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ON A CLASS OF RADICALS OF RINGS

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Abstract

Let λ be a property that a lattice of submodules of a module may possess and which is preserved under taking sublattices and isomorphic images of such lattices and is satisfied by the lattice of subgroups of the group of integer numbers. For a ring R the lower radical Λ generated by the class $\lambda(R)$ of R-modules whose lattice of submodules possesses the property λ is considered. This radical determines the unique ideal $\Lambda(R)$ of R, called the λ-radical of R. We show that Λ is a Hoehnke radical of rings. Although generally Λ is not a Kurosh-Amitsur radical, it has the ADS-property and the class of Λ-radical rings is closed under extensions. We prove that $\Lambda(M_n(R)) \subseteq M_n(\Lambda(R))$ and give some illustrative examples.

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1. Introduction

Let $\mathscr{C}$ denote the class of lattices of submodules of modules and isomorphic images of such lattices. Let λ be a certain property that an element of $\mathscr{C}$ may possess and which is preserved under isomorphisms of lattices, under taking sublattices (in $\mathscr{C}$) and is satisfied by the lattice of subgroups of the group of integers (for example λ may denote that a lattice is noetherian, distributive or its cardinality is not greater than a given infinite cardinal number). We shall say that the module M is a λ-*module* if the lattice of its submodules possesses the property λ. Let Λ be the lower radical of modules generated by the class of λ-modules (see Section 2 for particularities).

Many authors study modules which are λ-modules for a concrete property λ and, as a special case, rings whose lattice of one-sided ideals possesses the property λ. Sometimes the structural results obtained for such rings carry over to the module radical $\Lambda(R)$ of any ring R considered as a module over itself.

Motivated by the above, in the paper we investigate general properties of the module

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radical Λ restricted to the class of rings. In Section 3 we show that Λ is a Hoehnke radical of rings and call it the *λ-radical of rings*. We prove that Λ is a Kurosh-Amitsur radical only in the trivial case when every ring is Λ-radical. Although generally Λ is not a ring radical in the sense of Kurosh and Amitsur, it has two important properties of such radicals. Namely, we show that Λ has the ADS-property and the class of Λ-radical rings is closed under extensions.

In Section 4 some applications and examples are given. We prove among others that for every ring R with unity (the assumption about unity is essential), $\Lambda(M_n(R)) = M_n(\Lambda(R))$ where $M_n(R)$ is the ring of $n \times n$-matrices over R. Using this and the fact that every finite (even left artinian) ring is Λ-radical, we find that the endomorphism ring of a finitely generated abelian group is Λ-radical. An example is given to show that the result cannot be extended to infinitely generated abelian groups.

Throughout this paper all rings are associative, but do not necessarily have unity, and all modules are left modules. The fundamental properties of radicals may be found in [1, 3 and 5].

2. Preliminaries

In this paper we use the following notation. Let R be a ring. As usual, $I \lhd R (I <_\ell R)$ means that I is an ideal (left ideal) of R. The category of R-modules is denoted by $\mathscr{M}(R)$. If $M \in \mathscr{M}(R)$, then $\mathscr{L}({}_RM)$ stands for the lattice of R-submodules of M. If the additive group M is an S-module for another ring S, then we write $\mathscr{L}({}_RM) \subseteq \mathscr{L}({}_SM)$ to denote that every R-submodule of M is also an S-submodule of M; $\mathscr{L}({}_RM) = \mathscr{L}({}_SM)$ means that $\mathscr{L}({}_RM) \subseteq \mathscr{L}({}_SM)$ and $\mathscr{L}({}_SM) \subseteq \mathscr{L}({}_RM)$.

If $\mathscr{L}_1, \mathscr{L}_2$ are lattices and there exists a lattice monomorphism from $\mathscr{L}_1$ into $\mathscr{L}_2$, then we write $\mathscr{L}_1 \hookrightarrow \mathscr{L}_2$.

DEFINITION. To the end of this paper λ will denote a map defined on the class of rings, which to each ring R assigns a subclass $\lambda(R)$ of $\mathscr{M}(R)$ in such a way that the following conditions are satisfied:

(C1) If $\mathscr{L}({}_RM) \hookrightarrow \mathscr{L}({}_SN)$ and $N \in \lambda(S)$, then $M \in \lambda(R)$,
(C2) $\mathbb{Z} \in \lambda(\mathbb{Z})$, where $\mathbb{Z}$ is the group of integers.

Hence, if $\mathscr{C}$ denotes the class consisting of lattices of submodules of modules and isomorphic images of such lattices, then λ can be thought of as a property that an element $\mathscr{C}$ may possess and which is preserved under taking sublattices (in $\mathscr{C}$) and isomorphic images, and is satisfied by $\mathscr{L}({}_Z\mathbb{Z})$. Then $\lambda(R)$ is simply the class of R-modules whose lattice of submodules possesses the property λ.

It is an immediate consequence of (C1) and (C2) that for every ring R the class $\lambda(R)$ of R-modules is non-empty, homomorphically closed and closed under taking submodules.

Recall that a map $\rho : \mathscr{M}(R) \to \mathscr{M}(R)$ is a *preradical* if for every $M \in \mathscr{M}(R)$, $\rho(M)$ is a submodule of M and $f(\rho(M)) \subseteq \rho(N)$ for each homomorphism $f : M \to N$ in $\mathscr{M}(R)$. A preradical ρ is called a *radical* if $\rho(M/\rho(M)) = 0$ for all $M \in \mathscr{M}(R)$. A preradical is *hereditary* if $\rho(N) = N \cap \rho(M)$ whenever N is a submodule of $M \in \mathscr{M}(R)$.

For a ring R, let $\Lambda^{(R)}$ denote the lower radical on $\mathscr{M}(R)$ generated by $\lambda(R)$. The radical $\Lambda^{(R)}$ can be constructed by transfinite induction in the following way. For a module $M \in \mathscr{M}(R)$, let $\Lambda_0^{(R)}(M)$ denote the zero submodule of M and let $\Lambda_1^{(R)}(M)$ be the sum of all submodules of M which belong to $\lambda(R)$. Since the class $\lambda(R)$ is homomorphically closed, $\Lambda_1^{(R)}(R)$ is a preradical on $\mathscr{M}(R)$. Now, for every ordinal α which is not a limit ordinal, we define $\Lambda_\alpha^{(R)}(M)$ to be the submodule of M such that $\Lambda_\alpha^{(R)}(M)/\Lambda_{\alpha-1}^{(R)}(M) = \Lambda_1^{(R)}(M/\Lambda_{\alpha-1}^{(R)}(M))$. If α is a limit ordinal then let $\Lambda_\alpha^{(R)}(M) = \sum_{\beta<\alpha} \Lambda_\beta^{(R)}(M)$. In this way we obtain an ascending chain of submodules of M,

$$\Lambda_0^{(R)}(M) \subseteq \Lambda_1^{(R)}(M) \subseteq \cdots \subseteq \Lambda_\alpha^{(R)}(M) \subseteq \cdots ,$$

which must terminate, and $\Lambda^{(R)}(M) = \sum_\alpha \Lambda_\alpha^{(R)}(M)$.

PROPOSITION 2.1. *For every ring R, $\Lambda^{(R)}$ is a hereditary radical on $\mathscr{M}(R)$.*

PROOF. It is clear from the construction that $\Lambda^{(R)}$ is a radical on $\mathscr{M}(R)$. To show that $\Lambda^{(R)}$ is hereditary, let $M, N \in \mathscr{M}(R)$ and $N \subseteq M$. By Zorn's lemma there exists a module $K \in \mathscr{M}(R)$, maximal among modules $X \in \mathscr{M}(R)$ with $\Lambda^{(R)}(N) \subseteq X \subseteq M$ and $N \cap X = \Lambda^{(R)}(N)$. Since $(N+K)/K \cong N/(N \cap K) = N/\Lambda^{(R)}(N)$, $\Lambda^{(R)}((N+K)/K) = 0$. Now, since $(N+K)/K$ is an essential submodule of M/K and the class $\lambda(R)$ is closed under taking submodules, we get $\Lambda^{(R)}(M/K) = 0$. Hence $\Lambda^{(R)}(M) \subseteq K$ and consequently $N \cap \Lambda^{(R)}(M) \subseteq N \cap K = \Lambda^{(R)}(N)$. This proves the proposition.

Later on we will need the following property of radicals of modules.

LEMMA 2.2. *Let R be a ring, $M \in \mathscr{M}(R)$ and ρ a radical on $\mathscr{M}(R)$. Suppose $K \subseteq N$ are submodules of M such that $\rho(M/K) = N/K$ and $f \in \mathrm{End}_R M$. Then $f(K) \subseteq \rho(M)$ if and only if $f(N) \subseteq \rho(M)$.*

PROOF. Clearly, $f(N) \subseteq \rho(M)$ implies $f(K) \subseteq \rho(M)$. Suppose now that $f(K) \subseteq \rho(M)$ and consider the map $\bar{f} : M/K \to M/\rho(M)$ given by $\bar{f}(m+K) = f(m) + \rho(M)$. Then $\bar{f}$ is a homomorphism in $\mathscr{M}(R)$, so

$$(f(N) + \rho(M))/\rho(M) = \bar{f}(N/K) = \bar{f}(\rho(M/K)) \subseteq \rho(M/\rho(M)) = 0$$

and we obtain $f(N) \subseteq \rho(M)$.

3. The λ-radical of rings

Let R be a ring and let $\Lambda^{(R)}$ be the radical on $\mathscr{M}(R)$ defined in Section 2. The radical $\Lambda^{(R)}(R)$ of the R-module R will be called the *λ-radical of the ring R* and denoted by $\Lambda(R)$. We will also write simply $\Lambda_\alpha(R)$ instead of $\Lambda_\alpha^{(R)}(R)$.

PROPOSITION 3.1. *Let R be a ring. Then*

(i) *For every ordinal α, $\Lambda_\alpha(R)$ is an ideal of R,*

(ii) *If $f : R \to S$ is a homomorphism of rings, then $f(\Lambda(R)) \subseteq \Lambda(f(R))$,*

(iii) $\Lambda(R/\Lambda(R)) = 0$,

(iv) *If $a \in R$ and α is an ordinal such that $Ra \subseteq \Lambda_\alpha(R)$, then $a \in \Lambda_{\alpha+1}(R)$,*

(v) *Let $I \subseteq J$ be ideals of R such that $\Lambda(R/I) = J/I$. If $a \in R$ with $Ia \subseteq \Lambda(R)$, then $Ja \subseteq \Lambda(R)$.*

PROOF. (i) The case $\alpha = 0$ is obvious. Suppose now that the result is true for all $\beta < \alpha$. If α is not a limit ordinal, then by the induction assumption $\Lambda_{\alpha-1}(R) \lhd R$. Let $\tilde{R} = R/\Lambda_{\alpha-1}(R)$ be the factor ring. Then $\mathscr{L}({}_{\tilde{R}}\tilde{R}) = \mathscr{L}({}_R\tilde{R})$, and so the condition (C1) implies $\Lambda_\alpha(R)/\Lambda_{\alpha-1}(R) = \Lambda_1(\tilde{R})$. Since the class $\lambda(\tilde{R})$ is homomorphically closed, $La \in \lambda(\tilde{R})$ whenever $a \in \tilde{R}$ and $L <_\ell \tilde{R}$ with $L \in \lambda(\tilde{R})$. Hence $\Lambda_1(\tilde{R}) \lhd \tilde{R}$ and consequently $\Lambda_\alpha(R) \lhd R$. The case when α is a limit ordinal is clear.

(ii) Let us observe that $T = f(R)$ is an R-module with the multiplication $rf(a) = f(ra)$ for $r, a \in R$. Since the map $f : R \to T$ is a homomorphism of R-modules and $\Lambda^{(R)}$ is a radical on $\mathscr{M}(R)$ we get $f(\Lambda(R)) \subseteq \Lambda^{(R)}(T)$. Since $\mathscr{L}({}_RT) = \mathscr{L}({}_TT)$, the condition (C1) implies $\Lambda^{(R)}(T) = \Lambda(T)$ and consequently $f(\Lambda(R)) \subseteq \Lambda(T)$.

(iii) The statement (iii) is obvious.

(iv) Suppose that $Ra \subseteq \Lambda_\alpha(R)$. By (i), $\Lambda_\alpha(R) \lhd R$, so we can consider the factor ring $\tilde{R} = R/\Lambda_\alpha(R)$. Let M be the R-submodule of $\tilde{R}$ generated by $a + \Lambda_\alpha(R)$. Since $R \cdot M = 0$, $\mathscr{L}({}_RM) = \mathscr{L}({}_{\mathbb{Z}}M)$ and ${}_{\mathbb{Z}}M$ is a homomorphic image of $\mathbb{Z}$. By (C2) we have $\mathbb{Z} \in \lambda(\mathbb{Z})$ and thus ${}_{\mathbb{Z}}M \in \lambda(\mathbb{Z})$. Since $\mathscr{L}({}_RM) = \mathscr{L}({}_{\mathbb{Z}}M)$, (C1) implies ${}_RM \in \lambda(R)$. Consequently $M = \Lambda_1^{(R)}(M)$, and thus $a + \Lambda_\alpha(R) \in M = \Lambda_1^{(R)}(M) \subseteq \Lambda_1^{(R)}(\tilde{R}) = \Lambda_{\alpha+1}(R)/\Lambda_\alpha(R)$. Hence $a \in \Lambda_{\alpha+1}(R)$.

(v) For the ring $\tilde{R} = R/I$ we have $\mathscr{L}({}_{\tilde{R}}\tilde{R}) = \mathscr{L}({}_R\tilde{R})$, and so (C1) implies $\Lambda(\tilde{R}) = \Lambda^{(R)}(\tilde{R})$. Now Lemma 2.2 gives the result.

Recall that a *Hoehnke radical of rings* [3] is a mapping σ assigning to each ring R a uniquely determined ideal $\sigma(R)$ of R such that $\sigma(R/\sigma(R)) = 0$ and $f(\sigma(R)) \subseteq \sigma(f(R))$ for every ring homomorphism f defined on R. If furthermore $\sigma(\sigma(R)) = \sigma(R)$ and $I \subseteq \sigma(R)$ whenever $\sigma(I) = I \lhd R$, then σ is called a *Kurosh-Amitsur radical of rings*.

Statements (ii) and (iii) of Proposition 3.1 show that the map Λ assigning to each ring R the ideal $\Lambda(R)$ of R is a Hoehnke radical in the class of rings. In general, Λ

is not a radical of rings in the sense of Kurosh and Amitsur: this is a consequence of the following

PROPOSITION 3.2. *Λ is a Kurosh-Amitsur radical of rings if and only if $\Lambda(R) = R$ for every ring R.*

PROOF. Suppose Λ is a Kurosh-Amitsur radical of rings and $\Lambda(R) \neq R$ for some ring R. Put $S = R/\Lambda(R)$, then $S \neq 0$ and $\Lambda(S) = 0$. Now we form the matrix ring $T = \begin{pmatrix} S & S \\ 0 & 0 \end{pmatrix}$. Since $I = \begin{pmatrix} 0 & S \\ 0 & 0 \end{pmatrix}$ is an ideal of T with $T/I \cong S$, $\Lambda(T/I) = 0$ and so $\Lambda(T) \subseteq I$. By Proposition 2.1 the radical $\Lambda^{(T)}$ is hereditary and thus $\Lambda(T) = \Lambda^{(T)}(\Lambda(T)) \subseteq \Lambda^{(T)}(I)$. Let us observe that I is an S-module with multiplication $sy = \begin{pmatrix} s & 0 \\ 0 & 0 \end{pmatrix} y$ for $s \in S$, $y \in I$. Since $\mathscr{L}(_S I) = \mathscr{L}(_T I)$, (C1) implies $\Lambda^{(S)}(I) = \Lambda^{(T)}(I)$. Moreover, the S-modules I and S are isomorphic and $\Lambda(S) = 0$, and thus $\Lambda^{(S)}(I) = 0$. Consequently $\Lambda(T) = 0$. But $I^2 = 0$ and so $\Lambda(I) = I$ by Proposition 3.1(iv). Since Λ is a Kurosh-Amitsur radical, $0 \neq I \subseteq \Lambda(T) = 0$, a contradiction.

A ring R and a λ-radical Λ with $\Lambda(R) \neq R$ are presented in Example 4.5.
Although Λ is not a Kurosh-Amitsur radical of rings, it has the ADS-property:

THEOREM 3.3. *If R is a ring and I is an ideal of R, then $\Lambda(I)$ is an ideal of R.*

PROOF. Clearly $\Lambda(I)$ is a subgroup of the additive group R. Now let $a \in R$ and let $f : I \to I$ be the map defined by $f(x) = xa$ for $x \in I$. Since $f \in \mathrm{End}_I I$ and $\Lambda^{(I)}$ is a radical on $\mathscr{M}(I)$, we have $\Lambda(I)a = \Lambda^{(I)}(I)a = f(\Lambda^{(I)}(I)) \subseteq \Lambda^{(I)}(I) = \Lambda(I)$. Hence $\Lambda(I)$ is a right ideal of R. To show that $\Lambda(I)$ is also a left ideal of R, put $K = a\Lambda(I) + \Lambda(I)$. Then in the ring $I/\Lambda(I)$ we have $I/\Lambda(I) \cdot K/\Lambda(I) = 0$. Hence Proposition 3.1(iv) implies $K/\Lambda(I) \subseteq \Lambda(I/\Lambda(I)) = 0$. Thus $K \subseteq \Lambda(I)$ and consequently $a\Lambda(I) \subseteq \Lambda(I)$, which ends the proof.

LEMMA 3.4. *Let R be a ring and let I be an ideal of R. If $\Lambda(R/I) = R/I$, then $\Lambda(I) \subseteq \Lambda(R)$.*

PROOF. We will show inductively that $\Lambda_\alpha(I) \subseteq \Lambda(R)$ for every ordinal α. The only case where proof is required is when $\alpha = \beta + 1$ for some ordinal β and $\Lambda_\beta(I) \subseteq \Lambda(R)$. Put $J = \Lambda_\beta(I)$ and let L be a left ideal of I such that $J \subseteq L$ and $L/J \in \lambda(I)$ (observe that $\Lambda_\alpha(I)$ is the sum of such L's). Since $(IL + J)/J$ is an I-submodule of I/J contained in $L/J \in \lambda(I)$ and $\lambda(I)$ is closed under submodules, $(IL + J)/J \in \lambda(I)$. By the induction assumption $J \subseteq \Lambda(R)$ and so $(IL + J)/J$ can be homomorphically mapped onto the I-module $(IL + \Lambda(R))/\Lambda(R)$. Since the class

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$\lambda(I)$ is homomorphically closed, we get $(IL+\Lambda(R))/\Lambda(R) \in \lambda(I)$. Now $\mathscr{L}({}_R(IL+\Lambda(R))/\Lambda(R)) \subseteq \mathscr{L}({}_I(IL+\Lambda(R))/\Lambda(R))$ and (C1) give $(IL+\Lambda(R))/\Lambda(R) \in \lambda(R)$. Hence $(IL+\Lambda(R))/\Lambda(R) \subseteq \Lambda(R/\Lambda(R)) = 0$ and thus $IL \subseteq \Lambda(R)$. The assumption $\Lambda(R/I) = R/I$ and Proposition 3.1(v) imply $RL \subseteq \Lambda(R)$, so by Proposition 3.1(iv) $L \subseteq \Lambda(R)$. Since $\Lambda_\alpha(I)$ is the sum of such L's we obtain $\Lambda_\alpha(I) \subseteq \Lambda(R)$.

Now we prove that the class of Λ-radical rings is closed under extensions:

THEOREM 3.5. *Let R be a ring and I be an ideal of R. If $\Lambda(I) = I$ and $\Lambda(R/I) = R/I$, then $\Lambda(R) = R$.*

PROOF. Since $\Lambda(R/I) = R/I$, Lemma 3.4 implies $I = \Lambda(I) \subseteq \Lambda(R)$. Let a be an arbitrary element of R. Then $Ia \subseteq I \subseteq \Lambda(R)$ and so $Ra \subseteq \Lambda(R)$ by Proposition 3.1(v). Now Proposition 3.1(iv) gives $a \in \Lambda(R)$ and consequently $R \subseteq \Lambda(R)$. The proof is completed.

4. Examples and the λ-radical of matrix rings

We start this section with some examples of λ-radicals.

EXAMPLE 4.1. Let δ be the map which to each ring R assigns the class $\delta(R)$ of distributive R-modules, that is $M \in \delta(R)$ if and only if $\mathscr{L}({}_RM)$ is distributive. Obviously, δ satisfies the conditions (C1) and (C2). The δ-radical of rings is called the *distributive radical* and denoted by Δ (some results of Section 3 extend those obtained earlier for the distributive radical; see [4]).

EXAMPLE 4.2. Let R be a ring and let

$\nu(R)$ denote the class of noetherian R-modules,

$\alpha(R)$ be the class of such R-modules M that M/N is artinian for any non-zero submodule N of M,

$\gamma_\kappa(R)$ be the class of such R-modules M that $\mathscr{L}({}_RM)$ has cardinality equal to or smaller than a given infinite cardinal number κ.

$\tau(R)$ be the class of all R-modules, that is $\tau(R) = \mathscr{M}(R)$.

Then the maps $\nu, \alpha, \gamma_\kappa, \tau$ satisfy (C1) and (C2).

To get other examples, let us observe that if λ_1, λ_2 are maps satisfying (C1) and (C2), and we define

$$(\lambda_1 \vee \lambda_2)(R) = \lambda_1(R) \cup \lambda_2(R), \quad (\lambda_1 \wedge \lambda_2)(R) = \lambda_1(R) \cap \lambda_2(R),$$

then also $\lambda_1 \vee \lambda_2$ and $\lambda_1 \wedge \lambda_2$ satisfy (C1) and (C2).

Now we pass to the λ-radical of matrix rings. Let R be a ring and n a positive integer number. Then the $n \times n$ matrix ring over R is denoted by $M_n(R)$. We introduce the following symbols. If $1 \leq k \leq n$ and $a = (a_{ij}) \in M_n(R)$, then $\pi_k(a) \in M_n(R)$ is the matrix whose kth column is the same as a and zeros elsewhere, that is $\pi_k(a) = (b_{ij})$, where $b_{ij} = a_{ij}$ if $j = k$, and $b_{ij} = 0$ otherwise. For a left ideal L of R we define $\mu_k(L) = \{\pi_k(a) | a \in M_n(L)\}$, that is $(b_{ij}) \in \mu_k(L)$ if and only if $b_{ij} = 0$ for $j \neq k$ and $b_{ik} \in L$. It is easy to verify that if L is a left ideal of R and I is a left ideal of $M_n(R)$, then $\mu_k(L)$ and $\pi_k(I) = \{\pi_k(a) | a \in I\}$ are left ideals of $M_n(R)$, both contained in $\mu_k(R)$. Moreover, we have the following

LEMMA 4.3. *Let I be a left ideal of $M_n(R)$ and let $1 \leq k \leq n$.*

(i) *If $I \in \lambda(M_n(R))$, then $\pi_k(I) \in \lambda(M_n(R))$.*

(ii) *There exists a left ideal L of R such that $\mu_k(RL) \subseteq \pi_k(I) \subseteq \mu_k(L)$.*

Furthermore, if $I \in \lambda(M_n(R))$, then $RL \in \lambda(R)$.

PROOF. (i) Since $\pi_k \in \operatorname{End}_{M_n(R)} M_n(R)$ and the class $\lambda(M_n(R))$ is homomorphically closed, $I \in \lambda(M_n(R))$ implies $\pi_k(I) \in \lambda(M_n(R))$.

(ii) For $1 \leq l \leq n$, let $L_l = \{x \in R \mid \text{there exists } (a_{ij}) \in I \text{ with } a_{lk} = x\}$. Then L_l is a left ideal of R and $L = L_1 + \cdots + L_n$ satisfies $\mu_k(RL) \subseteq \pi_k(I) \subseteq \mu_k(L)$. To prove the second part of (ii), let $I \in \lambda(M_n(R))$. Then by (i) also $\pi_k(I) \in \lambda(M_n(R))$. Since $\mu_k(RL) \subseteq \pi_k(I)$ and the class $\lambda(M_n(R))$ is closed under submodules, $\mu_k(RL) \in \lambda(M_n(R))$. Since μ_k is a lattice monomorphism from $\mathscr{L}({}_R RL)$ into $\mathscr{L}({}_{M_n(R)}\mu_k(RL))$, (C1) gives $RL \in \lambda(R)$.

We have the following

PROPOSITION 4.4. *For every ring R, $\Lambda(M_n(R)) \subseteq M_n(\Lambda(R))$.*

PROOF. We claim that $\Lambda(R) = 0$ implies $\Lambda(M_n(R)) = 0$. For, let I be a left ideal of $M_n(R)$ with $I \in \lambda(M_n(R))$. If $1 \leq k \leq n$, then by Lemma 4.3(ii) $\mu_k(RL) \subseteq \pi_k(I) \subseteq \mu_k(L)$ for some $L <_l R$ with $RL \in \lambda(R)$. But $\Lambda(R) = 0$ and hence $RL = 0$. Consequently Proposition 3.1(iv) gives $L \subseteq \Lambda(R) = 0$. Thus $\pi_k(I) \subseteq \mu_k(L) = 0$. Since $I \subseteq \pi_1(I) + \cdots + \pi_n(I)$, we get $I = 0$. Our claim is proved.

We finish the proof as follows. Since $\Lambda(R/\Lambda(R)) = 0$, the preceding paragraph implies $\Lambda(M_n(R/\Lambda(R))) = 0$. Using proposition 3.1(ii) and the ring isomorphism $M_n(R/\Lambda(R)) \cong M_n(R)/M_n(\Lambda(R))$, we get $\Lambda(M_n(R)/M_n(\Lambda(R))) = 0$. Hence $\Lambda(M_n(R)) \subseteq M_n(\Lambda(R))$ and the proof is completed.

The following example shows that generally the inclusion in Proposition 4.4 is strict.

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EXAMPLE 4.5. Let R be the ring of even integer numbers, that is $R = 2\mathbb{Z}$ and let Δ be the distributive radical defined in Example 4.1. Since the group of integers has a distributive lattice of subgroups, $\Delta(R) = R$.

We will show that $\Delta(M_2(R)) = 0$. For, let I be a left ideal of $M_2(R)$ with $I \in \delta(M_2(R))$. Then by Lemma 4.3 $\pi_1(I)$ is a left ideal of $M_2(R)$ with $\pi_1(I) \in \delta(M_2(R))$ and there exists $L <_l R$ such that $\mu_1(RL) \subseteq \pi_1(I) \subseteq \mu_1(L)$. If we denote

$$A = \begin{pmatrix} RL & 0 \\ R^2L & 0 \end{pmatrix}, \qquad B = \begin{pmatrix} R^2L & 0 \\ RL & 0 \end{pmatrix},$$

$$C = \left\{ \begin{pmatrix} x+a & 0 \\ x+b & 0 \end{pmatrix} | x \in RL; a, b \in R^2L \right\},$$

then A, B, C are left ideals of $M_2(R)$, all contained in $\pi_1(I)$. Now, if we would have $L \neq 0$, then $R^2L \neq RL$ and

$$(A+B) \cap C = C \neq \begin{pmatrix} R^2L & 0 \\ R^2L & 0 \end{pmatrix} = (A \cap C) + (B \cap C),$$

a contradiction. Thus $L = 0$ and consequently $\pi_1(I) = 0$. Similarly we get $\pi_2(I) = 0$, and so $I \subseteq \pi_1(I) + \pi_2(I) = 0$. Hence $\Delta(M_2(R)) = 0 \neq M_2(R) = M_2(\Delta(R))$.

For every ring R with unity we have $\Lambda(M_n(R)) = M_n(\Lambda(R))$. It is a consequence of the following

PROPOSITION 4.6. *Let R be a ring. If $RL = L$ for every left ideal L of R, then* $\Lambda(M_n(R)) = M_n(\Lambda(R))$.

PROOF. Let R^1 denote the natural extension of R to a ring with unity (see [1, p. 122]). Then $M_n(R)$ is an ideal of $M_n(R^1)$, so by Theorem 3.3, $\Lambda(M_n(R))$ is an ideal of $M_n(R^1)$. Since R^1 is a ring with unity, any ideal of $M_n(R^1)$ is of the form $M_n(I)$ for some ideal I or R^1. Since moreover $\Lambda(M_n(R)) \subseteq M_n(R)$, we get $\Lambda(M_n(R)) = M_n(I)$ where I is an ideal of R. Let $\tilde{R} = R/I$. Since $M_n(\tilde{R}) \cong M_n(R)/\Lambda(M_n(R))$, it follows that $\Lambda(M_n(\tilde{R})) = 0$. Consequently, since $\mu_1(\tilde{R})$ is a left ideal of $M_n(\tilde{R})$, $\Lambda^{(M_n(\tilde{R}))}(\mu_1(\tilde{R})) \subseteq \Lambda(M_n(\tilde{R})) = 0$. By the assumption about R we obtain $\tilde{R}\tilde{L} = \tilde{L}$ for every left ideal $\tilde{L}$ of $\tilde{R}$. Hence by Lemma 4.3 the map μ_1 is an isomorphism of the lattices $\mathscr{L}({}_{\tilde{R}}\tilde{R})$ and $\mathscr{L}({}_{M_n(\tilde{R})}\mu_1(\tilde{R}))$. Therefore, since $\Lambda^{(M_n(\tilde{R}))}(\mu_1(\tilde{R})) = 0$, we get $\Lambda(\tilde{R}) = \Lambda(R/I) = 0$ and so $\Lambda(R) \subseteq I$. Thus $M_n(\Lambda(R)) \subseteq M_n(I) = \Lambda(M_n(R))$. The opposite inclusion holds in view of Proposition 4.4.

PROPOSITION 4.7. *If G is a finitely generated abelian group, then the ring* End G *is Λ-radical.*

PROOF. G is of the form $G = A \oplus B$, where A is a finite abelian group and B is a finitely generated torsion-free abelian group (the component B may be absent). Since $I = \{\phi \in \operatorname{End} G \mid \phi(G) \subseteq A\}$ is an ideal of $\operatorname{End} G$ and $\operatorname{End} G/I \cong \operatorname{End} B \cong M_k(\mathbb{Z})$ for some positive integer k, $\operatorname{End} G/I$ is Λ-radical by Proposition 4.6. Since I is finite, also I is Λ-radical. Now Theorem 3.5 implies that $\operatorname{End} G$ is Λ-radical.

The following example shows that Proposition 4.7 cannot be extended to infinitely generated abelian groups.

EXAMPLE 4.8. Let Δ be the distributive radical defined in Example 4.1 and let $R = \mathbb{Z}[X]$. Then R is a countable torsion-free ring and the additive group of R has no non-zero divisible subgroups. Thus by [2, Theorem 110.1], R is a ring of endomorphisms of an abelian group. However R is not a Δ-radical ring. Indeed, suppose I is a non-zero ideal of R with $I \in \delta(R)$ and let $0 \neq a \in I$. Since R and Ra are isomorphic R-modules and $Ra \in \delta(R)$, we get $R \in \delta(R)$. But the lattice of ideals of R is not distributive in view of [6, Theorem 3].

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