THE SWISS RE EXPOSURE CURVES AND THE MBBEFD[1] DISTRIBUTION CLASS

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ABSTRACT

A new two-parameter family of analytical functions will be introduced for the modelling of loss distributions and exposure curves. The curve family contains the Maxwell-Boltzmann, the Bose-Einstein and the Fermi-Dirac distributions, which are well known in statistical mechanics. The functions can be used for the modelling of loss distributions on the finite interval [0, 1] as well as on the interval [0, ∞]. The functions defined on the interval [0, 1] are discussed in detail and related to several Swiss Re exposure curves used in practice. The curves can be fitted to the first two moments μ and σ of a loss distribution or to the first moment μ and the total loss probability p.

1. INTRODUCTION

Whenever possible, the rating of non proportional (NP) reinsurance treaties should not only rely on the loss experience of the past, but also on actual exposure. For the case of per risk covers, exposure rating is based on risk profiles. All risks of similar size (SI, MPL or EML) belonging to the same risk category are summarized in a risk band. For the purpose of rating, all the risks belonging to one specific band are assumed to be homogeneous. They can thus be modelled with the help of one single loss distribution function.

The problem of exposure rating is how to divide the total premiums of one band between the ceding company and the reinsurer. The problem is solved in two steps. First, the overall risk premiums (per band) are estimated by applying an appropriate loss ratio to the gross premiums. In a second step, these risk premiums are divided into risk premiums for the retention and risk premiums for the cession. Due to the nature of NP reinsurance, this is possible only with the help of the loss distribution function.

However, the correct loss distribution function for an individual band of a risk profile is hardly known in practice. This lack of information is overcome with the help of distribution functions derived from large portfolios of similar risks. Such distribution functions are available in the form of so-called exposure curves. These curves directly permit the extraction of the risk premium ratio required by the reinsurer as a function of the deductible.

[1] Maxwell-Boltzmann, Bose-Einstein and Fermi-Dirac distribution

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Often, underwriters have only a finite number of discrete exposure curves at their disposal. These curves are available in graphical or tabulated form, and are also implemented in computerized underwriting tools. One of the curves must be selected for each risk band, but it is not always clear which curve should be used. In such cases, the underwriter might also want to use a virtual curve lying between two of the discrete curves available to him.

This can be achieved by replacing the discrete curves with analytical exposure curves. Each set of parameters then defines another curve. If a continuous set of parameters is available, the exposure curves can be varied smoothly within the whole range of available curves. However, the curves must fulfill certain conditions which restrict the range of the parameters. In addition, practical problems can arise if a curve family with many (more than two) parameters is used. It might then become very difficult to find a set of parameters which can be associated with the information available for a class of risks. This problem can be overcome if a curve family is restricted to a one- or two-parameter subclass and if new parameters are introduced which can easily be interpreted by the underwriters.

In the following, the MBBEFD class of analytical exposure curves will be introduced. As will be seen, this class is very well suited for the modelling of exposure curves used in practice. Before analysing the MBBEFD curves in detail, some general relations between a distribution function and its related exposure curve will be discussed in section 2. These relations permit the derivation of the conditions to be fulfilled by exposure curves. The new, two-parameter class of distribution functions will then be introduced in section 3. Finally, several practical aspects, and the link to the well known Swiss Re property exposure curves Y_i, will be discussed in section 4.

Conventions

Following the notation used by Daykin et al in [1], we will denote stochastic variables by bold letters, e.g. $\mathbf{X}$ or $\mathbf{x}$. Monetary variables are denoted by capital letters, for instance, X or M, while ratio variables are denoted by small letters, for instance, x = X/M.

2. DISTRIBUTION FUNCTION AND EXPOSURE CURVE

2.1. Definition of the exposure curve

In the following, the relation between the distribution function F(x) defined on the interval [0, 1] and its limited expected value function $L(d) = E[\min(d, \mathbf{x})]$ will be discussed. Here, d = D/M and $\mathbf{x} = \mathbf{X}/M$ represent the normalized deductible and the normalized loss, respectively. M is the maximum possible loss (MPL) and $\mathbf{X} \le M$ the gross loss. The deductible D is the cedent's maximum retention under a non proportional reinsurance treaty. $M \cdot L(d)$ is the expected value of the losses retained by the cedent while $M \cdot (L(1) - L(d))$ is the expected value of the losses paid by the reinsurer. Thus, the ratio of the pure risk premiums retained by the cedent is given by the relative limited expected value function G(d) = L(d)/L(1) [1]. The curve representing this function is also called the **exposure curve**:

$$G(d) = \frac{L(d)}{L(1)} = \frac{\int_0^d (1 - F(y))dy}{\int_0^1 (1 - F(y))dy} = \frac{\int_0^d (1 - F(y))dy}{E[x]} \tag{2.1}$$

Because of 1 – F(x) ≥ 0 and F'(x) = f(x) ≥ 0, G(d) is an increasing and concave function on the interval [0, 1]. In addition, G(0) = 0 and G(1) = 1 by definition.

2.2. Deriving the distribution function from the exposure curve

If the exposure curve G(x) is given, the corresponding distribution function F(x) can be derived from:

$$G'(d) = \frac{1 - F(d)}{E[x]} \tag{2.2}$$

With F(0) = 0 and G'(0) = 1/E[**x**] one obtains:

$$F(x) = \begin{cases} 1 & x = 1 \\ 1 - \dfrac{G'(x)}{G'(0)} & 0 \le x < 1 \end{cases} \tag{2.3}$$

Thus, F(x) and G(x) are equivalent representations of the loss distribution.

2.3. Total loss probability and expected value

The probability p for a total loss equals 1 – F(1^-) and the expected (or average) loss μ equals E[**x**]. These two functionals of the distribution function F(x) can be derived directly from the derivatives of G(x) at x = 0 and x = 1:

$$\begin{aligned} \mu &= E[x] = \frac{1}{G'(0)} \\ p &= 1 - F(1^-) = \frac{G'(1)}{G'(0)} \end{aligned} \tag{2.4}$$

The fact that G(x) is a concave and increasing function on the interval [0 ,1] with G(0) = 0 and G(1) = 1 implies:

$$G'(0) \ge 1 \ge G'(1) \ge 0 \tag{2.5}$$

This is also reflected in the relation:

$$0 \le p \le \mu \le 1 \tag{2.6}$$

2.4. Unlimited distributions

If the distribution function F(X) is defined on the interval [0, ∞], the above relations have to be slightly modified. In this case there is no finite maximum loss M. However, the deductible D and the losses **X** can be normalized with respect to an arbitrary reference loss X_0, i.e. **x** = **X**/X_0 and d = D/X_0. G(d) is still a concave and increasing function with G(0) = 0 and G(∞) = 1. The expected value μ = E[**x**] is also given by 1/G'(0), but there are no total losses, i.e. G'(∞) = 0.

3. THE MBBEFD CLASS OF TWO-PARAMETER EXPOSURE CURVES

3.1. Definition of the curve

In this section we will investigate the exposure curves and the related distribution functions defined by:

$$G(x) = \frac{\ln(a+b^x) - \ln(a+1)}{\ln(a+b) - \ln(a+1)} \tag{3.1 a}$$

The distribution function belonging to this exposure curve is given by:

$$F(x) = \begin{cases} 1 & x = 1 \\ 1 - \dfrac{(a+1)b^x}{a+b^x} & 0 \le x < 1 \end{cases} \tag{3.1 b}$$

The denominator and the term -ln(a + 1) in the nominator of (3.1 a) ensure that the boundary conditions G(0) = 0 and G(1) = 1 are fulfilled. As will be seen below, the cases a = {-1, 0, ∞} or b = {0, 1, ∞} have to be treated separately.

*Distribution functions of the type (3.1), defined on the interval [0, ∞] or [-∞, ∞], are very well known in statistical mechanics (**Maxwell-Boltzmann**, **Bose-Einstein**, **Fermi-Dirac** and **Planck** distribution). The implementation of these functions in risk theory does not mean that the distribution of insured losses can be derived from the theory of statistical mechanics. However, the MBBEFD distribution class defined in (3.1) shows itself to be very appropriate for the modelling of empirical loss distributions on the interval [0, 1].*

3.2. New parametrisation

The parameters {a, b} are restricted to those values, for which $G_{a,b}(x)$ is a real, increasing and concave function on the interval [0, 1]. It is easier to fulfill this condition by using the inverse g = 1/p of the total loss probability p as a curve parameter and to replace the parameter a in (3.1):

$$g = \frac{a+b}{(a+1)b}; \quad a = \frac{(g-1)b}{1-gb} \tag{3.2}$$

On the one hand, the condition $0 \le p \le 1$ is fulfilled only for $g \ge 1$. On the other hand, G(x) is a real function only for $b \ge 0$. It can be shown that no other restrictions regarding the set of parameters are necessary.

However, cases b = 1 (i.e. a = -1), b = 0 or g = 1 (i.e. a = 0) and $b \cdot g = 1$ (i.e. $a = \infty$) must be treated as special cases. The cases $b \cdot g = 1$ (i.e. $a = \infty$), $b \cdot g > 1$ (i.e. $a < 0$) and $b \cdot g < 1$ (i.e. $a > 0$) correspond to the MB, the BE and the FD distribution, respectively (cf. figure 4.1). By considering special cases b = 1, g = 1 and $b \cdot g = 1$ separately, all real, increasing and concave functions G(x) on the interval [0, 1] with G(0) = 0 and G(1) = 1 belonging to the MBBEFD class (3.1) can be represented as follows:

$$G_{b,g}(x) = \begin{cases} x & g = 1 \vee b = 0 \\ \dfrac{\ln(1+(g-1)x)}{\ln(g)} & b = 1 \wedge g > 1 \\ \dfrac{1-b^x}{1-b} & bg = 1 \wedge g > 1 \\ \dfrac{\ln\left(\dfrac{(g-1)b+(1-gb)b^x}{1-b}\right)}{\ln(gb)} & b > 0 \wedge b \ne 1 \wedge bg \ne 1 \wedge g > 1 \end{cases} \tag{3.3}$$

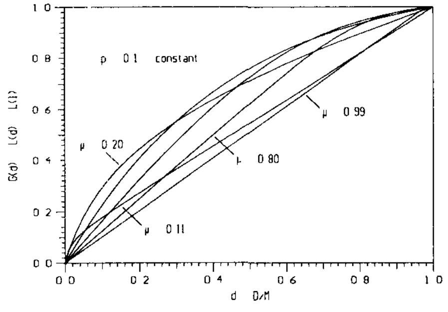

FIGURE 3 1 a) Set of MBBEFD exposure curves with constant parameter g = 1/p = 10 and μ = E[**x**] = 0 11, 0 2, 0 4, 0 6, 0 8, 0 99

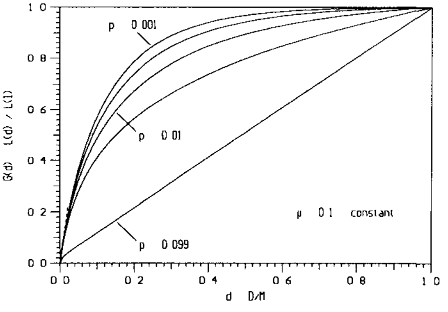

FIGURE 3 1 b) Set of MBBEFD exposure curves with constant μ = E[**x**] = 0 1 and p = 1/ g = 0.099, 0 031, 0 01, 0 0031, 0 001 The dashed line with slope 1/ μ represents the tangent at d = 0

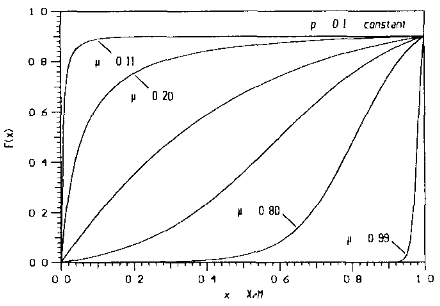

FIGURE 3 2 a) Distribution functions belonging to exposure curves of figure 3 1 a)

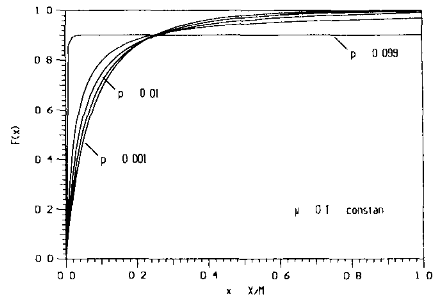

FIGURE 3 2 b) Distribution functions belonging to exposure curves of figure 3 1 b)

Examples of MBBEFD exposure curves are shown in figure 3.1. A set of curves with constant total loss probability p = 0.1 (i.e. g = 10) is represented in figure 3.1 a). Figure 3.1 b) contains a set of curves with constant expected value $\mu = 0.1$. The corresponding distribution functions are shown in figures 3.2 a) and b).

3.3. Derivatives

The derivatives of the exposure curves are given by:

$$G'(x) = \begin{cases} 1 & g = 1 \vee b = 0 \\ \dfrac{g-1}{\ln(g)\left(1+(g-1)x\right)} & b = 1 \wedge g > 1 \\ \dfrac{\ln(b)b^x}{b-1} & bg = 1 \wedge g > 1 \\ \dfrac{\ln(b)(1-gb)}{\ln(gb)\left((g-1)b^{1-x}+(1-gb)\right)} & b > 0 \wedge b \neq 1 \wedge bg \neq 1 \wedge g > 1 \end{cases} \tag{3.4}$$

with

$$G'(0) = \begin{cases} 1 & g = 1 \vee b = 0 \\ \dfrac{g-1}{\ln(g)} & b = 1 \wedge g > 1 \\ \dfrac{\ln(b)}{b-1} = \dfrac{\ln(g)g}{g-1} & bg = 1 \wedge g > 1 \\ \dfrac{\ln(b)(1-gb)}{\ln(gb)(1-b)} & b > 0 \wedge b \neq 1 \wedge bg \neq 1 \wedge g > 1 \end{cases} \tag{3.4 a}$$

and

$$G'(1) = \begin{cases} 1 & g = 1 \vee b = 0 \\ \dfrac{g-1}{\ln(g)g} & b = 1 \wedge g > 1 \\ \dfrac{\ln(b)b}{b-1} = \dfrac{\ln(g)}{g-1} & bg = 1 \wedge g > 1 \\ \dfrac{\ln(b)(1-gb)}{\ln(gb)g(1-b)} & b > 0 \wedge b \neq 1 \wedge bg \neq 1 \wedge g > 1 \end{cases} \tag{3.4 b}$$

The relation p = G'(1)/G'(0) = 1/g is obtained immediately from (3.4 a) and (3.4 b).

3.4. Expected value

According to (2.4) the expected value μ is given by:

$$\mu = E[x] = \frac{1}{G'(0)} = \begin{cases} 1 & g = 1 \vee b = 0 \\ \dfrac{\ln(g)}{g-1} & b = 1 \wedge g > 1 \\ \dfrac{b-1}{\ln(b)} = \dfrac{g-1}{\ln(g)g} & bg = 1 \wedge g > 1 \\ \dfrac{\ln(gb)(1-b)}{\ln(b)(1-gb)} & b > 0 \wedge b \neq 1 \wedge bg \neq 1 \wedge g > 1 \end{cases} \tag{3.5}$$

The expected value μ is represented as a function of the parameters b and g in figure 3 3 and discussed below in section 3.7.

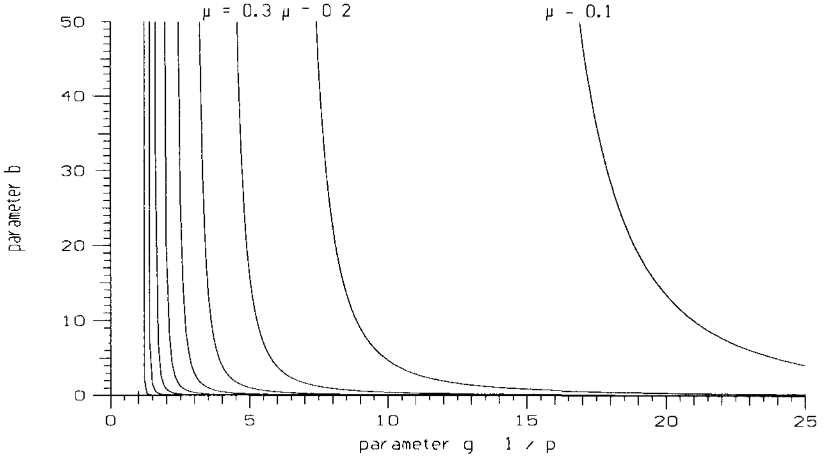

FIGURE 3 3 Parameter b as a function of g = 1/p for μ = E[x] = 0 1, 0 2, 0 9
The dashed line at g = 1 and the horizontal line at b = 0 represent
the parameter sets {b, g} with μ = 1

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3.5. Distribution function

According to (2.3), the distribution function belonging to the exposure curve $G_{b,g}(x)$ is given by:

$$F(x)=\begin{cases}1 & x=1\\ 0 & x<1\wedge(g=1\vee b=0)\\ 1-\dfrac{1}{1+(g-1)x} & x<1\wedge b=1\wedge g>1\\ 1-b^{x} & x<1\wedge bg=1\wedge g>1\\ 1-\dfrac{1-b}{(g-1)b^{1-x}+(1-gb)} & x<1\wedge b>0\wedge b\neq 1\wedge bg\neq 1\wedge g>1\end{cases} \qquad (3.6)$$

The distribution functions belonging to the exposure curves of figure 3.1 are represented in figure 3.2. The set of distribution functions with constant total loss probability p = 0.1 (g = 10) is shown in figure 3.2 a). Figure 3.2 b) contains the set of distribution functions with constant expected value $\mu = 0.1$.

3.6. Density function

Because of the finite probability p = 1/g for a total loss, the density function f(x) = F'(x) is defined only on the interval [0, 1):

$$f(x)=\begin{cases}0 & g=1\vee b=0\\ \dfrac{g-1}{(1+(g-1)x)^{2}} & b=1\wedge g>1\\ -\ln(b)b^{x} & bg=1\wedge g>1\\ \dfrac{(b-1)(g-1)\ln(b)b^{1-x}}{\left((g-1)b^{1-x}+(1-gb)\right)^{2}} & b>0\wedge b\neq 1\wedge bg\neq 1\wedge g>1\end{cases} \qquad (3.7)$$

3.7. Discussion

It is instructive to analyse the expected value $\mu = \mu(b, g)$ as a function of the parameters b and g (3.5). Figure 3.3. shows the range of permitted parameters in the {b, g} plane and the curves with constant expected value μ. One can see in figure 3.3 that $\mu_g(b)$ is a decreasing function of b (for g > 1 constant) and that $\mu_b(g)$ is a decreasing function of g (for b > 0 constant):

$$\begin{aligned}\frac{\partial}{\partial b}\mu_g(b)\leq 0\\ \frac{\partial}{\partial g}\mu_b(g)\leq 0\end{aligned} \qquad g>1\wedge b>0 \qquad (3.8)$$

The expected value μ is related as follows to the extreme values of the parameters b and g:

$$\lim_{b \to 0} \mu_g(b) = 1; \qquad \lim_{b \to \infty} \mu_g(b) = 1/g = p$$
$$\lim_{g \to 1} \mu_b(g) = 1; \qquad \lim_{g \to \infty} \mu_b(g) = 0 \tag{3.9}$$

3.8. Unlimited distributions

So far, only distributions defined on the interval [0, 1] have been discussed. However, as the MB, the BE and the FD distributions are defined on the interval [-∞, ∞] or [0, ∞], the MBBEFD distribution class can also be used for the modelling of loss distributions on the interval [0, ∞]. If the losses **X** and the deductible D are normalized with respect to an arbitrary reference loss X_0, then $\mathbf{x} = \mathbf{X}/X_0$ and $d = D/X_0$. The above formula can now be modified as follows:

$$G_{b,g}(x) = \begin{cases} 1 - b^x & bg = 1 \wedge g > 1 \\ \dfrac{\ln\left(\dfrac{(g-1)b + (1-gb)b^x}{1-b}\right)}{\ln\left(\dfrac{(g-1)b}{1-b}\right)} & 0 < b < 1 \wedge bg \neq 1 \wedge g > 1 \end{cases} \tag{3.10}$$

$$G'(x) = \begin{cases} -\ln(b)b^x & bg = 1 \wedge g > 1 \\ \dfrac{\ln(b)(1-gb)}{\ln\left(\dfrac{(g-1)b}{1-b}\right)\left((g-1)b^{1-x} + (1-gb)\right)} & 0 < b < 1 \wedge bg \neq 1 \wedge g > 1 \end{cases} \tag{3.11}$$

$$G'(0) = \begin{cases} -\ln(b) & bg = 1 \wedge g > 1 \\ \dfrac{\ln(b)(1-gb)}{\ln\left(\dfrac{(g-1)b}{1-b}\right)(1-b)} & 0 < b < 1 \wedge bg \neq 1 \wedge g > 1 \end{cases} \tag{3.11 a}$$

$$G'(1) = \begin{cases} -\ln(b)b & bg = 1 \wedge g > 1 \\ \dfrac{\ln(b)(1-gb)}{\ln\left(\dfrac{(g-1)b}{1-b}\right)g(1-b)} & 0 < b < 1 \wedge bg \neq 1 \wedge g > 1 \end{cases} \tag{3.11 b}$$

$$G'(\infty) = 0 \tag{3.11 c}$$

$$F(x) = \begin{cases} 1 - b^x & bg = 1 \wedge g > 1 \\ 1 - \dfrac{1-b}{(g-1)b^{1-x} + (1-gb)} & 0 < b < 1 \wedge bg \neq 1 \wedge g > 1 \end{cases} \tag{3.12}$$

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The restriction $0 < b < 1$ is obtained immediately from (3.12) and the condition $F(\infty) = 1$, while the restriction $g > 1$ is obtained from (3.10), where the argument of the logarithm in the denominator must be greater than 0. The same restriction is also obtained from the relation $p = G'(1)/G'(0) = 1/g$, which is still valid. The parameter g is thus the inverse of the probability p of having a loss **X** exceeding the reference loss X_0.

4. CURVE FITTING

4.1. Expected value μ and total loss probability p

Because of (3.8) and (3.9), there exists exactly one distribution function belonging to the MBBEFD class for each given pair of functionals p and μ (cf. figure 3.3), provided that p and μ fulfill the conditions (2.6). The curve parameter $g = 1/p$ is obtained directly. The second curve parameter b can be calculated with the help of (3.5). Here, the following cases must be distinguished:

$$\begin{array}{lll} a) & \mu = 1 & \Rightarrow b = 0 \\ b) & \mu = \dfrac{g-1}{\ln(g)g} & \Rightarrow b = 1/g \\ c) & \mu = \dfrac{\ln(g)}{g-1} & \Rightarrow b = 1 \\ d) & \mu = 1/g & \Rightarrow b = \infty \\ e) & else & \Rightarrow 0 < b < \infty \wedge b \neq 1/g \wedge b \neq 1 \end{array} \tag{4.1}$$

In the general case e), the parameter b has to be calculated iteratively by solving the equation:

$$\mu = \frac{\ln(gb)(1-b)}{\ln(b)(1-gb)} \tag{4.2}$$

Because $\mu_g(b)$ is a decreasing function of b (3.8), the iteration causes no problems. An upper and a lower limit for b can be derived directly from (4.1).

4.2. Expected value μ and standard deviation σ

It is also possible to find a MBBEFD distribution assuming the first two moments (e.g. μ and σ) are known, provided the moments fulfill certain conditions. The first two moments of a distribution function with total loss probability p are given by:

$$\begin{aligned} \mu &= E[x] = p + \int_0^{1^-} x f(x)dx \\ \mu^2 + \sigma^2 &= E[x^2] = p + \int_0^{1^-} x^2 f(x)dx \le \mu \end{aligned} \tag{4.3}$$

According to (4.3) the first two moments of F(x) and p must fulfill the following conditions:

$$\mu^2 \le E[x^2] \le \mu$$
$$p \le E[x^2] \tag{4.4}$$

Calculation of g and b

Basic idea:
1. Start with $p^* = E[\mathbf{x}^2] \ge p$ as a first estimate (upper limit) for p, and calculate b^* and g^* for the given functionals μ and p^* with the method described in 4.1 above.
2. Compare the second moment $E^*[\mathbf{x}^2]$ with the given moment $E[\mathbf{x}^2]$ and find a new estimate for p^*.
3. Repeat until $E^*[\mathbf{x}^2]$ is close enough to $E[\mathbf{x}^2]$.

If the first moment μ is kept constant, then the second moment $E^*[\mathbf{x}^2]$ will be an increasing function of p^*. Thus the parameters g and b can be calculated without complications.

Remark: The second moment of the MBBEFD distribution has to be calculated numerically. This is best done by replacing F(x) with a discrete distribution function which has the same upper tail area $L(x_{i+1}) - L(x_i)$ as F(x) on each discretized interval $[x_i, x_{i+1}]$.

4.3. The MBBEFD distribution class and the Swiss Re Y_i property exposure curves

The Swiss Re Y_i exposure curves (i = 1 … 4) are very well known and widely used by non proportional property underwriters. As will be shown in this section, all these curves can be approximated very well with the help of a subclass of the MBBEFD exposure curves. In a first step, the parameters b_i and g_i have been evaluated for each curve i. By plotting the points belonging to these pairs of parameters in the {b, g} plane, we found that the points were lying on a smooth curve in the plane. In a next step, this curve was modelled as a function of a single curve parameter c. Finally, the parameters c_i representing the curves Y_i were evaluated.

The subclass of the one-parameter MBBEFD exposure curves is defined as follows:

$$G_c(x) = G_{b_c, g_c}(x) \tag{4.5}$$

with:

$$b_c = b(c) = e^{3.1-0.15(1+c)c}$$
$$g_c = g(c) = e^{(0.78+0.12c)c} \tag{4.6}$$

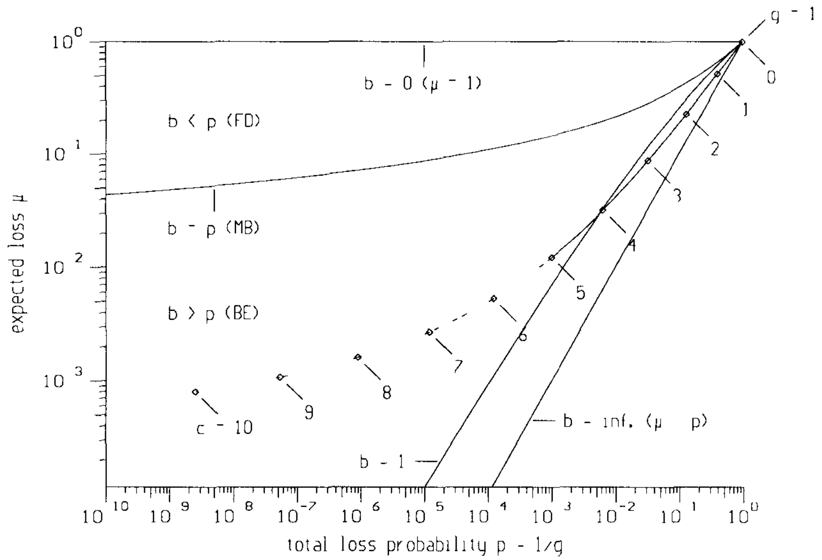

FIGURE 4 1 Range of parameters of the exposure curves $G_{b,g}(x)$ The expected value μ is shown as a function of p = 1/g for special cases b = 0, b = p, b = 1 and b = ∞ In addition, p and μ are shown as a function of the curve parameter c for c = 0 … 10 The dashed part of this curve has no empirical counterparts

The position of the curves c = 0 … 10 in the {p, μ} plane is shown in figure 4.1 Here, the special cases b = 0, p, 1, ∞ and g = 1 are also shown.

The curves defined by c = 0.0, …, 5.0, which are shown in figure 4.2, are related as follows to several exposure curves used in practice:

- The curve c = 0 represents a distribution of total losses only because of g(0) = 1.
- The four curves defined by c = {1.5, 2.0, 3.0 and 4.0} coincide very well with the Swiss Re curves {Y_1, Y_2, Y_3, Y_4}.
- The curve defined by c = 5.0 coincides very well with a Lloyd's curve used for the rating of industrial risks.

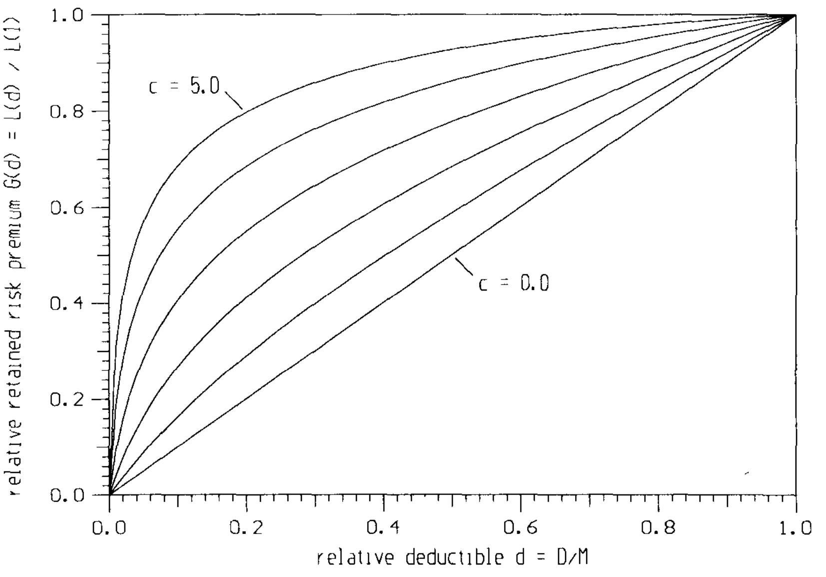

FIGURE 4.2: One-parameter subclass of the MBBEFD exposure curves, shown for c = 0.0, 1.0, 2.0, 3.0, 4.0 and 5.0.

Thus, the exposure curves defined in (4.6) are very well suited for practical purposes. The underwriter can use curve parameters which are very familiar to him. In addition, the class of exposure curves defined by (4.6) is continuous and the underwriter has at his disposal all curves lying between the individual curves Y_i, too.

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