INEQUALITIES FOR CERTAIN CYCLIC SUMS

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1. Introduction

Let M be a positive integer, let $a_1, a_2, \ldots, a_M$ be non-negative reals, and put $a_{M+i} = a_i$ for $i = 1, 2, 3$. Further let each of v_1, v_2, v_3 and $\delta_1, \delta_2, \delta_3$ be 0 or 1, giving 2^6 possibilities. This note is concerned with the problem of finding bounds for each of the non-trivial cases out of the 2^6 cyclic sums

$$S_M = \sum_{i=1}^{M} \frac{v_1 a_{i+1} + v_2 a_{i+2} + v_3 a_{i+3}}{\delta_1 a_{i+1} + \delta_2 a_{i+2} + \delta_3 a_{i+3}}. \tag{1}$$

Of course we do not allow zero denominators. Known results (**1**), (**5**) are

Theorem 1. *If* $\delta_1 + \delta_2 + \delta_3 = 1$ *then* $(v_1 + v_2 + v_3)M \leqq S_M < \infty$.

Theorem 2. *If* $v_1 = \delta_1 = \delta_2 = 1$ *and* $v_3 = 0$ *then*

$$1 + v_2 \leqq S_M \leqq M - 1 + v_2 - \delta_3.$$

To simplify the notation let

$$\Sigma \frac{a+c}{a+b} \quad \text{mean} \quad \sum_{i=1}^{M} \frac{a_{i+1} + a_{i+3}}{a_{i+1} + a_{i+2}}$$

and so on. Then all non-trivial cases of (1), not covered by Theorems 1 and 2, can be dealt with by means of one of the following:

$$0{\cdot}461238M \leqq A = \Sigma \frac{a}{b+c} < \infty, \tag{2}$$

$$M \leqq B = \Sigma \frac{a+b}{b+c} < \infty, \tag{3}$$

$$\min\{2, \tfrac{1}{2}M\} \leqq C = \Sigma \frac{b}{a+c} < \infty, \tag{4}$$

$$1 \leqq D = \Sigma \frac{b}{a+b+c} \leqq [\tfrac{1}{2}M] \text{ for } 3 \leqq M, \tag{5}$$

$$\min\{4, M\} \leqq E = \Sigma \frac{a+b}{a+c} < \infty, \tag{6}$$

$$[\tfrac{1}{2}(M+1)] \leqq F = \Sigma \frac{a+c}{a+b} < \infty. \tag{7}$$

The most famous case of (1) is the sum A in (2) above. Clearly $A_M = \frac{1}{2}M$ when all a_i are equal, so $\inf A_M \leqq \frac{1}{2}M$. It was Shapiro who suggested the verification that $\inf A_M = \frac{1}{2}M$. Diananda (3) proved that

$$\inf A_{M+2} \leqq 1+\inf A_M,$$

and that if M is odd then $\inf A_{M+1} \leqq \frac{1}{2}+\inf A_M$. It has been shown by examples of a_i that $\inf A_M<\frac{1}{2}M$ for $M = 14$ by Lighthill and Zulauf, and for $M = 25$ by the author. On the other hand Nowosad (4) proved that

$$\inf A_M = \tfrac{1}{2}M \text{ for } M = 10.$$

Thus the only cases for which it is still not known whether $\inf A_M<\frac{1}{2}M$ are $M = 12$ and $11 \leqq M$ odd $\leqq 23$. The lower bound in (2) was obtained by Diananda (2), by developing a method of Rankin. However, the infimum of A and F are not known, and they seem to be hard to find. The other bounds given above are best possible, the infinite ones are trivial and will not be mentioned further.

It is easy (1) to prove (3). The four remaining inequalities (4)-(7) appear to be new and will be proved below. It will be shown that $\inf F_M$ behaves in the same way as $\inf A_M$, which we just described. Other new inequalities presented here are

$$-M\varepsilon^{-1} \leqq G_M = \Sigma\left(\frac{a}{a+b}\right)\log\left(\frac{a}{a+b}\right) \leqq 0, \tag{8}$$

$$-M\varepsilon^{-1} \leqq H_M = \Sigma\left(\frac{2a}{a+b}\right)\log\left(\frac{2a}{a+b}\right) \leqq 2(M-1)\log 2, \tag{9}$$

where $\varepsilon>1$ is the base of the logarithms. Again $\inf G_M$ and $\inf H_M$ are not known, but it will be shown that they are both near $-M\varepsilon^{-1}$.

One can fairly easily obtain the corresponding results for $\Sigma t \log t$ when t has any one of the following forms

$$\frac{a}{a+b+c},\quad \frac{3a}{a+b+c},\quad \frac{a+b}{a+b+c},\quad \frac{3(a+b)}{2(a+b+c)},$$
$$\frac{b}{a+c},\quad \frac{2b}{a+c},\quad \frac{b}{a+b+c},\quad \frac{3b}{a+b+c},\quad \frac{a+b}{a+c}.$$

Here are some conjectures:

$$0 \leqq \Sigma\left(\frac{2a}{b+c}\right)\log\left(\frac{2a}{b+c}\right), \tag{10}$$

$$\tfrac{1}{2}M \leqq \Phi(x) = \Sigma\left(\frac{2a}{b+c}\right)^x \text{ provided } a_i>0, \text{ for any real } x, \tag{11}$$

$$0 \leqq \Sigma\left(\frac{a+c}{a+b}\right)\log\left(\frac{a+c}{a+b}\right), \tag{12}$$

$$M \leqq \Sigma\left(\frac{a+c}{a+b}\right)^2. \tag{13}$$

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Conjecture (10) is the logarithmic analogue of Shapiro's conjecture, and the fact that Shapiro's conjecture turned out to be wrong makes it necessary to say something about (10). In (1) the function $\Phi(x)$ in (11) was discussed, and it was shown that $\dfrac{\partial\Phi(0)}{\partial x} \leqq 0$ and $\Phi(0) = M$ and $\Phi(2) \geqq M$. Hence because $\Phi(x)$ is a convex function of x, for any given a_i, the minimum value of $\Phi(x)$ occurs with $0 \leqq x \leqq 2$. In fact computer studies of $\Phi(x)$ indicate that the minimum is near $x = 0$, and they gave rise to conjecture (11). We can get $\Phi(x)$ as close as we like to $\frac{1}{2}M$ for M even by letting $\{a_i\} = \delta, 1, \delta, 1, \ldots$ when δ and x are both small. Since $\Phi(x)$ is convex and $\Phi(0) = M$, if a particular set of a_i were to make (10) false then they would make $\frac{1}{2}\Phi(1) = A_M < \frac{1}{2}M$, and this is impossible for $M \leqq 10$ by Nowosad's theorem, so this proves (10) is true for $M \leqq 10$ and $a_i > 0$. The proof is easier for $a_i \geqq 0$ and it seems that (11) holds also in this case. Replacing A by F in all this gives a corresponding set of conjectures, the most interesting ones perhaps are (12) and (13), both of which have been proved true for $M \leqq 4$ and tested for higher M on a computer. Inequality (6) shows why we don't have something for F corresponding to (11).

2. Proof of Inequality (4)

The cases $M = 1, 2, 4$ are easy, and $\frac{3}{2} \leqq A_3 = C_3$. We now use induction on M. Let $4 < M$ and, without loss of generality, assume a_M has the smallest value out of the a_i. Then

$$0 \leqq C_M - C_{M-1} = C_M(a_1, a_2, \ldots, a_M) - C_{M-1}(a_1, a_2, \ldots, a_{M-1})$$
$$= \left\{\frac{a_{M-1}}{a_{M-2}+a_M} - \frac{a_{M-1}}{a_{M-2}+a_1}\right\} + \left\{\frac{a_M}{a_{M-1}+a_1}\right\} + \left\{\frac{a_1}{a_M+a_2} - \frac{a_1}{a_{M-1}+a_2}\right\}$$

because $0 \leqq \{\ldots\}$ for each of the three brackets. Thus $2 \leqq \inf C_{M-1} \leqq \inf C_M$. To see that 2 is the best lower bound let $\{a_i\} = \lambda^1, \lambda^2, \lambda^3, \ldots, \lambda^{M-1}, \lambda^{M-1}$, where λ is large.

3. Proof of Inequality (5)

First we deal with the left hand bound. To attain it let $a_i = \lambda^i$ for $1 \leqq i \leqq M$ where λ is large. The case $M = 3$ is trivial. That $\inf D_{M-1} \leqq \inf D_M$ when $3 < M$ is proved in exactly the same way as for C_M in section 2 above, and the result follows inductively.

To attain the upper bound $[\frac{1}{2}M]$ let $\{a_i\} = 0, 1, 0, 1, \ldots$. Next note that $1 \leqq$ the sum of any two adjacent terms of D_M. This establishes the inequality for M even. Suppose $5 \leqq M$ odd and write $a, b, c, \ldots$ for $a_1, a_2, a_3, \ldots$. Then

$$\frac{\partial D}{\partial c} = -\frac{b}{(a+b+c)^2} + \frac{b+d}{(b+c+d)^2} - \frac{d}{(c+d+e)^2}$$

$$\begin{cases} \leqq 0 \text{ if } a \leqq d \text{ and } e \leqq b, & \text{(i)} \\ \geqq 0 \text{ if } a \geqq d \text{ and } e \geqq b. & \text{(ii)} \end{cases}$$

If condition (i) holds then D increases as $c \to 0$. When $c = 0$ one term of D is zero, and summing the remaining pairs of adjacent terms shows that $D \leqq [\frac{1}{2}M]$. If condition (ii) holds then D increases as $c \to \infty$, and in the limit we again have at least one zero term in D. The result now follows because one of conditions (i) and (ii) must hold for some renumbering of the a_i's. The renumbering must of course preserve the cycle of the a_i's.

4. Proof of Inequality (6)

We again use induction on M. The cases $M \leqq 4$ are easy, so suppose $5 \leqq M$ and that a_M is the smallest a_i. Then

$$E_M - E_{M-1} = t_1 + t_2 + t_3 - t_4 - t_5$$

where

$$t_1 = \frac{a_{M-2}+a_{M-1}}{a_{M-2}+a_M}, \quad t_2 = \frac{a_{M-1}+a_M}{a_{M-1}+a_1}, \quad t_3 = \frac{a_M+a_1}{a_M+a_2}$$

and

$$t_4 = \frac{a_{M-2}+a_{M-1}}{a_{M-2}+a_1}, \quad t_5 = \frac{a_{M-1}+a_1}{a_{M-1}+a_2}.$$

Now $0 \leqq t_1 - t_4$. If $a_2 \leqq a_1$ then $0 \leqq t_3 - t_5$ and since $0 \leqq t_2$ we have

$$0 \leqq E_M - E_{M-1}.$$

On the other hand if $a_1 < a_2$ then

$$0 \leqq E_M - E_{M-1} = (t_1 - t_4) + \frac{a_M}{a_{M-1}+a_1} + \frac{a_M}{a_M+a_2}$$
$$+ a_{M-1}\left(\frac{1}{a_{M-1}+a_1} - \frac{1}{a_{M-1}+a_2}\right) + a_1\left(\frac{1}{a_M+a_2} - \frac{1}{a_{M-1}+a_2}\right).$$

Thus we have proved $4 \leqq \inf E_{M-1} \leqq \inf E_M$. To attain the bound 4 let $\{a_i\} = \lambda^1, \lambda^2, \lambda^3, \ldots, \lambda^{M-1}, \lambda^{M-1}$ where λ is large.

5. The Sum in (7)

The bound $[\frac{1}{2}(M+1)]$ holds because F has at least one term $\geqq 1$, and $1 \leqq$ the sum of any two adjacent terms of F, as is easily verified. Following Diananda's trick (3) we note that

$$F_{M+2}(a_1, a_2, \ldots, a_M, a_1, a_2) = 2 + F_M(a_1, a_2, \ldots, a_M)$$

and so $\inf F_{M+2} \leqq 2 + \inf F_M$. Also

$$F_{M+1}(a_1, a_2, \ldots, a_{r-1}, a_r, a_r, a_{r+1}, \ldots, a_M) - F_M(a_1, a_2, \ldots, a_M) - 1$$
$$= \frac{(a_{r-1}-a_r)(a_{r+1}-a_r)}{2a_r(a_{r-1}+a_r)}$$

which is $\leqq 0$ for some r provided M is odd. Hence $\inf F_{M+1} \leqq 1 + \inf F_M$

for M odd. Since $\inf F_{RM} \leqq R \inf F_M$ for any positive integer R, it follows that $M^{-1} \inf F_M$ tends to a limit as $M \to \infty$. Now

$$F_4 = (a+c)\left\{\frac{1}{a+b}+\frac{1}{c+d}\right\}+(b+d)\left\{\frac{1}{b+c}+\frac{1}{d+a}\right\}$$

$$\geqq (a+c)\left\{\frac{4}{(a+b)+(c+d)}\right\}+(b+d)\left\{\frac{4}{(b+c)+(d+a)}\right\} = 4,$$

so $M \leqq F_M$ for $1 \leqq M \leqq 4$. However $F_6 = 5{\cdot}99902 < 6$ when

$$\{a_i\} = 381, 0, 334, 29, 340, 49$$

and $F_{13} = 12{\cdot}9623 < 13$ when

$$\{a_i\} = 41, 0, 28, 0, 19, 4, 17, 10, 18, 18, 20, 29, 18.$$

It would be interesting to know if $\inf F_M = M$ for $M = 5, 7, 9, 11$, and it seems that this could be determined by means of Nowosad's technique. The lower bound in (7) does not appear to be best possible. When F_M is as small as possible the $\{a_i\}$ follow a pattern illustrated by the following example which has $M = 110$ and $F_M = 108{\cdot}735$,

$\{a_i\} =$	0	18	0	16	0	14	0	10	2	12	3	12	4	12	5
	13	7	14	9	15	10	16	12	17	14	18	16	21	20	23
	22	26	24	28	27	32	30	36	32	40	36	44	44	52	55
	64	70	84	94	112	126	154	168	216	222	315	278	478	266	683
	0	502	0	376	0	286	0	220	0	170	0	132	0	104	0
	82	0	66	0	56	0	50	0	46	0	42	0	40	0	38
	0	34	0	32	0	30	0	28	0	26	0	24	0	22	0
	21	0	20	0	19.										

If $\{a_i\} = a, b, c, \ldots$ then

$$\frac{\partial F}{\partial c} = \frac{1}{a+b} - \frac{b+d}{(b+c)^2} + \frac{d-e}{(c+d)^2},$$

and when F is at its minimum value this derivative is 0 for $c \neq 0$, and is positive for $c = 0$. Given any positive a_1, a_2, a_3, a_4 we can choose a_5 to make $\dfrac{\partial F}{\partial a_3} = 0$, and then a_6 so that $\dfrac{\partial F}{\partial a_4} = 0$, and so on. In this way we can determine

$$a_5, a_6, \ldots, a_{M+4} \text{ from } a_1, a_2, a_3, a_4,$$

but generally we will not have $a_{M+i} = a_i$ for $i = 1, 2, 3, 4$. However, by iteration we can change a_1, a_2, a_3, a_4 until $a_{M+i} = a_i$ for $i = 1, 2, 3, 4$. The corresponding sum F_M will be a local minimum. Roughly speaking this was the technique used on a computer to find the examples given here. It is much faster than the method described in (1).

6. The Inequalities (8) and (9)

Every term of G_M is non-positive, and if $\{a_i\} = \lambda^i$ then $G_M \to 0$ as $\lambda \to \infty$. This proves the right hand side of (8). Each term of H_M is $\leqq 2 \log 2$, and at least one term is $\leqq 0$, because we must have $a_i \leqq a_{i+1}$ for some i. Thus $H_M \leqq 2(M-1) \log 2$, and to see that this is the best possible bound let $a_i = \lambda^{M-i}$ with λ large. Since $-\varepsilon^{-1} \leqq x \log_\varepsilon x$ for $0 \leqq x$ the left hand sides of (8) and (9) are trivial. It seems that inf $G_M \to -(M-1)\varepsilon^{-1}$ and

$$\inf H_M \to (2 \log_\varepsilon 2) - (M-1)\varepsilon^{-1} \text{ as } M \to \infty.$$

The examples $a_i = (\varepsilon - 1)^{i-1}$ and $a_i = (2\varepsilon - 1)^{i-1}$ respectively support these ideas.

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