Seventh Meeting, May 8th 1885.

A. J. G. Barclay, Esq., M.A., President, in the Chair.

On certain formulae for Repeated Differentiation.

By Professor Chrystal.

In many questions of analysis relating to the theory of plane curves, it is convenient to be able to obtain quickly the expansions of $\left(\frac{d}{dx}\right)^m(y^n)$, and of $\left(\frac{d}{dx}\right)^m(x^p y^n)$. These may be obtained as follows:—

By an obvious extension of the theorem of Leibnitz we have

$$\left(\frac{d}{dx}\right)^m(y_1 y_2 \ldots y_n) = (d_1 + d_2 + \ldots + d_n)^m y_1 y_2 \ldots y_n,$$

where $d_1, d_2, \ldots d_n$ differentiate $y_1, y_2, \ldots y_n$ respectively.

Hence by the multinomial theorem

$$\left(\frac{d}{dx}\right)^m(y_1 y_2 \ldots y_n) = m!\Sigma\left(\frac{y_{1r_1} y_{2r_2} \ldots y_{nr_n}}{r_1!\, r_2! \ldots r_n!}\right),$$

where $r_1 \nless 0 \ldots r_n \nless 0$; $r_1 + r_2 \ldots + r_n = n$.

Now in this formula let $y_1 = y_2 \ldots = y_n$, each $= y$, and observe that the term $\frac{(y_{r_1})^{\rho_1}(y_{r_2})^{\rho_2}\ldots\ldots}{(r_1!)^{\rho_1}(r_2!)^{\rho_2}\ldots\ldots}$ will occur as often as there are permutations of n things taken all together, ρ_1 of which are all alike, ρ_2 all alike, &c.; that is $n!/\rho_1!\rho_2!\ldots\ldots$ times; we then obtain

$$\left(\frac{d}{dx}\right)^m(y^n) = m!n!\Sigma\left(\frac{(y_{r_1})^{\rho_1}(y_{r_2})^{\rho_2}\ldots\ldots}{(r_1!)^{\rho_1}(r_2!)^{\rho_2}\ldots\rho_1!\rho_2!\ldots\ldots}\right);$$

where

$$r_1 \nless 0 \ngtr m,\ r_2 \nless 0 \ngtr m, \ldots\ldots;$$
$$\rho_1 \nless 1 \ngtr n,\ \rho_2 \nless 1 \ngtr n, \ldots\ldots;$$
$$r_1\rho_1 + r_2\rho_2 + \ldots\ldots = m;$$
$$\rho_1 + \rho_2 + \ldots\ldots = n.$$

Similarly from

$$\left(\frac{d}{dx}\right)^m(x^p y_1 y_2 \ldots y_n) = (d + d_1 + d_2 \ldots + d_n)^m(x^p y_1 y_2 \ldots y_n)$$ we derive

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$$\left(\frac{d}{dx}\right)^m(x^p y^n)$$

$$= m!\,\Sigma\left((n-r)!\,\frac{p(p-1)\ldots(p-r+1)}{r!}x^{p-r}\,\frac{(y_{r_1})^{\rho_1}(y_{r_2})^{\rho_2}\ldots\ldots}{(r_1!)^{\rho_1}(r_2!)^{\rho_2}\ldots\rho_1!\,\rho_2!\ldots\ldots}\right);$$

where
$$r \nless 0 \ngtr m,\ r_1 \nless 0 \ngtr m, \ldots\ldots;$$
$$\rho_1 \nless 1 \ngtr n,\ \rho_2 \nless 1 \ngtr n, \ldots\ldots;$$
$$r + \rho_1 r_1 + \rho_2 r_2 + \ldots\ldots = m;$$
$$\rho_1 + \rho_2 + \ldots\ldots = n.$$

Example

$$\left(\frac{d}{dx}\right)^9(y^3) = 9!\,3!\left[\frac{y_9 y^2}{9!\,2!} + \frac{y_8 y_1 y}{8!} + \frac{y_7 y_1^2}{7!\,2!} + \frac{y_7 y_2 y}{7!\,2!} + \frac{y_6 y_3 y}{6!\,3!} + \frac{y_6 y_2 y_1}{6!\,2!} + \frac{y_5 y_4 y}{5!\,4!}\right.$$
$$\left. + \frac{y_5 y_3 y_1}{5!\,3!} + \frac{y_5 y_2^2}{5!(2!)^3} + \frac{y_4^2 y_1}{(4!)^2 2!} + \frac{y_4 y_3 y_2}{4!\,3!\,2!} + \frac{y_3^3}{(3!)^4}\right]$$
$$= 3y_9 y^2 + 54 y_8 y_1 y + 216 y_7 y_1^2 + 216 y_7 y_2 y + 504 y_6 y_3 y_1 + 1512 y_6 y_2 y_1$$
$$+ 756 y_5 y_4 y + 3024 y_5 y_3 y_1 + 2268 y_5 y_2^2 + 1890 y_4^2 y_1 + 7560 y_4 y_3 y_2 + 1680 y_3^3.$$

On a method for obtaining the differential equation to an Algebraical Curve.

By Professor CHRYSTAL.

1. Consider the conic represented by the general equation

$$a_0 + b_0 x + b_1 y + c_0 x^2 + c_1 xy + c_2 y^2 = 0. \quad \ldots \quad \ldots \quad (1)$$

Differentiating three times with respect to a we get

$$b_1(y)_3 + c_2(y^2)_3 + c_1(xy)_3 = 0 \ldots \quad \ldots \quad \ldots \quad (2)$$

where $(y)_3$ stands for $\left(\frac{d}{dx}\right)^3(y)$.

Again, from (2) by successive differentiation we derive

$$b_1(y)_4 + c_2(y^2)_4 + c_1(xy)_4 = 0 \quad \ldots \quad \ldots \quad \ldots \quad (3)$$

$$b_1(y)_5 + c_2(y^2)_5 + c_1(xy)_5 = 0 \quad \ldots \quad \ldots \quad \ldots \quad (4)$$

From (2) (3) (4), eliminating the remaining constants, we have

$$\begin{vmatrix} (y)_3 & (y^2)_3 & (xy)_3 \\ (y)_4 & (y^2)_4 & (xy)_4 \\ (y)_5 & (y^2)_5 & (xy)_5 \end{vmatrix} = 0 \quad \ldots \quad \ldots \quad \ldots \quad (6)$$

which is one form of the differential equation to the conic (1).