SEMI-ALGEBRAS IN $C(T)$

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Let $C(T)$ be the Banach algebra of all complex-valued continuous functions on the compact set $T$ of all complex numbers with modulus one. As usual we may suppose that $A$ is embedded in $C(T)$, where $A$ is the disc algebra, i.e., the algebra of all complex-valued functions $f(\lambda)$ continuous for $|\lambda| \leq 1$ and analytic for $|\lambda| < 1$. We set $M_\lambda = \{f \in A : f(\lambda) = 0\}$ and $M_\lambda^+ = \{f \in A : f(\lambda) \geq 0\}$.

Following Bonsall [1], we call a subset $S$ of $C(T)$ a semi-algebra if, whenever $f, g \in S$ and $t$ is a non-negative number, we have $f + g \in S$, $tg \in S$ and $tf \in S$. In connection with the semi-algebra $S$, we consider the real subalgebra $S_b = S \cap (-S)$ of $C(T)$ and the complex subalgebra $S_c = S_b + iS_b$. It is convenient to let $e = e(\lambda)$ stand for the function identically one. Our theorem shows that all these items are intimately related.

**Theorem 1.** Let $S$ be a semi-algebra in $C(T)$, where $-e \notin S$. Then either $S_c$ is dense in $C(T)$ or no $M_\lambda^+$, with $|\lambda| < 1$, is properly contained in $S$.

**Proof.** Suppose that $S$ properly contains some $M_\lambda^+$, with $|\lambda| < 1$. Without loss of generality, we may take $\lambda = 0$ in the ensuing argument. We must show that $S_c$ is dense in $C(T)$.

Consider the subalgebra

$$B = S_c + Ce,$$  \hspace{1cm} (1)

where $C$ is the field of complex numbers. Since $S_c$ contains the maximal ideal $M_0$ of $A$, we get $B \supseteq A$. Hence, by Wermer's maximality theorem [5], the closure of $B$ is either $C(T)$ or $A$.

If the closure of $B$ is $C(T)$, there exist a sequence $\{p_n(\lambda)\}$ in $S_c$ and a sequence $\{\alpha_n\}$ in $C$ such that, in the metric of $C(T)$, $p_n(\lambda) + \alpha_n e(\lambda) \to \lambda^{-1}$. Notice that the functions $\lambda p_n(\lambda)$ and $\alpha_n \lambda$, as functions of $\lambda$, all lie in $S_c$ and that, in $C(T)$, $\lambda p_n(\lambda) + \alpha_n \lambda \to e(\lambda)$. Therefore, by (1), the closure of $S_c$ is the closure of $B$, which is here $C(T)$.

Our conclusion would then follow if we could show that the closure of $B$ cannot be $A$. Suppose that the closure of $B$ is $A$. By (1) and the fact that $S_c$ contains the maximal ideal $M_0$ of $A$, we see that

$$A = S_c + Ce.$$  \hspace{1cm} (2)

Next we show that $e \notin S_c$. For otherwise we could write $e = f + ig$, where $f$ and $g$ lie in $S_b$. Then we could write

$$-e = f^2 + g^2 - 2f.$$

Since the right side lies in $S_b \subseteq S$, we get a contradiction.

It now follows from (2) that $S_c$ is a proper ideal in $A$ containing $M_0$. Therefore $S_c = M_0$.

Now take $g \in S$. The function $\lambda g(\lambda)$ lies in $S_b \subset M_0$ and is therefore an element of $A$ vanishing.

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at zero. Hence there exists \( w \in A \) such that \( \lambda g(\lambda) = \lambda w(\lambda), \quad |\lambda| = 1 \). Therefore \( g \in A \) and so \( M_0^+ \subset S \subset A \), where \(-e \notin S\).

We shall show from this that \( S = M_0^+ \). For let \( v \in S \). First we show that \( v(0) = -a, a > 0 \), is impossible. For suppose otherwise and set \( w = a^{-1}v \). Now \( M_0 \) is a maximal linear subspace of \( A \); so there is a scalar \( \lambda \) and \( f \in M_0 \) such that \(-e = f + \lambda w \). Evaluating at zero, we see that \( \lambda = 1 \), so that \(-e = f + w \in S \), which is impossible. It follows that \( v(0) = bi, b \) real, \( b \neq 0 \) is impossible, for otherwise \( v^2 \in S \) and \( v^2(0) = -b^2 \). Next we show that \( v(0) = a + bi \) with \( a, b \) real, \( a < 0, b \neq 0 \) is impossible, for otherwise \( w = -ae + v \in S \) and \( w(0) = bi \). Next we rule out \( v(0) = a + bi, a, b \) real, \( a > 0, b \neq 0 \). For if this holds, then \( v(0) \) must lie in the open left-hand plane for some positive integer and \( v \in S \). By elimination we see finally that \( v(0) \geq 0 \) or \( v \in M_0^+ \). Therefore \( S = M_0^+ \).

However this is in conflict with the hypothesis that \( S \) properly contains \( M_0^+ \) and the proof of the theorem is completed.

The choice \( S = A \) shows that the requirement that \(-e \notin S\) cannot be dropped from the hypothesis. Also, \( S_e \) may fail to be dense and, simultaneously, \( S \) can properly contain some \( M_1^+ \), with \( |\lambda| = 1 \). For consider \( g \in C(T), \) where \( g \notin A \) and \( g(1) = 0 \). The semi-algebra \( S \) generated by \( M_1^+ \) and \( g \) properly contains \( M_1^+ \) and fails to contain \(-e \), but has the property that \( S_e \) is at a distance of one from \(-e \).

The following special case of Theorem 1 is, to the author, somewhat surprising.

**Corollary 1.** Let \( g \in C(T), \) where \( g \neq 0 \) and \( g \) vanishes on a subset \( T_0 \) of \( T \) of positive Lebesgue measure. Let \( \lambda \) be a complex number with \( |\lambda| < 1 \). If \( S \) is the semi-algebra generated by \( M_1^+ \) and \( g \), then \( S_e \) is dense in \( C(T) \).

**Proof.** A well-known theorem of F. and M. Riesz [2, p. 50] shows that \( g \notin A \), so that \( S \) properly contains \( M_1^+ \). The conclusion follows from Theorem 1 if we verify that \(-e \notin S_e \).

Suppose that \(-e \in S \). Then there exists a finite subset \( f_0, f_1, \ldots, f_n \) of \( M_1^+ \) such that

\[
-e = f_0 + \sum_{k=1}^{n} f_k g^k. \tag{3}
\]

Notice that, from (3), \( e + f_0 \) is identically zero on \( T_0 \). The F. and M. Riesz theorem then gives \( f_0 = -e \), which is impossible.

For a ring \( R \) with identity 1, Harrison [4] defines a preprime as a nonvoid set closed under addition and multiplication and not containing \(-1 \). He calls a maximal preprime a prime. Civin and White [3, p. 243] showed that, if \( P \) is a closed prime in a Banach algebra \( B \) with identity 1, then \( 1 \in P \) and \( P \) is a semi-algebra. If further, \( B \) is a complex and commutative Banach algebra, then \( iP_b \subset P_b \) [3, Proposition 1.11]. They also point out [3, p. 245] that \( M_1^+ \) with \( |\lambda| < 1 \) is not a prime in \( C(T) \). By using Theorem 1, more can be shown along these lines.

**Corollary 2.** Let \( S \) be closed semi-algebra in \( C(T) \) where \(-e \notin S \) and \( S \) contains some \( M_1^+ \) with \( |\lambda| < 1 \). Then \( S \) is not a prime in \( C(T) \).

**Proof.** Suppose that \( S \) is a prime in \( C(T) \). As noted above, this implies that \( iS_b \subset S_b \). Consequently \( S_e \subset S \), so that \( S_e \) cannot be dense in \( C(T) \). Theorem 1 shows that \( S \) cannot
properly contain any $M_2^+$ with $|\alpha| < 1$. Therefore $S = M_2^+$. But in this situation the proof of Corollary 1 provides the existence of a preprime properly containing $S$. This is a contradiction.

REFERENCES


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