. Hence, we may consider the case $(h)^j_i = \kappa_p I$, so that $(h_{ij}) = \kappa_p(g_{ij})$. Then we put $\boldsymbol{v}_0 = \lambda \boldsymbol{x}(u_0) + \mu \boldsymbol{e}(u_0) \in H^n_+(-1) \setminus M$, where

$$\lambda = \pm\frac{\kappa_p(u_0)}{\sqrt{\kappa_p^2(u_0) - 1}}, \qquad \mu = \pm\frac{1}{\sqrt{\kappa_p^2(u_0) - 1}}.$$

In this case the Hessian matrix

$$\mathcal{H}(h^{\mathrm{T}}_{v_0})(u_0) = (-\lambda g_{ij} + \mu h_{ij}) = (-\lambda + \mu\kappa_p(u_0))(g_{ij}) = 0.$$

On the other hand, if $-\lambda g_{ij} + \mu h_{ij} = 0$ for all i, j, then $(h_{ij}) = \kappa_p(g_{ij})$ $(\kappa_p = \lambda/\mu)$. This is equivalent to the condition $((h)^j_i) = \kappa_p I$.

The proof of (2) is also given by direct calculation similar to (1). □

We say that u_0 is a *timelike ridge point* (respectively, *spacelike ridge point*) if h^{T}_v (respectively, h^{S}_v) has the $A_{k\geqslant 3}$-type singular point at u_0, where $\boldsymbol{v} \in \mathcal{B}_{H^{\mathrm{T}}}$ (respectively, $\boldsymbol{v} \in \mathcal{B}_{H^{\mathrm{S}}}$).

For a function germ $f : (\mathbb{R}^{n-1}, \tilde{u}_0) \to \mathbb{R}$, f has A_k-type singular point at $\tilde{u}_0$ if f is $\mathcal{R}^+$-equivalent to the germ $u_1^{k+1} \pm u_2^2 \pm \cdots \pm u_{n-1}^2$. We say that two function germs $f_i : (\mathbb{R}^{n-1}, \tilde{u}_i) \to \mathbb{R}$ $(i = 1, 2)$ are *$\mathcal{R}^+$-equivalent* if there exists a diffeomorphism germ $\Phi : (\mathbb{R}^{n-1}, \tilde{u}_1) \to (\mathbb{R}^{n-1}, \tilde{u}_2)$ and a real number c such that $f_2 \circ \Phi(u) = f_2(u) + c$.

We now consider the geometric meaning of timelike ridge points. Let $F : H^n_+(-1) \to \mathbb{R}$ be a submersion and let $\boldsymbol{x} : U \to H^n_+(-1)$ be a hypersurface. We say that $\boldsymbol{x}$ and $F^{-1}(0)$ have a *corank-r contact* at $p_0 = \boldsymbol{x}(u_0)$ if the Hessian of the function $g(u) = F \circ \boldsymbol{x}(u)$ has corank r at u_0. We also say that $\boldsymbol{x}$ and $F^{-1}(0)$ have an *A_k-type contact* at $p_0 = \boldsymbol{x}(u_0)$ if the function $g(u) = F \circ \boldsymbol{x}(u)$ has the A_k-type singularity at u_0. By definition, if $\boldsymbol{x}$ and $F^{-1}(0)$ have an *A_k-type contact* at $p_0 = \boldsymbol{x}(u_0)$, then these have a corank-1 contact. For any $r \in \mathbb{R}$ and $\boldsymbol{a}_0 \in H^n_+(-1)$ (respectively, $\boldsymbol{a}_0 \in S^n_1$), we consider a function $F : H^n_+(-1) \to \mathbb{R}$ defined by $F(\boldsymbol{u}) = \langle \boldsymbol{u}, \boldsymbol{a}_0 \rangle - r$. We define

$$\mathrm{PS}^{n-1}(\boldsymbol{a}_0, r) = F^{-1}(0) = \{\boldsymbol{u} \in H^n_+(-1) \mid \langle \boldsymbol{u}, \boldsymbol{a}_0 \rangle = r\}.$$

It follows that $\mathrm{PS}^{n-1}(\boldsymbol{a}_0, r)$ is a hypersphere (respectively, equidistant hyperplane) with centre $\boldsymbol{a}_0$ if $\boldsymbol{a}_0$ is in $H^n_+(-1)$ (respectively, S^n_1).

We put $\boldsymbol{a}_0 = \mathrm{HE}^{\pm}_{\kappa}(u_0)$ (respectively, $\boldsymbol{a}_0 = \mathrm{SE}^{\pm}_{\kappa}(u_0)$) and

$$r_0 = \mp\frac{\kappa(u_0)}{\sqrt{|\kappa^2(u_0) - 1|}},$$

where we fix a de Sitter principal curvature $\kappa(u)$ on U at u_0. We then have the following simple proposition.

Proposition 3.3. *With the above notation, there exists an integer ℓ with $1 \leqslant \ell \leqslant n-1$ such that $\boldsymbol{x}(U) = M$ and $\mathrm{PS}^{n-1}(\boldsymbol{a}_0, r_0)$ have corank-ℓ contact at u_0.*

In the above proposition, $\mathrm{PS}^{n-1}(\boldsymbol{a}_0, r_0)$ is called an *osculating hypersphere* (respectively, *osculating equidistant hyperplane*) of $M = \boldsymbol{x}(U)$ if $\boldsymbol{a}_0 \in H^n_+(-1)$ (respectively, $\boldsymbol{a}_0 \in S^n_1$). We also call $\boldsymbol{a}_0$ the *centre of de Sitter principal curvature* $\kappa(u_0)$. By Proposition 3.2, $\boldsymbol{x}(U) = M$ and the osculating hypersphere (respectively, equidistant hyperplane) has corank-$(n-1)$ contact at an umbilic point. Therefore, the hyperbolic (respectively, de Sitter) ridge point is not an umbilic point.

By the general theory of unfoldings of function germs, the bifurcation set $\mathcal{B}_F$ is non-singular at the origin if and only if the function $f = F \mid \mathbb{R}^n \times \{0\}$ has the A_2-type singularity (i.e. the fold-type singularity). Therefore, we have the following proposition.

Proposition 3.4. *With the same notation as in the previous proposition, the total evolute $\mathrm{TE}^{\pm}_M$ is non-singular at $\boldsymbol{a}_0 = \mathrm{TE}^{\pm}_\kappa(u_0)$ if and only if $\boldsymbol{x}(U) = M$ and $\mathrm{PS}^{n-1}(\boldsymbol{a}_0, r_0)$ have A_2-type contact at u_0.*

Here, $\mathrm{TE}^{\pm}_\kappa(u_0) = \mathrm{HE}^{\pm}_\kappa(u_0)$ if $\boldsymbol{a}_0 \in H^n_+(-1)$ and $\mathrm{TE}^{\pm}_\kappa(u_0) = \mathrm{SE}^{\pm}_\kappa(u_0)$ if $\boldsymbol{a}_0 \in S^n_1$.

4. Evolutes as caustics

In this section we naturally interpret the hyperbolic (de Sitter) evolute of hypersurface in hyperbolic space as a caustic in the framework of symplectic geometry and consider the geometric meaning of singularities. In the appendix we give a brief survey of the theory of Lagrangian singularities. For notions and basic results on the theory of Lagrangian singularities, please refer to the appendix.

For a hypersurface $\boldsymbol{x} : U \to H^n_+(-1)$, we consider the hyperbolic timelike height function H^{T} and the hyperbolic spacelike height function H^{S} (cf. § 3). We have the following propositions.

Proposition 4.1. *Both the hyperbolic timelike height function $H^{\mathrm{T}} : U \times H^n_+(-1) \to \mathbb{R}$ and the hyperbolic spacelike height function $H^{\mathrm{S}} : U \times S^n_1 \to \mathbb{R}$ on $\boldsymbol{x}$ are Morse families.*

Proof. First we consider the hyperbolic timelike height function.

For any $\boldsymbol{v} = (v_0, v_1 \dots, v_n) \in H^n_+(-1)$, we have $v_0 = \sqrt{v_1^2 + \dots + v_n^2 + 1}$, so that

$$H^{\mathrm{T}}(u, \boldsymbol{v}) = -x_0(u)\sqrt{v_1^2 + \dots + v_n^2 + 1} + x_1(u)v_1 + \dots + x_n(u)v_n,$$

where $\boldsymbol{x}(u) = (x_0(u), \dots, x_n(u))$. We will prove that the mapping

$$\Delta H^{\mathrm{T}} = \left(\frac{\partial H^{\mathrm{T}}}{\partial u_1}, \dots, \frac{\partial H^{\mathrm{T}}}{\partial u_{n-1}}\right)$$

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is non-singular at any point. The Jacobian matrix of ΔH^{T} is given as follows:

$$\begin{pmatrix} \langle \boldsymbol{x}_{u_1u_1}, \boldsymbol{v}\rangle & \cdots & \langle \boldsymbol{x}_{u_1u_{n-1}}, \boldsymbol{v}\rangle & -x_{0u_1}\dfrac{v_1}{v_0}+x_{1u_1} & \cdots & -x_{0u_1}\dfrac{v_n}{v_0}+x_{nu_1} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \langle \boldsymbol{x}_{u_{n-1}u_1}, \boldsymbol{v}\rangle & \cdots & \langle \boldsymbol{x}_{u_{n-1}u_{n-1}}, \boldsymbol{v}\rangle & -x_{0u_{n-1}}\dfrac{v_1}{v_0}+x_{1u_{n-1}} & \cdots & -x_{0u_{n-1}}\dfrac{v_n}{v_0}+x_{nu_{n-1}} \end{pmatrix},$$

where $\boldsymbol{x}_{u_iu_j} = \partial^2\boldsymbol{x}/\partial u_i \partial u_j(u)$. We will show that the rank of the matrix

$$X = \begin{pmatrix} -x_{0u_1}\dfrac{v_1}{v_0}+x_{1u_1} & \cdots & -x_{0u_1}\dfrac{v_n}{v_0}+x_{nu_1} \\ \vdots & \vdots & \vdots \\ -x_{0u_{n-1}}\dfrac{v_1}{v_0}+x_{1u_{n-1}} & \cdots & -x_{0u_{n-1}}\dfrac{v_n}{v_0}+x_{nu_{n-1}} \end{pmatrix}$$

is $n-1$ at $(u, \boldsymbol{v}) \in C(H^{\mathrm{T}})$. It is enough to show that the rank of the matrix

$$A = \begin{pmatrix} -x_0\dfrac{v_1}{v_0}+x_1 & \cdots & -x_0\dfrac{v_n}{v_0}+x_n \\ -x_{0u_1}\dfrac{v_1}{v_0}+x_{1u_1} & \cdots & -x_{0u_1}\dfrac{v_n}{v_0}+x_{nu_1} \\ \vdots & \vdots & \vdots \\ -x_{0u_{n-1}}\dfrac{v_1}{v_0}+x_{1u_{n-1}} & \cdots & -x_{0u_{n-1}}\dfrac{v_n}{v_0}+x_{nu_{n-1}} \end{pmatrix}$$

is n at $(u, \boldsymbol{v}) \in C(H^{\mathrm{T}})$. We define

$$\boldsymbol{a}_i = \begin{pmatrix} x_i \\ x_{iu_1} \\ \vdots \\ x_{iu_{n-1}} \end{pmatrix}$$

for $i = 0, \dots, n$.

Then we have

$$A = \left(-\boldsymbol{a}_0\frac{v_1}{v_0} + \boldsymbol{a}_1, \dots, -\boldsymbol{a}_0\frac{v_n}{v_0} + \boldsymbol{a}_n\right)$$

and

$$\det A = \frac{v_0}{v_0}\det(\boldsymbol{a}_1, \dots, \boldsymbol{a}_n) - \frac{v_1}{v_0}\det(\boldsymbol{a}_0, \boldsymbol{a}_2, \dots, \boldsymbol{a}_n) - \cdots - \frac{v_n}{v_0}\det(\boldsymbol{a}_1, \dots, \boldsymbol{a}_{n-1}, \boldsymbol{a}_0).$$

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On the other hand, we have

$$\boldsymbol{x} \wedge \boldsymbol{x}_{u_1} \wedge \cdots \wedge \boldsymbol{x}_{u_{n-1}} = (-\det(\boldsymbol{a}_1, \dots, \boldsymbol{a}_n), -\det(\boldsymbol{a}_0, \boldsymbol{a}_2, \dots, \boldsymbol{a}_n), \dots, (-1)^n \det(\boldsymbol{a}_0, \dots, \boldsymbol{a}_{n-1})).$$

Therefore, we have

$$\begin{aligned}\det A &= \left\langle \left(\frac{v_0}{v_0}, \dots, \frac{v_n}{v_0} \right), \boldsymbol{x} \wedge \boldsymbol{x}_{u_1} \wedge \cdots \wedge \boldsymbol{x}_{u_{n-1}} \right\rangle \\ &= \frac{1}{v_0} \langle \lambda \boldsymbol{x} + \mu \boldsymbol{e}, \|\boldsymbol{x} \wedge \boldsymbol{x}_{u_1} \wedge \cdots \wedge \boldsymbol{x}_{u_{n-1}}\| \boldsymbol{e} \rangle \\ &= \frac{1}{v_0} \|\boldsymbol{x} \wedge \boldsymbol{x}_{u_1} \wedge \cdots \wedge \boldsymbol{x}_{u_{n-1}}\| \mu \neq 0\end{aligned}$$

for $(u, \boldsymbol{v}) \in C(H^{\mathrm{T}})$.

Next we consider the hyperbolic spacelike height function. The proof is also given by direct calculation but a bit more carefully than in the previous case. We use the same notation as in the previous case (e.g. $\boldsymbol{x}$ and $\boldsymbol{a}_i$, etc.). For any $\boldsymbol{v} \in S^n_1$, we have $-v_0^2 + v_1^2 + \cdots + v_n^2 = 1$. Without loss of the generality we might assume that $v_n \neq 0$. We have

$$v_n = \pm\sqrt{1 + v_0^2 - v_1^2 - \cdots - v_{n-1}^2},$$

so that

$$H^{\mathrm{S}}(u, \boldsymbol{v}) = -x_0(u)v_0 + x_1(u)v_1 + \cdots + x_{n-1}(u)v_{n-1} \pm x_n(u)\sqrt{1 + v_0^2 - v_1^2 - \cdots - v_{n-1}^2}.$$

We also prove that the mapping

$$\Delta H^{\mathrm{S}} = \left(\frac{\partial H^{\mathrm{S}}}{\partial u_1}, \dots, \frac{\partial H^{\mathrm{S}}}{\partial u_{n-1}} \right)$$

is non-singular at any point. The Jacobian matrix of ΔH^{S} is given as follows:

$$\begin{pmatrix} \langle \boldsymbol{x}_{u_1 u_1}, \boldsymbol{v} \rangle & \cdots & \langle \boldsymbol{x}_{u_1 u_{n-1}}, \boldsymbol{v} \rangle & -x_{0u_1} + x_{nu_1}\dfrac{v_0}{v_n} & \cdots & x_{n-1 u_1} - x_{nu_1}\dfrac{v_{n-1}}{v_n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \langle \boldsymbol{x}_{u_{n-1} u_1}, \boldsymbol{v} \rangle & \cdots & \langle \boldsymbol{x}_{u_{n-1} u_{n-1}}, \boldsymbol{v} \rangle & -x_{0u_{n-1}} + x_{nu_{n-1}}\dfrac{v_0}{v_n} & \cdots & x_{n-1 u_{n-1}} - x_{nu_{n-1}}\dfrac{v_{n-1}}{v_n} \end{pmatrix}.$$

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We will also show that the rank of the matrix

$$\tilde{X} = \begin{pmatrix} -x_{0u_1} + x_{nu_1}\dfrac{v_0}{v_n} & x_{1u_1} - x_{nu_1}\dfrac{v_1}{v_n} & \cdots & x_{n-1u_1} - x_{nu_1}\dfrac{v_{n-1}}{v_n} \\ \vdots & \vdots & \vdots & \vdots \\ -x_{0u_{n-1}} + x_{nu_{n-1}}\dfrac{v_0}{v_n} & x_{1u_{n-1}} - x_{nu_{n-1}}\dfrac{v_1}{v_n} & \cdots & x_{n-1u_{n-1}} - x_{nu_{n-1}}\dfrac{v_{n-1}}{v_n} \end{pmatrix}$$

is $n-1$ at $(u, \boldsymbol{v}) \in C(H^{\mathrm{S}})$. It can be proved that the rank of the matrix

$$\tilde{A} = \left(-\boldsymbol{a}_0 + \boldsymbol{a}_n\frac{v_0}{v_n}, \boldsymbol{a}_1 - \boldsymbol{a}_n\frac{v_1}{v_n}, \dots, \boldsymbol{a}_{n-1} - \boldsymbol{a}_n\frac{v_{n-1}}{v_n}\right)$$

is n at $(u, \boldsymbol{v}) \in C(H^{\mathrm{S}})$.

Therefore, we have

$$\begin{aligned} \det\tilde{A} &= (-1)^{n-1}\left\{\frac{v_0}{v_n}\det(\boldsymbol{a}_1, \dots, \boldsymbol{a}_n) - \frac{v_1}{v_n}\det(\boldsymbol{a}_0, \boldsymbol{a}_2, \dots, \boldsymbol{a}_n)\right. \\ &\qquad\qquad\left. + \cdots + (-1)^n\frac{v_n}{v_n}\det(\boldsymbol{a}_0, \dots, \boldsymbol{a}_{n-1})\right\} \\ &= (-1)^{n-1}\left\langle\left(\frac{v_0}{v_n}, \dots, \frac{v_n}{v_n}\right), \boldsymbol{x}\wedge\boldsymbol{x}_{u_1}\cdots\wedge\boldsymbol{x}_{u_{n-1}}\right\rangle \\ &= \frac{(-1)^{n-1}}{v_n}\langle\lambda\boldsymbol{x} + \mu\boldsymbol{e}, \|\boldsymbol{x}\wedge\boldsymbol{x}_{u_1}\wedge\cdots\wedge\boldsymbol{x}_{u_{n-1}}\|\boldsymbol{e}\rangle \\ &= \frac{(-1)^{n-1}}{v_n}\|\boldsymbol{x}\wedge\boldsymbol{x}_{u_1}\wedge\cdots\wedge\boldsymbol{x}_{u_{n-1}}\|\mu \neq 0 \end{aligned}$$

for $(u, \boldsymbol{v}) \in C(H^{\mathrm{S}})$. This completes the proof of the proposition. □

By the method for constructing the Lagrangian immersion germ from the Morse family (see the appendix), we can define a Lagrangian immersion germ whose generating family is the hyperbolic timelike height function or the hyperbolic spacelike height function of $M = \boldsymbol{x}(U)$ as follows.

For a hypersurface $\boldsymbol{x} : U \to H^n_+(-1)$, $\boldsymbol{x}(u) = (x_0(u), \dots, x_n(u))$, we define a smooth mapping

$$L(H^{\mathrm{T}}) : C(H^{\mathrm{T}}) \to T^*H^n_+(-1)$$

by

$$L(H^{\mathrm{T}})(u, \boldsymbol{v}) = \left(\boldsymbol{v}, -x_0(u)\frac{v_1}{v_0} + x_1(u), \dots, -x_0(u)\frac{v_n}{v_0} + x_n(u)\right),$$

where $\boldsymbol{v} = (v_0, \dots, v_n) \in H^n_+(-1)$ and $v_0 = \sqrt{v_1^2 + \cdots + v_n^2 + 1}$. Here we have used the triviality of the cotangent bundle $T^*H^n_+(-1)$. For the de Sitter space S^n_1, we consider the local coordinate $U_i = \{\boldsymbol{v} = (v_0, \dots, v_n) \in S^n_1 \mid v_i \neq 0\}$. Since $T^*S^n_1 \mid U_i$ is a trivial bundle, we define a map

$$L_i(H^{\mathrm{S}}) : C(H^{\mathrm{S}}) \to T^*S^n_1 \mid U_i \quad (i = 0, 1, \dots, n)$$

by

$$L_i(H^{\mathrm{S}})(u, \boldsymbol{v}) = \left(\boldsymbol{v}, -x_0(u) + x_i(u)\frac{v_0}{v_i}, x_1(u) - x_i(u)\frac{v_1}{v_i}, \dots, \widehat{x_i(u) - x_i(u)}\frac{v_i}{v_i}, \dots, x_n(u) - x_i(u)\frac{v_n}{v_i}\right),$$

where $v = (v_0, \dots, v_n) \in S_1^n$ and we define $(x_0, \dots, \hat{x}_i, \dots, x_n)$ as a point in n-dimensional space such that the ith component x_i is removed. We can show that if $U_i \cap U_j \neq \emptyset$ for $i \neq j$, then $L_i(H^{\mathrm{S}})$ and $L_j(H^{\mathrm{S}})$ are Lagrangian equivalent such that the corresponding Lagrangian equivalence is given by the local coordinate change of S_1^n and the Lagrangian lift of it. Indeed, we define the local coordinate change of S_1^n for $i < j$; $\varphi_{ij} : U_i \to U_j$, by

$$\varphi_{ij}(v_0, \dots, \hat{v}_i, \dots, v_n) = \left(v_0, \dots, v_i = \sqrt{1 + v_0^2 - v_1^2 - \dots - \hat{v}_i^2 - \dots - v_n^2}, \dots, \hat{v}_j, \dots, v_n\right),$$

and $\tilde{\varphi}_{ij} : T^*S_1^n \to T^*S_1^n$ are Lagrangian lifts of φ_{ij} that are defined by $\tilde{\varphi_{ij}}(\xi) = (\varphi_{ij*}^{-1})^*\xi$. Then $\tilde{\varphi}_{ij}$ are symplectic diffeomorphism germs (cf. [**1**]). Also we define diffeomorphism germs $\hat{\sigma}_{ij} : U \times U_i \to U \times U_j$ by $\hat{\sigma}_{ij}(u, \boldsymbol{v}) = (u, \varphi_{ij}(\boldsymbol{v}))$ and $\sigma_{ij} = \hat{\sigma}_{ij}|_{C(H^{\mathrm{S}})}$, then $\tilde{\varphi}_{ij} \circ L_i(H^{\mathrm{S}}) = L_j(H^{\mathrm{S}}) \circ \sigma_{ij}$ and $\varphi_{ij} \circ \pi = \pi \circ \tilde{\varphi}_{ij}$. Therefore, we can define a global Lagrangian immersion: $L(H^{\mathrm{S}}) : C(H^{\mathrm{S}}) \to T^*S_1^n$.

By definition, we have the following corollary of the above proposition.

Corollary 4.2. *With the above notation, $L(H^{\mathrm{T}})$ (respectively, $L(H^{\mathrm{S}})$) is a Lagrangian immersion such that the hyperbolic timelike height function $H^{\mathrm{T}} : U \times H_+^n(-1) \to \mathbb{R}$ (respectively, hyperbolic spacelike height function $H^{\mathrm{S}} : U \times S_1^n \to \mathbb{R}$) of $\boldsymbol{x}$ is a generating family of $L(H^{\mathrm{T}})$ (respectively, $L(H^{\mathrm{S}})$).*

Therefore, we have the Lagrangian immersion $L(H^{\mathrm{T}})$ (respectively, $L(H^{\mathrm{S}})$) whose caustic is the hyperbolic evolute (respectively, de Sitter evolute) of $\boldsymbol{x}$. We call $L(H^{\mathrm{T}})$ (respectively, $L(H^{\mathrm{S}})$) the*Lagrangian lift* of the hyperbolic evolute (respectively, de Sitter evolute) of $\boldsymbol{x}$.

5. Contact with families of hyperspheres and equidistant hyperplanes

Before we start to consider the contact between a hypersurface and a family of hyperspheres or equidistant hyperplanes, we briefly describe the theory of contact with foliations. Montaldi [**13**] considered that the relationship between the contact of submanifolds and singularity type (more precisely, the $\mathcal{K}$-class; cf. [**11**]) of maps. Here we consider the relationship between the contact of submanifolds with foliations and the $\mathcal{R}^+$-class of functions. Let X_i $(i = 1, 2)$ be submanifolds of $\mathbb{R}^n$ with $\dim X_1 = \dim X_2$, let $g_i : (X_i, \bar{x}_i) \to (\mathbb{R}^n, \bar{y}_i)$ be immersion germs, and let $f_i : (\mathbb{R}^n, \bar{y}_i) \to (\mathbb{R}, 0)$ be submersion germs. For a submersion germ $f : (\mathbb{R}^n, 0) \to (\mathbb{R}, 0)$, we let $\mathcal{F}_f$ be the regular foliation defined by f, i.e. $\mathcal{F}_f = \{f^{-1}(c) \mid c \in (\mathbb{R}, 0)\}$. We say that *the contact of X_1 with the regular foliation $\mathcal{F}_{f_1}$* at $\bar{y}_1$ is the *same type* as *the contact of X_2 with the regular foliation $\mathcal{F}_{f_2}$*

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at $\bar{y}_2$ if there is a diffeomorphism germ $\Phi : (\mathbb{R}^n, \bar{y}_1) \to (\mathbb{R}^n, \bar{y}_2)$ such that $\Phi(X_1) = X_2$ and $\Phi(Y_1(c)) = Y_2(c)$, where $Y_i(c) = f_i^{-1}(c)$ for each $c \in (\mathbb{R}, 0)$. In this case we write $K(X_1, \mathcal{F}_{f_1}; \bar{y}_1) = K(X_2, \mathcal{F}_{f_2}; \bar{y}_2)$. It is clear that in the definition $\mathbb{R}^n$ could be replaced by any manifold. We apply the method of Goryunov [**5**] to the case for $\mathcal{R}^+$-equivalences among function germs, so that we have the following.

Proposition 5.1 (see the appendix in [5]). *Let $X_i (i = 1, 2)$ be submanifolds of $\mathbb{R}^n$ with $\dim X_1 = \dim X_2 = n - 1$ (i.e. hypersurfaces), let $g_i : (X_i, \bar{x}_i) \to (\mathbb{R}^n, \bar{y}_i)$ be immersion germs, and let $f_i : (\mathbb{R}^n, \bar{y}_i) \to (\mathbb{R}, 0)$ be submersion germs. We assume that $\bar{x}_i$ are singularities of function germs $f_i \circ g_i : (X_i, \bar{x}_i) \to (\mathbb{R}, 0)$. Then $K(X_1, \mathcal{F}_{f_1}; \bar{y}_1) = K(X_2, \mathcal{F}_{f_2}; \bar{y}_2)$ if and only if $f_1 \circ g_1$ and $f_2 \circ g_2$ are $\mathcal{R}^+$-equivalent.*

On the other hand, Golubitsky and Guillemin [**4**] have given an algebraic characterization for the $\mathcal{R}^+$-equivalence among function germs. We denote by $C_0^\infty(X)$ the set of function germs $(X, 0) \to \mathbb{R}$. Let J_f be the Jacobian ideal in $C_0^\infty(X)$ (i.e. $J_f = \langle \partial f/\partial x_1, \dots, \partial f/\partial x_n \rangle_{C_0^\infty(X)}$). Let $\mathcal{R}_k(f) = C_0^\infty(X)/J_f^k$ and let $\bar{f}$ be the image of f in this local ring. We say that f satisfies the *Milnor Condition* if $\dim_{\mathbb{R}} \mathcal{R}_1(f) < \infty$.

Proposition 5.2 (see Proposition 4.1 in [4]). *Let f and g be germs of functions at 0 in X satisfying the Milnor Condition with $\partial f/\partial x_i(0) = \partial g/\partial x_i(0) = 0$ $(i = 1, \dots, n)$. Then f and g are $\mathcal{R}^+$-equivalent if*

(1) *the rank and signature of the Hessians $\mathcal{H}(f)(0)$ and $\mathcal{H}(g)(0)$ are equal; and*

(2) *there is an isomorphism $\gamma : \mathcal{R}_2(f) \to \mathcal{R}_2(g)$ such that $\gamma(\bar{f}) = \bar{g}$.*

We consider a function $\mathfrak{H}^{\mathrm{T}} : H_+^n(-1) \times (H_+^n(-1) \backslash M) \to \mathbb{R}$ defined by $\mathfrak{H}^{\mathrm{T}}(\boldsymbol{x}, \boldsymbol{v}) = \langle \boldsymbol{x}, \boldsymbol{v} \rangle$. For any $\boldsymbol{v}_0 \in H_+^n(-1) \setminus M$, we define $\mathfrak{h}_{v_0}^{\mathrm{T}}(\boldsymbol{x}) = \mathfrak{H}^{\mathrm{T}}(\boldsymbol{x}, \boldsymbol{v}_0)$ and we have a hypersphere

$$(\mathfrak{h}_{v_0}^{\mathrm{T}})^{-1}(\lambda) = \mathrm{HP}(\boldsymbol{v}_0, \lambda) \cap H_+^n(-1) = \mathrm{PS}^{n-1}(\boldsymbol{v}_0, \lambda).$$

It is easy to show that $\mathfrak{h}_{v_0}^{\mathrm{T}}$ is a submersion. For any $\bar{u}_0 \in U$, we consider a timelike vector (i.e. in hyperbolic n-space) $\boldsymbol{v}_0 = \lambda \boldsymbol{x}(\bar{u}_0) + \mu \boldsymbol{e}(\bar{u}_0) \in H_+^n(-1)$, then we have

$$\mathfrak{h}_{v_0}^{\mathrm{T}} \circ \boldsymbol{x}(\bar{u}_0) = \mathcal{H} \circ (\boldsymbol{x} \times \mathrm{id}_{H_+^n(-1)})(\bar{u}_0, \boldsymbol{v}_0) = \lambda,$$

and

$$\frac{\partial \mathfrak{h}_{v_0}^{\mathrm{T}} \circ \boldsymbol{x}}{\partial u_i}(\bar{u}_0) = \frac{\partial H^{\mathrm{T}}}{\partial u_i}(\bar{u}_0, \boldsymbol{v}_0) = 0,$$

for $i = 1, \dots, n-1$. This means that the hypersphere $(\mathfrak{h}_{v_0}^{\mathrm{T}})^{-1}(\lambda) = \mathrm{PS}^{n-1}(\boldsymbol{v}_0, \lambda)$ is tangent to $M = \boldsymbol{x}(U)$ at $p = \boldsymbol{x}(\bar{u}_0)$. In this case we call $p = \boldsymbol{x}(\bar{u}_0)$ and $\mathrm{PS}^{n-1}(\boldsymbol{v}_0, \lambda)$ a *tangent hypersphere* with the centre $\boldsymbol{v}_0$. However, there are infinitely many tangent hyperspheres at a general point $p = \boldsymbol{x}(\bar{u}_0)$ depending on the real number λ. If $\boldsymbol{v}_0$ is a point of the hyperbolic evolute, the tangent hypersphere with the centre $\boldsymbol{v}_0$ is called the *osculating hypersphere* at $p = \boldsymbol{x}(\bar{u}_0)$, which is uniquely determined. Let $\boldsymbol{x}_i : (U, \bar{u}_i) \to (H_+^n(-1), \boldsymbol{x}_i(\bar{u}_i))$ $(i = 1, 2)$ be hypersurface germs. We consider hyperbolic timelike height functions $H_i^{\mathrm{T}} : (U \times H_+^n(-1), (\bar{u}_i, \boldsymbol{v}_i)) \to \mathbb{R}$ of $\boldsymbol{x}_i$, where $\boldsymbol{v}_i$ are points of hyperbolic evolutes of $\boldsymbol{x}_i$, respectively. We define $h_{i,v_i}^{\mathrm{T}}(u) = H_i^{\mathrm{T}}(u, \boldsymbol{v}_i)$, then we have $h_{i,v_i}^{\mathrm{T}}(u) = \mathfrak{h}_{v_i}^{\mathrm{T}} \circ \boldsymbol{x}_i(u)$. Then we have the following theorem.

Theorem 5.3. *Let*

$$\boldsymbol{x}_i : (U, \bar{u}_i) \to (H^n_+(-1), \boldsymbol{x}_i(\bar{u}_i))$$

be hypersurface germs such that the corresponding Lagrangian immersion germs

$$L(H_i^{\mathrm{T}}) : (C(H_i^{\mathrm{T}}), (\bar{u}_i, \boldsymbol{v}_i)) \to (T^* H^n_+(-1), \bar{z}_i)$$

are Lagrangian stable, where $\boldsymbol{v}_i$ are centres of the osculating hyperspheres of $\boldsymbol{x}_i$, respectively. Then the following conditions are equivalent.

(1) $K(\boldsymbol{x}_1(U), \mathcal{F}_{\mathfrak{h}^{\mathrm{T}}_{v_1}}; \boldsymbol{x}(\bar{u}_1)) = K(\boldsymbol{x}_2(U), \mathcal{F}_{\mathfrak{h}^{\mathrm{T}}_{v_2}}; \boldsymbol{x}(\bar{u}_2))$.

(2) h^{T}_{1,v_1} *and* h^{T}_{2,v_2} *are* $\mathcal{R}^+$*-equivalent.*

(3) H_1^{T} *and* H_2^{T} *are* P–$\mathcal{R}^+$*-equivalent.*

(4) $L(H_1^{\mathrm{T}})$ *and* $L(H_2^{\mathrm{T}})$ *are Lagrangian equivalent.*

(5) (a) *The rank and signature of the* $\mathcal{H}(h^{\mathrm{T}}_{1,v_1})(\bar{u}_1)$ *and* $\mathcal{H}(h^{\mathrm{T}}_{2,v_2})(\bar{u}_2)$ *are equal.*
(b) *There is an isomorphism* $\gamma : \mathcal{R}_2(h^{\mathrm{T}}_{1,v_1}) \to \mathcal{R}_2(h^{\mathrm{T}}_{2,v_2})$ *such that* $\gamma(\overline{h^{\mathrm{T}}_{1,v_1}}) = \overline{h^{\mathrm{T}}_{2,v_2}}$.

Proof. By Proposition 5.1, condition (1) is equivalent to condition (2). Since both of $L(H_i^{\mathrm{T}})$ are Lagrangian stable, both of H_i^{T} are $\mathcal{R}^+$-versal unfoldings of h^{T}_{i,v_i}, respectively. By the uniqueness theorem on the $\mathcal{R}^+$-versal unfolding of a function germ, condition (2) is equivalent to condition (3). By Proposition A 3, condition (3) is equivalent to condition (4). It also follows from Proposition A 3 that both of h_i^{T} satisfy the Milnor Condition. Therefore, we can apply Proposition 5.2 to our situation, so that condition (2) is equivalent to condition (5). This completes the proof. □

We remark that if $L(H_1^{\mathrm{T}})$ and $L(H_2^{\mathrm{T}})$ are Lagrangian equivalent, then the corresponding hyperbolic evolutes are diffeomorphic. Since the hyperbolic evolute of a hypersurface $\boldsymbol{x}(U) = M$ is considered to be the caustic of $L(H^{\mathrm{T}})$, the above theorem gives a symplectic interpretation for the contact of hypersurfaces with a family of hyperspheres (cf. the appendix).

Similarly, we can construct the *osculating equidistant hyperplane* of a hypersurface $\boldsymbol{x} : U \to H^n_+(-1)$ by using a function $\mathfrak{H}^{\mathrm{S}} : H^n_+(-1) \times S^n_1 \to \mathbb{R}$ defined by $\mathfrak{H}^{\mathrm{S}}(\boldsymbol{x}, \boldsymbol{v}) = \langle \boldsymbol{x}, \boldsymbol{v} \rangle$. For any $\boldsymbol{v}_0 \in S^n_1$, we also define $\mathfrak{h}^{\mathrm{S}}_{v_0}(\boldsymbol{x}) = \mathfrak{H}^{\mathrm{S}}(\boldsymbol{x}, \boldsymbol{v}_0)$ and we have $h^{\mathrm{S}}_{0,v_0}(u) = \mathfrak{h}^{\mathrm{S}}_{v_0} \circ \boldsymbol{x}(u)$. Then we have the following theorem.

Theorem 5.4. *Let*

$$\boldsymbol{x}_i : (U, \bar{u}_i) \to (H^n_+(-1), \boldsymbol{x}_i(\bar{u}_i))$$

be hypersurface germs such that the corresponding Lagrangian immersion germs

$$L(H_i^{\mathrm{S}}) : (C(H_i^{\mathrm{T}}), (\bar{u}_i, \boldsymbol{v}_i)) \to (T^* H^n_+(-1), \bar{z}_i)$$

are Lagrangian stable, where $\boldsymbol{v}_i$ are centres of osculating equidistant hyperplanes of $\boldsymbol{x}_i$, respectively. Then the following conditions are equivalent.

(1) $K(\boldsymbol{x}_1(U), \mathcal{F}_{\mathfrak{h}^{\mathrm{S}}_{v_1}}; \boldsymbol{x}(\bar{u}_1)) = K(\boldsymbol{x}_2(U), \mathcal{F}_{\mathfrak{h}^{\mathrm{S}}_{v_2}}; \boldsymbol{x}(\bar{u}_2))$.

(2) h^{S}_{1,v_1} *and* h^{S}_{2,v_2} *are* $\mathcal{R}^+$*-equivalent.*

(3) H^{S}_1 *and* H^{S}_2 *are* P–$\mathcal{R}^+$*-equivalent.*

(4) $L(H^{\mathrm{S}}_1)$ *and* $L(H^{\mathrm{S}}_2)$ *are Lagrangian equivalent.*

(5) (a) *The rank and signature of the* $\mathcal{H}(h^{\mathrm{S}}_{1,v_1})(\bar{u}_1)$ *and* $\mathcal{H}(h^{\mathrm{S}}_{2,v_2})(\bar{u}_2)$ *are equal.*
 (b) *There is an isomorphism* $\gamma : \mathcal{R}_2(h^{\mathrm{S}}_{1,v_1}) \to \mathcal{R}_2(h^{\mathrm{S}}_{2,v_2})$ *such that* $\gamma(\overline{h^{\mathrm{S}}_{1,v_1}}) = \overline{h^{\mathrm{S}}_{2,v_2}}$.

The proof follows by direct analogy with the proof for Theorem 5.3 so we omit it.

6. Generic properties

In this section we consider generic properties of hypersurfaces in $H^n_+(-1)$. The main tool is a kind of transversality theorem. We consider the space of embeddings $\mathrm{Emb}(U, H^n_+(-1))$ with Whitney C^∞-topology. We define two functions:

(a) $\mathfrak{H}^{\mathrm{T}} : H^n_+(-1) \times H^n_+(-1) \to \mathbb{R}$; $\mathfrak{H}^{\mathrm{T}}(\boldsymbol{u}, \boldsymbol{v}) = \langle \boldsymbol{u}, \boldsymbol{v} \rangle$; and

(b) $\mathfrak{H}^{\mathrm{S}} : H^n_+(-1) \times S^n_1 \to \mathbb{R}; \mathfrak{H}^{\mathrm{S}}(\boldsymbol{u}, \boldsymbol{v}) = \langle \boldsymbol{u}, \boldsymbol{v} \rangle$.

We claim that $\mathfrak{h}^{\mathrm{T}}_v$ (respectively, $\mathfrak{H}^{\mathrm{S}}_v$) is a submersion for any $\boldsymbol{v} \in H^n_+(-1)$ (respectively, $\boldsymbol{v} \in S^n_1$), where $\mathfrak{h}^{\mathrm{T}}_v(\boldsymbol{u}) = \mathfrak{H}^{\mathrm{T}}(\boldsymbol{u}, \boldsymbol{v})$ (respectively, $\mathfrak{h}^{\mathrm{S}}_v(\boldsymbol{u}) = \mathfrak{H}^{\mathrm{S}}(\boldsymbol{u}, \boldsymbol{v})$). For any $\boldsymbol{x} \in \mathrm{Emb}(U, H^n_+(-1))$, we have $H^{\mathrm{T}} = \mathfrak{H}^{\mathrm{T}} \circ (\boldsymbol{x} \times \mathrm{id}_{H^n_+(-1)})$ and $H^{\mathrm{S}} = \mathfrak{H}^{\mathrm{S}} \circ (\boldsymbol{x} \times \mathrm{id}_{S^n_1})$. We also have the ℓ-jet extensions

$$j^\ell_1 H^{\mathrm{T}} : U \times H^n_+(-1) \to J^\ell(U, \mathbb{R}) \quad (\text{respectively, } j^\ell_1 H^{\mathrm{S}} : U \times S^n_1 \to J^\ell(U, \mathbb{R}))$$

defined by $j^\ell_1 H^{\mathrm{T}}(u, \boldsymbol{v}) = j^\ell h^{\mathrm{T}}_v(u)$ (respectively, $j^\ell_1 H^{\mathrm{S}}(u, \boldsymbol{v}) = j^\ell h^{\mathrm{S}}_v(u)$). We consider the trivialization $J^\ell(U, \mathbb{R}) \equiv U \times \mathbb{R} \times J^\ell(n-1, 1)$. For any submanifold $Q \subset J^\ell(n-1, 1)$, we define $\tilde{Q} = U \times \mathbb{R} \times Q$. Then we have the following proposition as a corollary of Lemma 6 in Wassermann [**16**] (see also [**14**] and [**10**]).

Proposition 6.1. *Let* Q *be a submanifold of* $J^\ell(n-1, 1)$*. Then the set*

$$T^X_Q = \{\boldsymbol{x} \in \mathrm{Emb}(U, H^n_+(-1)) \mid j^\ell_1 H^X \text{ is transversal to } \tilde{Q}\}$$

is a residual subset of $\mathrm{Emb}(U, H^n_+(-1))$*. If* Q *is a closed subset, then* T_Q *is open.*
Here, X *is* T *or* S.

In the case when $n \leqslant 6$, we have finitely many $\mathcal{R}$-orbits in $J^\ell(n, 1)$ consisting of the jet $z = j^\ell f(0)$ with $\dim \mathcal{R}_1(f) \leqslant n$. Let $\Sigma^\ell_0(n, 1)$ be the union of $\mathcal{R}$-orbits consisting of the jet $z = j^\ell f(0)$ with $\dim \mathcal{R}_1(f) > n$. It is known that $\Sigma^\ell_0(n, 1)$ is a semi-algebraic subset of $J^\ell(n, 1)$ with $\operatorname{codim} \Sigma^\ell_0(n, 1) > 2n + 1$ for sufficiently large ℓ. Therefore, we have a stratification of $(J^\ell(n, 1) \setminus \Sigma^\ell_0(n, 1)) \cup \Sigma^\ell_0(n, 1)$ by finitely many $\mathcal{R}$-orbits in $(J^\ell(n, 1) \setminus \Sigma^\ell_0(n, 1))$ and semi-algebraic stratum of $\Sigma^\ell_0(n, 1)$ with codimension greater than $2n + 1$. By the above proposition, the appendix and the characterization of $\mathcal{R}^+$-versal unfolding (cf. [**1**]), we have the following theorem.

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Theorem 6.2. *Suppose that $n \leqslant 6$, then there is an open dense subset $\mathcal{O} \subset \mathrm{Emb}(U, H^n_+(-1))$ such that for any $\boldsymbol{x} \in \mathcal{O}$, the germ of the Lagrangian lift of the hyperbolic (respectively, de Sitter) evolute of $\boldsymbol{x}$ at each point is Lagrangian stable.*

7. Surfaces in hyperbolic 3-space

In this section we stick to the case when $n = 3$. Let $\boldsymbol{x} : U \to H^3_+(-1)$ be a surface, $H^{\mathrm{T}} : U \times H^3_+(-1) \to \mathbb{R}$ be a hyperbolic timelike height function and $H^{\mathrm{S}} : U \times S^3_1 \to \mathbb{R}$ be a hyperbolic spacelike height function. We consider the hyperbolic evolutes $\mathrm{HE}^{\pm}_M$ (respectively, de Sitter evolutes $\mathrm{SE}^{\pm}_M$). In the case $n = 3$, if $p_0 = \boldsymbol{x}(u_0, v_0)$ is not an umbilic point, then $h^{\mathrm{T}}_{a_0}$ (respectively, $h^{\mathrm{S}}_{a_0}$) has the $A_{k \geqslant 2}$-type singularity at p_0 by Proposition 3.2, where $\boldsymbol{a}_0 = \mathrm{HE}^{\pm}_M(u_0, v_0)$ (respectively, $\boldsymbol{a}_0 = \mathrm{SE}^{\pm}_M(u_0, v_0)$). Since $\mathrm{HE}^{\pm}_M$ (respectively, $\mathrm{SE}^{\pm}_M$) is the bifurcation set of H^{T} (respectively, H^{S}), it is non-singular if and only if $h^{\mathrm{T}}_{a_0}$ (respectively, $h^{\mathrm{S}}_{a_0}$) has the A_2-type singularity (i.e. the fold-type singularity) at p_0. Therefore, we have the following proposition.

Proposition 7.1.

(1) $p_0 = \boldsymbol{x}(u_0, v_0)$ *is a timelike ridge point if and only if* p_0 *is a non-umbilic point and the corresponding point* $\boldsymbol{a}_0 = \mathrm{HE}^{\pm}_M(u_0, v_0)$ *is a singular point of* $\mathrm{HE}^{\pm}_M(u, v)$.

(2) $p_0 = \boldsymbol{x}(u_0, v_0)$ *is a spacelike ridge point if and only if* p_0 *is a non-umbilic point and the corresponding point* $\boldsymbol{a}_0 = \mathrm{SE}^{\pm}_M(u_0, v_0)$ *is a singular point of* $\mathrm{SE}^{\pm}_M(u, v)$.

A *line of de Sitter principal curvature* on M is a curve which is everywhere tangent to these de Sitter principal directions. Let $\boldsymbol{x}(t) = \boldsymbol{x}(u(t), v(t))$ be a regular curve on $\boldsymbol{x}(U) = M$ and consider the corresponding curve on $\mathrm{HE}^{\pm}_M$ (respectively, $\mathrm{SE}^{\pm}_M$), which is given by

$$\boldsymbol{a}(t) = \pm \frac{\kappa(t)}{\sqrt{|\kappa^2(t) - 1|}} \biggl(\boldsymbol{x}(u(t), v(t)) + \frac{1}{\kappa(t)} \boldsymbol{e}(u(t), v(t)) \biggr),$$

where $\kappa(u, v)$ is a corresponding de Sitter principal curvature with $\kappa^2(u, v) > 1$ (respectively, $\kappa^2(u, v) < 1$) on U and $\kappa(t) = \kappa(u(t), v(t))$. We have the following characterization of the ridge point.

Corollary 7.2. *Let $p_0 = \boldsymbol{x}(u_0, v_0)$ be a non-umbilic point of M. Then we have the following assertions.*

(1) *We assume that $\kappa^2(t) > 1$, in which case $p_0 = \boldsymbol{x}(u(t_0), v(t_0))$ is a timelike ridge point if and only if there exists a line of de Sitter principal curvature $\boldsymbol{x}(t) = \boldsymbol{x}(u(t), v(t))$ with $p_0 = \boldsymbol{x}(u(t_0), v(t_0))$ and $\dot{\kappa}(t_0) = 0$.*

(2) *We assume that $\kappa^2(t) < 1$, in which case $p_0 = \boldsymbol{x}(u(t_0), v(t_0))$ is a spacelike ridge point if and only if there exists a line of de Sitter principal curvature $\boldsymbol{x}(t) = \boldsymbol{x}(u(t), v(t))$ with $p_0 = \boldsymbol{x}(u(t_0), v(t_0))$ and $\dot{\kappa}(t_0) = 0$.*

Proof. (1) Suppose that $p_0 = \boldsymbol{x}(u(t_0), v(t_0))$ is a timelike ridge point. By Proposition 7.1, there exists a regular curve $\boldsymbol{x}(t) = \boldsymbol{x}(u(t), v(t))$ with $p_0 = \boldsymbol{x}(u(t_0), v(t_0))$ and $\dot{\boldsymbol{a}}(t_0) = \mathbf{0}$. We can calculate that

$$\dot{\boldsymbol{a}}(t) = \pm \frac{-\dot{\kappa}(t)}{(\kappa^2(t)-1)^{3/2}} \left(\boldsymbol{x}(u(t), v(t)) + \frac{1}{\kappa(t)} \boldsymbol{e}(u(t), v(t)) \right) \\ \pm \frac{\kappa(t)}{\sqrt{\kappa^2(t)-1}} \left(\frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t}(u(t), v(t)) + \frac{1}{\kappa(t)} \frac{\mathrm{d}\boldsymbol{e}}{\mathrm{d}t}(u(t), v(t)) - \frac{\dot{\kappa}(t)}{\kappa(t)^2} \boldsymbol{e}(u(t), v(t)) \right).$$

Since $\mathrm{d}\boldsymbol{x}/\mathrm{d}t + (1/\kappa)\,\mathrm{d}\boldsymbol{e}/\mathrm{d}t$ is a tangent vector and $\boldsymbol{x}$, $\boldsymbol{e}$ are linearly independent normal vectors, $\dot{\boldsymbol{a}}(t) = \mathbf{0}$ if and only if $\dot{\kappa}(t) = 0$ and $\mathrm{d}\boldsymbol{x}/\mathrm{d}t + (1/\kappa)\,\mathrm{d}\boldsymbol{e}/\mathrm{d}t = \mathbf{0}$. Therefore, we have $\dot{\kappa}(t_0) = 0$ and $\mathrm{d}\boldsymbol{x}/\mathrm{d}t(t_0) + (1/\kappa)\,\mathrm{d}\boldsymbol{e}/\mathrm{d}t(t_0) = \mathbf{0}$. This means that the tangent vector $\mathrm{d}\boldsymbol{x}/\mathrm{d}t(t_0)$ gives a de Sitter principal direction at p_0. We can choose the curve $\boldsymbol{x}(u(t), v(t))$ as a line of de Sitter principal curvature with $\boldsymbol{x}(u(t_0), v(t_0)) = p_0$. The converse assertion follows by straightforward calculations.

The proof of (2) is also given by similar arguments to those for the proof of (1). $\square$

By Theorems 6.2 and A 2 and the classification of function germs under $\mathcal{R}^+$-codimension less than or equal to 3, we have the following classification theorem.

Theorem 7.3. *There exists an open dense subset $\mathcal{O} \subset \mathrm{Emb}(U, H^3_+(-1))$ such that for any $\boldsymbol{x} \in \mathcal{O}$, the corresponding Lagrangian immersion germ $L(H^{\mathrm{T}})$ at any point $((u_0, v_0), \boldsymbol{a}_0) \in U \times (H^3_+(-1) \setminus M)$ is Lagrangian equivalent to a Lagrangian immersion germ $L(F) : (C(F), 0) \to T^*\mathbb{R}^3$ whose generating family $F(u, v, q)$ ($q = (q_1, q_2, q_3) \in \mathbb{R}^3$) is one of the germs in the following list.*

(1) $u^3 + v^2 + q_1 u$ (*fold*).

(2) $\pm u^4 + v^2 + q_1 u + q_2 u^2$ ($\pm$*cusp*).

(3) $u^5 + v^2 + q_1 u + q_2 u^2 + q_3 u^3$ (*swallowtail*).

(4) $u^3 - uv^2 + q_1 u + q_2 v + q_3(u^2 + v^2)$ (*pyramid*).

(5) $u^3 + v^3 + q_1 u + q_2 v + q_3 uv$ (*purse*).

We also have exactly the same result for de Sitter evolutes. However, we only change the notation in the above theorem; we omit the detailed statement for de Sitter evolutes here.

We now apply Theorem 5.3 to the above classification theorem. Let $F(u, v, q)$ be one of the germs in the above list. We write $f(u, v) = F(u, v, 0)$, then we define $\mathcal{F}(f)$ as the singular foliation germ in $(\mathbb{R}^2, 0)$ defined by f (i.e. $\{f^{-1}(c)\}_{c \in (\mathbb{R}, 0)}$). As a corollary of the above classification theorem and Theorem 5.3, we have the following.

Corollary 7.4. *There exists an open dense subset $\mathcal{O} \subset \mathrm{Emb}(U, H^3_+(-1))$ such that for any $\boldsymbol{x} \in \mathcal{O}$ and any point $((u_0, v_0), \boldsymbol{a}_0) \in U \times (H^3_+(-1) \setminus M)$, the osculating spherical foliation germ $(\boldsymbol{x}^{-1}(\mathcal{F}_{\mathfrak{h}^{\mathrm{T}}_{a_0}}), (u_0, v_0))$ is diffeomorphic to a foliation germ $(\mathcal{F}(f), 0)$, where $F(u, v, q)$ is one of the germs in the list of Theorem 7.3.*

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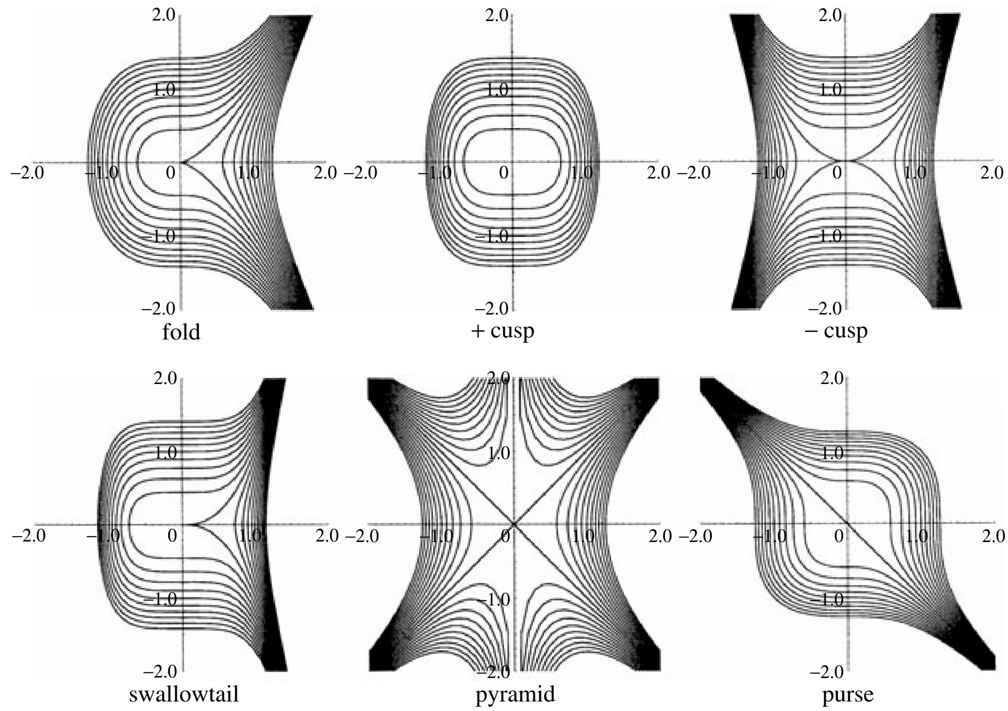

Figure 1. Generic osculating spherical foliation germs ($n = 3$).

We can draw the corresponding pictures of the foliation germs as in Figure 1.

8. Examples

In the last part of the paper, we give some examples and draw their pictures. Since we only consider the local situation, we use the notion of the hyperbolic Monge (H-Monge) form of a surface, which was introduced in [**6**]. Let $f(u,v)$ be a function with $f(0)=0$ and $f_u(0)=f_v(0)=0$. Then we have a surface in $H^3_+(-1)$ defined by

$$\boldsymbol{x}_f(u,v)=(\sqrt{f^2(u,v)+u^2+v^2+1}, f(u,v), u, v).$$

We can easily calculate that $\boldsymbol{e}(0)=(0,-1,0,0)$. We call $\boldsymbol{x}_f$ a *hyperbolic Monge form* (briefly, *H-Monge form*). If the point $\boldsymbol{x}_f(0,0)=(1,0,0,0)$ is umbilic with principal curvature $\kappa(0)=\kappa$, the function f can be written in the form $f(u,v)=(\kappa/2)(u^2+v^2)+g(u,v)$, where $\mathcal{H}f(0)=0$. Under the assumption that $0<\kappa^2\neq 1$, the centre of the osculating sphere is given by $\boldsymbol{a}_0=(\kappa/\sqrt{|\kappa^2-1|})(1,1/\kappa,0,0)$. Therefore, the osculating spherical foliation is given by

$$\left\{(u,v)\;\middle|\;\frac{\kappa}{\sqrt{|\kappa^2-1|}}\left\{\frac{1}{\kappa}f(u,v)+\sqrt{f^2(u,v)+u^2+v^2+1}\right\}=\frac{\kappa}{\sqrt{|\kappa^2-1|}}+c\right\}_{c\in(\mathbb{R},0)}.$$

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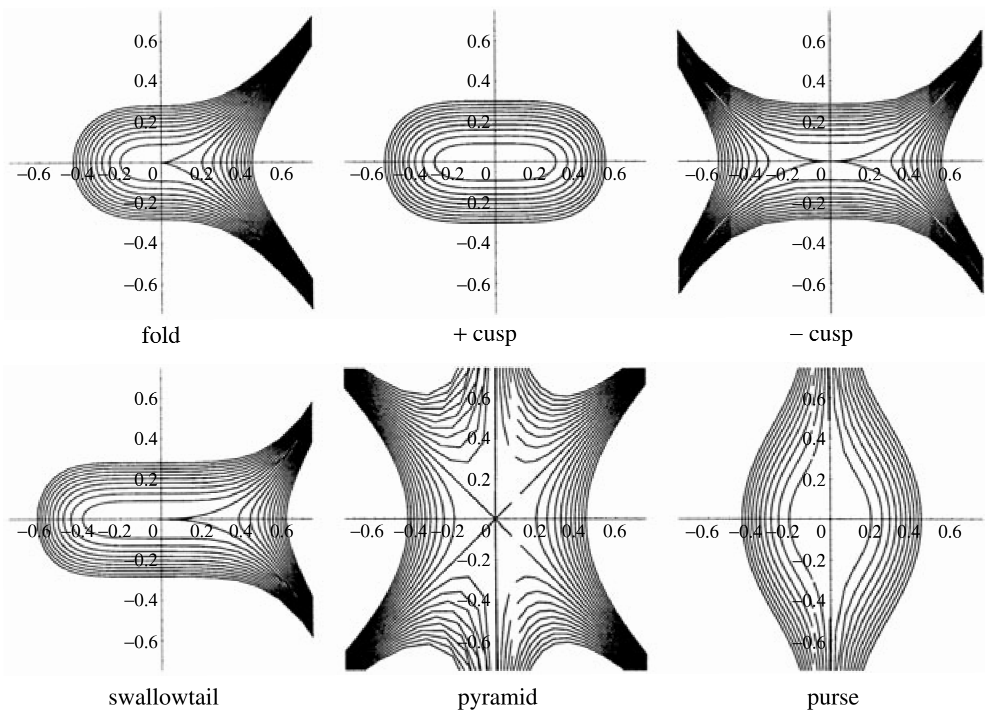

Figure 2. Osculating spherical foliation germs for examples (1)–(5).

For example, we consider the case where $\kappa = \frac{1}{2}$ and the $g(u, v)$ are given as follows.

(1) $g(u, v) = 3(u^3 - v^2)$ (fold).

(2) $g(u, v) = 3(\pm u^4 + v^2)$ ($\pm$cusp).

(3) $g(u, v) = 3(u^5 - v^2)$ (swallowtail).

(4) $g(u, v) = 3(u^3 - uv^2)$ (pyramid).

(5) $g(u, v) = 3(u^3 + v^3)$ (purse).

Since $\kappa = \frac{1}{2}$, the centre of the osculating sphere is located on S^3_1 and the total evolute is the de Sitter evolute. It might be very hard to draw the picture of the de Sitter evolute for each surface. However, we can easily draw the picture of the osculating spherical foliation for each surface by using the package ImplicitPlot of Mathematica as in Figure 2.

Appendix A. Lagrangian singularities and unfoldings of functions

In this section we give a brief review of the theory of Lagrangian singularities given in [**1**]. We consider the cotangent bundle $\pi : T^*\mathbb{R}^r \to \mathbb{R}^r$ over $\mathbb{R}^r$. Let $(u, p) =$

$(u_1, \dots, u_r, p_1, \dots, p_r)$ be the canonical coordinate on $T^*\mathbb{R}^r$. Then the canonical symplectic structure on $T^*\mathbb{R}^r$ is given by the *canonical* 2-*form* $\omega = \sum_{i=1}^r \mathrm{d}p_i \wedge \mathrm{d}u_i$. Let $i : L \to T^*\mathbb{R}^r$ be an immersion. We say that i is a *Lagrangian immersion* if $\dim L = r$ and $i^*\omega = 0$. In this case the critical value of $\pi \circ i$ is called the *caustic* of $i : L \to T^*\mathbb{R}^r$ and it is denoted by C_L. The main result in the theory of Lagrangian singularities is to describe Lagrangian immersion germs by using families of function germs. Let $F : (\mathbb{R}^n \times \mathbb{R}^r, (\mathbf{0}, \mathbf{0})) \to (\mathbb{R}, 0)$ be an r-parameter unfolding of function germs. We call

$$C(F) = \left\{(x, u) \in (\mathbb{R}^n \times \mathbb{R}^r, (\mathbf{0}, \mathbf{0})) \;\middle|\; \frac{\partial F}{\partial x_1}(x, u) = \dots = \frac{\partial F}{\partial x_n}(x, u) = 0\right\},$$

the *catastrophe set* of F and

$$\mathcal{B}_F = \left\{u \in (\mathbb{R}^r, 0) \;\middle|\; \text{there exist } (x, u) \in C(F) \text{ such that } \operatorname{rank}\left(\frac{\partial^2 F}{\partial x_i \partial x_j}(x, u)\right) < n\right\}$$

the *bifurcation set* of F.

Let $\pi_r : (\mathbb{R}^n \times \mathbb{R}^r, 0) \to (\mathbb{R}^r, 0)$ be the canonical projection, then we can easily show that the bifurcation set of F is the critical value set of $\pi_r|_{C(F)}$. We say that F is a *Morse family* if the map germ

$$\Delta F = \left(\frac{\partial F}{\partial u_1}, \dots, \frac{\partial F}{\partial u_r}\right) : (\mathbb{R}^n \times \mathbb{R}^r, 0) \to (\mathbb{R}^r, 0)$$

is non-singular, where $(x, u) = (x_1, \dots, x_n, u_1, \dots, u_r) \in (\mathbb{R}^n \times \mathbb{R}^r, 0)$. In this case we have a smooth submanifold germ $C(F) \subset (\mathbb{R}^n \times \mathbb{R}^r, 0)$ and a map germ $L(F) : (C(F), 0) \to T^*\mathbb{R}^r$ defined by

$$L(F)(x, u) = \left(u, \frac{\partial F}{\partial u_1}, \dots, \frac{\partial F}{\partial u_r}\right).$$

We can show that $L(F)$ is a Lagrangian immersion. Then we have the following fundamental theorem [**1**, p. 300].

Proposition A 1. *All Lagrangian submanifold germs in $T^*\mathbb{R}^r$ are constructed by the above method.*

Using the above notation, we call F a *generating family* of $L(F)$.

We define an equivalence relation among Lagrangian immersion germs. Let $i : (L, x) \to (T^*\mathbb{R}^r, p)$ and $i' : (L', x') \to (T^*\mathbb{R}^r, p')$ be Lagrangian immersion germs. Then we say that i and i' are *Lagrangian equivalent* if there exist a diffeomorphism germ $\sigma : (L, x) \to (L', x')$, a symplectic diffeomorphism germ $\tau : (T^*\mathbb{R}^r, p) \to (T^*\mathbb{R}^r, p')$, and a diffeomorphism germ $\bar{\tau} : (\mathbb{R}^r, \pi(p)) \to (\mathbb{R}^r, \pi(p'))$ such that $\tau \circ i = i' \circ \sigma$ and $\pi \circ \tau = \bar{\tau} \circ \pi$, where $\pi : (T^*\mathbb{R}^r, p) \to (\mathbb{R}^r, \pi(p))$ is the canonical projection and a symplectic diffeomorphism germ is a diffeomorphism germ which preserves symplectic structure on $T^*\mathbb{R}^r$. In this case, the caustic C_L is diffeomorphic to the caustic $C_{L'}$ by the diffeomorphism germ $\bar{\tau}$.

A Lagrangian immersion germ into $T^*\mathbb{R}^r$ at a point is said to be *Lagrangian stable* if for every map with the given germ there is a neighbourhood in the space of Lagrangian

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immersions (in the Whitney C^∞-topology) and a neighbourhood of the original point such that each Lagrangian immersion belonging to the first neighbourhood has in the second neighbourhood a point at which its germ is Lagrangian equivalent to the original germ.

We can interpret the Lagrangian equivalence by using the notion of generating families. We denote by $\mathcal{E}_m$ the local ring of function germs $(\mathbb{R}^m, 0) \to \mathbb{R}$ with the unique maximal ideal $\mathfrak{M}_m = \{h \in \mathcal{E}_m \mid h(0) = 0\}$. Let $F, G : (\mathbb{R}^n \times \mathbb{R}^r, 0) \to (\mathbb{R}, 0)$ be function germs. We say that F and G are *P–$\mathcal{R}^+$-equivalent* if there exists a diffeomorphism germ $\Phi : (\mathbb{R}^n \times \mathbb{R}^r, 0) \to (\mathbb{R}^n \times \mathbb{R}^r, 0)$ of the form $\Phi(x, u) = (\Phi_1(x, u), \phi(u))$ and a function germ $h : (\mathbb{R}^r, 0) \to \mathbb{R}$ such that $G(x, u) = F(\Phi(x, u)) + h(u)$. For any $F_1 \in \mathfrak{M}_{n+r}$ and $F_2 \in \mathfrak{M}_{n'+r}$, F_1, F_2 are said to be *stably P–$\mathcal{R}^+$-equivalent* if they become P–$\mathcal{R}^+$-equivalent after the addition to the arguments to x_i of new arguments y_i and to the functions F_i of non-degenerate quadratic forms Q_i in the new arguments (i.e. $F_1 + Q_1$ and $F_2 + Q_2$ are P–$\mathcal{R}^+$-equivalent).

Let $F : (\mathbb{R}^n \times \mathbb{R}^r, 0) \to (\mathbb{R}, 0)$ be a function germ. We say that F is an *$\mathcal{R}^+$-versal deformation* of $f = F|_{\mathbb{R}^n \times \{0\}}$ if

$$\mathcal{E}_n = J_f + \left\langle \frac{\partial F}{\partial u_1}\bigg|\mathbb{R}^n \times \{0\}, \dots, \frac{\partial F}{\partial u_r}\bigg|\mathbb{R}^n \times \{0\} \right\rangle_{\mathbb{R}} + \langle 1 \rangle_{\mathbb{R}},$$

where

$$J_f = \left\langle \frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n} \right\rangle_{\mathcal{E}_n}.$$

Theorem A 2. *Let $F_1 \in \mathfrak{M}_{n+r}$ and $F_2 \in \mathfrak{M}_{n'+r}$ be Morse families. Then we have the following.*

(1) *$L(F_1)$ and $L(F_2)$ are Lagrangian equivalent if and only if F_1, F_2 are stably P–$\mathcal{R}^+$-equivalent.*

(2) *$L(F)$ is Lagrangian stable if and only if F is a $\mathcal{R}^+$-versal deformation of $F|\mathbb{R}^n \times \{0\}$.*

For the proof of the above theorem, see pp. 304 and 325 of [**1**]. The following proposition describes the well-known relationship between bifurcation sets and equivalence among unfoldings of function germs.

Proposition A 3. *Let $F, G : (\mathbb{R}^n \times \mathbb{R}^r, 0) \to (\mathbb{R}, 0)$ be function germs. If F and G are P–$\mathcal{R}^+$-equivalent then there exists a diffeomorphism germ $\phi : (\mathbb{R}^r, 0) \to (\mathbb{R}^r, 0)$ such that $\phi(\mathcal{B}_F) = \mathcal{B}_G$.*

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