THE ITERATED PROJECTION SOLUTION FOR THE FREDHOLM INTEGRAL EQUATION OF SECOND KIND

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Abstract

We are concerned with the solution of the second kind Fredholm equation (and eigenvalue problem) by a projection method, where the projection is either an orthogonal projection on a set of piecewise polynomials or an interpolatory projection at the Gauss points of subintervals.

We study these cases of superconvergence of the Sloan iterated solution: global superconvergence for a smooth kernel, and superconvergence at the partition points for a kernel of "Green's function" type. The mathematical analysis applies for the solution of the inhomogeneous equation as well as for an eigenvector.

1. Introduction

We consider some projection methods for the solution of second kind integral equations of the form

\[(Tx)(s) - zx(s) = f(s), \quad 0 < s < 1, \tag{1}\]

where \(T\) is the operator defined by

\[x(s) \mapsto \int_0^1 k(s, t)x(t) \, dt, \quad 0 < s < 1.\]

Along with (1), we consider the eigenvalue problem

\[(T\phi)(s) = \lambda \phi(s), \quad 0 < s < 1, \phi \neq 0. \tag{2}\]

(1) and (2) are regarded as equations in an appropriate subspace \(X\) of the complex Banach space \(L^\infty(0, 1)\) with the norm \(\| \cdot \|_\infty\). \(T\) is supposed to be
compact and \( z \in \rho(T) \), the resolvent set of \( T \), so that \((T - z)^{-1}\) is bounded with domain \( X \). Let \( X_n \) be a finite dimensional subspace of \( X \) and let \( \Pi_n \) be a projection onto \( X_n \). Then the projection method consists in approximating (1) and (2) respectively by

\[
\begin{align*}
(\Pi_n T - z) x_n &= \Pi_n f, \quad x_n \in X_n, \\
\Pi_n T \phi_n &= \lambda_n \phi_n, \quad 0 \neq \phi_n \in X_n,
\end{align*}
\]

where \( x_n \) (resp. \( \phi_n \)) is the projection solution (resp. eigenvector), corresponding to the approximation \( T^p_n = \Pi_n T \) of \( T \) (\( P \) for projection).

Given a projection \( \Delta = (t_i)_{i=0}^n \) of \([0, 1]\), \( t_0 = 0, t_n = 1 \), let \( X_n \) be a space \( S_\Delta \) of piecewise polynomials of degree \( < r \) on each subinterval \( \Delta_i = [t_{i-1}, t_i] \), \( i = 1, \ldots, n \). We set \( h = \max_{1 \leq i \leq n} (t_i - t_{i-1}) \). We shall consider two types of projection methods:

(a) \( \Pi_n \) is the orthogonal projection (in \( L^2(0, 1) \)) on \( S_\Delta \).

(b) \( \Pi_n \) is an interpolatory projection defined so that \( \Pi_n x \) is the piecewise polynomial of degree \( < r \) which interpolates \( x \) at \( r + 1 \) points \( \{t_i^j\}_{j=1}^{r+1} \), on each \( \Delta_i, i = 1, \ldots, n \).

Case (a) corresponds to a Galerkin method, and case (b) to a collocation method at the collocation points \( \{t_i^j\} \).

If \( z \neq 0 \) (resp. \( \lambda_n \neq 0 \)) we consider the iterated projection solution \( \tilde{x}_n \) (resp. eigenvector \( \tilde{\phi}_n \)) introduced by Sloan [13], [14] and given by the formulae:

\[
\begin{align*}
\tilde{x}_n &= \frac{1}{z} (T x_n - f), \\
\tilde{\phi}_n &= \frac{1}{\lambda_n} T \phi_n,
\end{align*}
\]

where \( \tilde{x}_n \) and \( \tilde{\phi}_n \) are solutions of the equations

\[
(T \Pi_n - z) \tilde{x}_n = f, \quad T \Pi_n \tilde{\phi}_n = \lambda_n \tilde{\phi}_n,
\]

and

\[
(1, 0)
\]

corresponding to the approximation \( T^S_n = T \Pi_n \) of \( T \) (\( S \) for Sloan). Now \( \Pi_n \tilde{x}_n = x_n \) and \( \Pi_n \tilde{\phi}_n = \phi_n \), so that in case (b), the iterated solutions and the solutions themselves agree at the collocation points.

If \( k \) and \( f \) are smooth enough, it is known that \( \| x_n - x \|_\infty = O(h^{r+1}) \), while \( \| \tilde{x}_n - x \|_\infty = O(h^{2r+2}) \) for case (b) for example, provided that the \( \{t_i^j\} \) are the \( r + 1 \) Gauss points on \( \Delta_i \), \( i = 1, \ldots, n \). The optimal rate of convergence, relative to \( S_\Delta \), which is \( \inf_{y \in S_\Delta} \| x - y \|_\infty = O(h^{r+1}) \), is then overshot by \( \tilde{x}_n \notin S_\Delta \), when \( k \) and \( f \) are smooth. Such fast convergence is often called superconvergence.

When \( k \) is the Green’s function of an ordinary differential equation (o.d.e.) of order \( p \) with smooth coefficients, \( \tilde{x}_n \) is still superconvergent at the partition points \( \{t_i\}_{i=0}^n \), but not globally: the global rate of convergence is now \( O(h^{r+1+p}) \).
Similar results hold for \( \tilde{\phi}_n \). This problem is studied for the equation (1) and the Galerkin method in Chandler’s thesis [6]. The collocation method for a nonlinear o.d.e. has been looked at by de Boor-Swartz (see [1] for the solution of (1), and [2], [3] for the linear eigenvalue problem (2)), where \( T \) is the associated differential operator. In de Boor-Swartz [4] the “essential” least squares method (or local moment method) for an o.d.e. is also studied.

We present in this paper an analysis of the convergence rates which is a blend of the techniques of Chandler and of de Boor-Swartz. It applies for the iterated solution \( \tilde{x}_n \) as well as for the iterated eigenvector \( \tilde{\phi}_n \) (the result seems to be new for the eigenvector in the most general case). It is based on a study of the error at the point \( t \) of \([0, 1]\) in terms of the scalar product \( \langle l'_i, (1 - \Pi_n)\tilde{x}_n \rangle \) (resp. \( \langle l'_i, (1 - \Pi_n)\tilde{\phi}_n \rangle \)) where \( l_i \) (resp. \( l'_i \)) is a function having the same smoothness properties as \( k_i(\cdot) := k(t, \cdot) \), and where \( \langle f, g \rangle = \int_0^1 fg \).

In case (a), we use the orthogonality of \( \Pi_n \):

\[
\langle l'_i, (1 - \Pi_n)\tilde{x}_n \rangle = \langle (1 - \Pi_n)l_i, (1 - \Pi_n)\tilde{x}_n \rangle.
\]

In case (b) we use firstly that the function \((1 - \Pi_n)\tilde{x}_n\) vanishes at the collocation points \( \tau_j^i \), and secondly that the \( \{\tau_j^i\} \) being the \( r + 1 \) Gauss points in \( \Delta_i \), then

\[
\int_{\Delta_i} p(s)\pi_{j+1}^i(s - \tau_j^i) \, ds = 0 \text{ for all polynomials } p \text{ of degree } < r.
\]

The superconvergence in case (a) is proved under the assumption that \( \Delta \) is quasi-uniform. In case (b), \( \Delta \) is arbitrary but more smoothness properties are required for \( k \) and \( f \).

2. The setting of the problem

2.1. Piecewise continuous functions

Let be given \( \Delta = \{t_j\}_0^n \), a strict partition of \([0, 1]\), \( 0 = t_0 < t_1 < \cdots < t_n = 1 \).

It is quasi-uniform if there exists \( \sigma > 0 \): \( \max(t_i - t_{i-1})/\min(t_i - t_{i-1}) < \sigma \) for \( n = 1, 2, \ldots \). Then \( nh \leq \sigma \). \( \Delta_i := [t_{i-1}, t_i], \; i = 1, 2, \ldots, n \). We define \( C_\Delta := \prod_{i=1}^n C_{\Delta_i} \); \( f \in C_\Delta \) consists of \( n \) components \( f_i \in C_{\Delta_i} \), \( f \) is a piecewise continuous function having (possibly) different left and right values at the partition points \( t_i \). With the norm \( \| \cdot \|_\Delta \) defined by \( \| f \|_\Delta = \max_{i=1, \ldots, n} \| f_i \|_\infty \), \( C_\Delta \) is a Banach space. \( C_\Delta \subset L^\infty(0, 1) \) by \( \| f \|_\infty \leq \| f \|_\Delta \) and if \( f \) is continuous on \([0, 1]\), then \( \| f \|_\infty = \| f \|_\Delta \). We define, more generally, \( C^l_\Delta \) for positive integer \( l \) by \( C^l_\Delta := \prod_{i=1}^n C^l_{\Delta_i} \) where \( f_i \in C^l_{\Delta_i} \) iff its \( l \)th derivative \( f_i^{(l)} \) is continuous on \( \Delta_i \). Clearly \( S_\Delta \subset C_\Delta \) and the projection \( \Pi_n \) is defined \( C_\Delta \rightarrow S_\Delta \) with \( f = (f_1, \ldots, f_n) \mapsto \Pi_n f = (\Pi f_1, \ldots, \Pi f_n) \), where \( \Pi f_i \) is the projection of \( f_i \in C_{\Delta_i} \) on the polynomials of degree \(< r \) on \( \Delta_i \).
2.2. Spectral definitions

$T$ is supposed to be compact in the complex Banach space $X = C_{a^*}$. $L(X)$ is the algebra of bounded operators on $X$. The resolvent set of $T$ is $\rho(T) = \{z \in \mathbb{C}; (T - z)^{-1} \in L(X)\}$ where $z$ stands for $z_1$. For $z$ in $\rho(T)$, $R(z) = (T - z)^{-1}$ is the resolvent of $T$ and $TR(z) = R(z)T$. The unique solution of (1) is then $x = R(z)f$.

Let $\lambda \neq 0$ be an isolated eigenvalue of $T$ with algebraic (resp. geometric) multiplicity $m$ (resp. $g$), and ascent $\mu$, $1 < \mu < m$, $1 < g < m$. The associated eigenspace is $E = \ker(T - \lambda)$, the null space of $T - \lambda$ so $\dim E = g$; the invariant subspace is

$$M = \ker(T - \lambda)^m, \quad \dim M = m, \quad \text{and} \quad \ker(T - \lambda)^\mu \equiv \ker(T - \lambda)^m.$$

Let $\Gamma$ be a Jordan curve in $\rho(T)$, around $\lambda$, which contains neither 0 nor any other eigenvalue of $T$. $P := -1/2 \pi i \int_{\Gamma} R(z) \, dz$ is the spectral projection associated with $\lambda$. $M = PX$. Let $T_n$ be a sequence of operators in $L(X)$ such that $T_n$ converge to $T$ pointwise. $T_n$ will be either $T_n^p = \Pi_n T$ or $T_n^S = T \Pi_n$. If $\Gamma \subset \rho(T_n)$, we may define for $T_n$ the resolvent $R_n(z)$ for $z \in \Gamma$ and the spectral projection $P_n := -1/2 \pi i \int_{\Gamma} R_n(z) \, dz$. If $T_n$ is strongly stable inside $\Gamma$ (Chatelin [8], [9]), there are, for $n$ large enough, exactly $m$ eigenvalues $\{\lambda_{n,i}\}_{i=1}^m$ of $T_n$ inside $\Gamma$ (counting their algebraic multiplicities), $\hat{\lambda}_n$ is their arithmetic mean, and $\lambda_n$ is any one of them.

For the projections $\Pi_n$ under consideration, both $T_n^p$ and $T_n^S$ are strongly stable around any non-zero eigenvalue of $T$ (Chatelin [7], [9]). The solution $x_n$ of (3) is such that $x_n = R_n^p(z) \Pi_n f$, and $\tilde{x}_n = R_n^S(z)f$. Similarly $\phi_n$ is an eigenvector of $T_n^p$ and $\bar{\phi}_n$ of $T_n^S$, associated with the same eigenvalue $\lambda_n$.

2.3. The errors $x_n - x$, $\phi_n - P\phi_n$, $\tilde{x}_n - x$, $\bar{\phi}_n - P\bar{\phi}_n$ and $\lambda - \hat{\lambda}_n$

$C$ is a generic constant, which may depend on $r$ and $\sigma$, but is otherwise independent of $\Delta$.

2.3.1. The projection method

We recall the following equality:

$$x_n - x = z R_n^p(z)(1 - \Pi_n)x, \quad \text{then} \quad \|x - x_n\|_\infty \leq C \|(1 - \Pi_n)x\|_\infty.$$

As for the resolvents,

$$(R_n^p(z) - R(z))\phi_n = R(z)(T - T_n^p)R_n^p(z)\phi_n = \frac{R(z)}{\lambda_n - z}(1 - \Pi_n)T\phi_n,$$
because \( R_n^\phi(z)\phi_n = \phi_n/\lambda_n - z \). To integrate on \( \Gamma \), we distinguish whether \( \lambda_n = \lambda \) or not. If \( \lambda_n = \lambda \), then \(-1/2i\pi \int_\Gamma (R(z)/\lambda - z)\,dz = S = \lim_{z \to \lambda} R(z)(1 - P); \)
\( S \) is the reduced resolvent with respect to \( \lambda \). If \( \lambda_n \neq \lambda \), \( R(z) - R(\lambda_n) = (z - \lambda_n)R(\lambda_n)R(z), \) and
\[ \frac{-1}{2i\pi} \int_\Gamma \frac{R(z)}{\lambda_n - z} \,dz = R(\lambda_n) \left\{ \frac{-1}{2i\pi} \int_\Gamma \frac{dz}{\lambda_n - z} + \frac{1}{2i\pi} \int_\Gamma R(z) \,dz \right\} = R(\lambda_n)(1 - P). \]
\( \lambda \) is the only pole of \( R(z) \) inside \( \Gamma \), \( R(\lambda_n)(1 - P) \) is well defined and when \( n \to \infty \), \( \lambda_n \to \lambda \), \( R(\lambda_n)(1 - P) \to S \). \( R(\lambda_n)(1 - P) \) is then uniformly bounded in \( n \), for \( n \) large enough. To have a unique formula for the cases \( \lambda_n = \lambda \) and \( \lambda_n \neq \lambda \), we set \( R(\lambda)(1 - P) = S \).

By integration in \( z \) on \( \Gamma \), we get \( \phi_n - P\phi_n = R(\lambda_n)(1 - P)(1 - \Pi_n)T\phi_n \), and
\[ \text{dist}(\phi_n, M) = \inf_{\phi \in M} \|\phi_n - \phi\|_\infty < \|\phi_n - P\phi_n\|_\infty < C\|1 - \Pi_n\|T\phi_n\|_\infty. \]

2.3.2. The Sloan method

1) \( \bar{x}_n - x = (R_n^S(z) - R(z))f = R(z)(T - T_n^S)R_n^Sf = R(z)T(1 - \Pi_n)\bar{x}_n = TR(z)(1 - \Pi_n)\bar{x}_n. \)

Then for any fixed \( t \) in \([0, 1]\), and any fixed \( z \) in \( \rho(T) \),
\[ (\bar{x}_n - x)(t) = \int_0^1 k(t, s)[R(z)(1 - \Pi_n)\bar{x}_n](s) \,ds = \langle k_t, R(z)(1 - \Pi_n)\bar{x}_n \rangle \]
\[ = \langle (R(z))^*k_t, (1 - \Pi_n)\bar{x}_n \rangle = \langle l_t, (1 - \Pi_n)\bar{x}_n \rangle. \]

Because \( R^*(z) := (R(z))^* = (T^* - \bar{z})^{-1}, l_t := R^*(z)k_t \) is the solution of \( (T^* - \bar{z})l_t = k_t \); the solution \( l_t \) (which depends on \( z \)) is unique since \( z \in \rho(T^*) \).

2) Similarly
\[ (R_n^S(z) - R(z))\tilde{\phi}_n = R(z)(T - T_n^S)R_n^S(z)\tilde{\phi}_n = \frac{R(z)}{\lambda_n - z} T(1 - \Pi_n)\tilde{\phi}_n. \]

By integration on \( \Gamma \), we get for any fixed \( t \) on \([0, 1]\)
\[ \tilde{\phi}_n(t) - (P\tilde{\phi}_n)(t) = \left[ T\left( \frac{-1}{2i\pi} \int_\Gamma \frac{R(z)}{\lambda_n - z} \,dz \right)(1 - \Pi_n)\tilde{\phi}_n \right](t) \]
\[ = \left[ TR(\lambda_n)(1 - P)(1 - \Pi_n)\tilde{\phi}_n \right](t). \]

We define \( l_t' := R^*(\lambda_n)(1 - P^*)k_t \), that is \( l_t' \) is the unique solution of \( (T^* - \bar{x}_n)l_t' = (1 - P^*)k_t \). We define accordingly \( R^*(\lambda)(1 - P^*) := S^* \). Then \( \tilde{\phi}_n(t) - (P\tilde{\phi}_n)(t) = \langle l_t', (1 - \Pi_n)\tilde{\phi}_n \rangle \). We have just proved that the error \( (x - \bar{x}_n)(t) \) (resp. \( (\phi_n - P\phi_n)(t) \)) at \( t \in [0, 1] \) can be expressed in terms of the scalar product \( \langle l_t', (1 - \Pi_n)\bar{x}_n \rangle \) (resp. \( \langle l_t', (1 - \Pi_n)\bar{x}_n \rangle \)).
REMARK. Another way to bound

\[(\tilde{\phi}_n - P\tilde{\phi}_n)(t) = \left[ T(-1/2i\pi \int_\Gamma (R(z)/\lambda_n - z) \, dz)(1 - \Pi_n)\tilde{\phi}_n \right](t) \]

is the following (Lebbar [10]). Let \( \Gamma' \) be the circle centered at \( \lambda_n \), with radius \( r \), containing \( \lambda \) and contained in \( \Gamma \) (for \( n \) large enough, there exists such a circle). We set \( z = \lambda_n + re^{i\theta} \), \( 0 < \theta < 2\pi \), for \( z \in \Gamma' \).

\[
\int_{\Gamma'} \frac{R(z)}{\lambda_n - z} \, dz = \int_{\Gamma} \frac{R(z)}{\lambda_n - z} \, dz \\
= \frac{-1}{2i\pi} \int_0^{2\pi} \frac{R(\lambda_n + re^{i\theta})}{r} \, rie^{i\theta} \, d\theta.
\]

Then

\[
|/(\tilde{\phi}_n - P\tilde{\phi}_n)(t)| = \frac{1}{2\pi} \left| \int_0^{2\pi} [TR(\lambda_n + re^{i\theta})(1 - \Pi_n)\tilde{\phi}_n](t) \, d\theta \right| \\
\leq \sup_{0 < \theta < 2\pi} \left| [TR(\lambda_n + re^{i\theta})(1 - \Pi_n)\tilde{\phi}_n](t) \right| \\
= \sup_{z \in \Gamma'} \left| [TR(z)(1 - \Pi_n)\tilde{\phi}_n](t) \right|.
\]

For \( z \in \Gamma' \), we define \( l(z) := R^*(z)k_r \). Then

\[
|/(\tilde{\phi}_n - P\tilde{\phi}_n)(t)| \leq \sup_{z \in \Gamma'} |\langle l(z), (1 - \Pi_n)\tilde{\phi}_n \rangle|.
\]

As for the global bounds on \([0, 1] \), they are easy to get:

\[
r\tilde{x}_n - x = R(z)T(1 - \Pi_n)\tilde{x}_n
\]

implies \( \|\tilde{x}_n - x\|_\infty \leq C \|T(1 - \Pi_n)\tilde{x}_n\|_\infty \), and

\[
\|T(1 - \Pi_n)\tilde{x}_n\|_\infty = \sup_{t \in [0, 1]} |\langle k_i, (1 - \Pi_n)\tilde{x}_n \rangle|.
\]

\( \tilde{\phi}_n - P\tilde{\phi}_n = R(\lambda_n)(1 - P)T(1 - \Pi_n)\tilde{\phi}_n \) implies

\[
\text{dist}(\tilde{\phi}_n, M) := \inf_{\phi \in M} \|\tilde{\phi}_n - \phi\|_\infty \leq \|\tilde{\phi}_n - P\tilde{\phi}_n\|_\infty \leq C \|T(1 - \Pi_n)\tilde{\phi}_n\|_\infty,
\]

and

\[
\|T(1 - \Pi_n)\tilde{\phi}_n\|_\infty = \sup_{t \in [0, 1]} |\langle k_i, (1 - \Pi_n)\tilde{\phi}_n \rangle|.
\]

3) Now we set \( M_n := P_nX \). For \( n \) large enough, \( P_{1_{\lambda_n}} \) has a bounded inverse \( m(\lambda - \lambda_n) = \Sigma_{i=1}^m \langle x_i^*, (1 - \Pi_n)T(P_{1_{\lambda_n}})^{-1}x_i \rangle \) where \( \{x_i\}^m_1 \) (resp. \( \{x_i^*\}^m_1 \)) is a basis of \( M \) (resp. the adjoint basis of \( M^* \)) (see de Boor-Swartz [2]). The error \( \lambda - \lambda_n \) is then of the same type as the errors \( (\tilde{x}_n - x)(t) \) and \( (\tilde{\phi}_n - P\tilde{\phi}_n)(t) \).

4) Let \( Q \) be the eigenprojection on \( E = \ker(T - \lambda) \), along a supplementary subspace \( F \).
\[ \tilde{\phi}_n - Q\tilde{\phi}_n = [(T - \lambda)(1 - Q)]^{-1}(T - \lambda)(1 - Q)\tilde{\phi}_n, \]

and

\[ (T - \lambda)(1 - Q)\tilde{\phi}_n = (T - \lambda)\tilde{\phi}_n = T(1 - \Pi_n)\tilde{\phi}_n + (\lambda_n - \lambda)\tilde{\phi}_n. \]

Therefore

\[ \text{dist}(\tilde{\phi}_n, E) = \inf_{\phi \in E} \|\tilde{\phi}_n - \phi\|_\infty < \|\tilde{\phi}_n - Q\tilde{\phi}_n\|_\infty \]

\[ < C(\|T(1 - \Pi_n)\tilde{\phi}_n\|_\infty + |\lambda_n - \lambda|). \]

Note that this method does not provide a pointwise estimate for \( \tilde{\phi}_n - Q\tilde{\phi}_n \). This is due to the fact that, unlike the spectral projection \( P \), the eigenprojection \( Q \) has no expression in terms of the resolvent.

### 3. Two basic results

We shall be concerned with two types of continuous kernels \( k \) that we define now.

i) \( k \) is smooth (of order \( l > 0 \)) if \( k \in C^l([0, 1] \times [0, 1]) \), that is \( k_{ij} \in C^l([\Delta, \Delta]) \) for \( 1 \leq i, j \leq n \), and \( k \) is continuous on \([0, 1] \times [0, 1]\).

ii) \( k \) is a Green's kernel (of order \( l > 1 \), and continuity \( \delta, 0 < \delta < l \)) if

\[ k(t, s) = \begin{cases} 
  k_1(t, s) & \text{for } t > s, \\
  k_2(t, s) & \text{for } t < s,
\end{cases} \]

is such that

\[ k_1 \in C^l([0 < s < t < 1]), \]

\[ k_2 \in C^l([0 < t < s < 1]), \]

\[ k \in C^\delta([0, 1] \times [0, 1]). \]

An obvious example of case ii) is the Green's function of an o.d.e. of order \( \delta + 2 \).

For any \( z \) in \( \rho(T) \), we consider the solution \( x = R(z)f \) of (1), along with \( \tilde{x}_n \) and \( \tilde{\phi}_n \), solutions of (5) and (6).

**Lemma 1.** Let \( T \) be an integral operator with a kernel \( k \) of order \( l \), of type i) or ii). If \( f \in C^l_\Delta \) then, in both cases, \( x, \tilde{x}_n \) and \( \tilde{\phi}_n \) are in \( C^l_\Delta \).

Now with \( k_\zeta(\cdot) := k(t, \cdot) \) for \( t \) fixed in \([0, 1]\), we consider the equation

\[ (T^* - z)l = k_\zeta, \]

for \( z \in \rho(T) \).
Lemmal. When $k$ is a smooth kernel of order $l$, then $l_i \in C^l_\Delta(0, 1)$ for any $t$ in $[0, 1]$. When $k$ is a Green's kernel of order $l$ and continuity $\delta$, then $l_i \in C^l_\Delta(0, 1)$ for $t_i \in \Delta$, $i = 0, \ldots, n$, and $l_i \in C^\delta(0, 1)$ for $t \notin \Delta$.

It is left to the reader to check the two lemmas (see Lebbar [10]). Note that when $k$ is a Green's kernel, $l_i$ is defined by the functions $l_i, l_i \in C^l(0, t), l_{2i} \in C^l(t, 1)$.

Lemma 2 shows that $l_i$ has the same smoothness properties as $k_i$. The same is true for $l'_i$.

We define $\alpha := \min(l, r + 1)$ and $\alpha^* := \min(l, r + 1, \delta + 2)$.

3.1. $\Pi_n$ is an orthogonal projection

Theorem 3. Let $\Delta$ be quasi-uniform. With the above definitions, then for $f \in C^\alpha_\Delta$, and $z$ in $\rho(T)$:

i) if $k$ is a smooth kernel of order $l$, then for $t \in [0, 1]$: $|\langle l_i, (1 - \Pi_n)f \rangle| \leq Ch^{2\alpha}\|l_i^{(\alpha)}\|_\Delta$, and globally $\|T(1 - \Pi_n)f\|_\infty \leq Ch^{2\alpha}$.

ii) if $k$ is a Green's kernel of order $l$ and continuity $\delta$, $0 \leq \delta < l$, then for $t_i \in \Delta$,

$|\langle l_i, (1 - \Pi_n)f \rangle| \leq Ch^{2\alpha}\|l_i^{(\alpha)}\|_\Delta, \quad i = 0, \ldots, n,$

for $t \notin \Delta$,

$|\langle l_i, (1 - \Pi_n)f \rangle| \leq Ch^{\alpha + \alpha^*}\max(\|l_i^{(\delta + 1)}\|_\infty, \|l_{2i}^{(\delta + 1)}\|_\infty),$

and globally, $\|T(1 - \Pi_n)f\|_\infty \leq Ch^{\alpha + \alpha^*}$.

Proof. It is adapted from Chandler [6]. Since $\Pi_n$ is an orthogonal projection:

$\langle l_i, (1 - \Pi_n)f \rangle = \langle (1 - \Pi_n)l_i, (1 - \Pi_n)f \rangle.$ And

$\int_0^1 (1 - \Pi_n)l_i(s)(1 - \Pi_n)f(s) \, ds = \sum_{i=1}^n \int_\Delta (1 - \Pi_i)l_i(s)(1 - \Pi_i)f_i(s) \, ds.$

Given $f_i \in C^{(\alpha)}(\Delta)$, $\Pi f_i$ is the orthogonal projection of $f_i$ on the set of polynomials of degree $\leq r$ on $\Delta$. When $f, l_i \in C^\alpha(\Delta)$, $f_i, l_{it} \in C^l(\Delta)$, and

$\|(1 - \Pi)f\|_l \leq Ch^{\alpha + 1}\|f^{(\alpha)}\|_\infty, \|(1 - \Pi)l_i\|_\infty \leq Ch^\alpha\|l_{it}^{(\alpha)}\|_\infty.$

When $l_i \in C^\delta(0, 1)$, with $t_{i-1} < t < t_i$, then on $\Delta$, if $\delta < r$,

$\|(1 - \Pi)l_{it}\|_\infty \leq Ch^{\delta + 1}\max(\|l_{it}^{(\delta + 1)}\|_\infty, \|l_{2it}^{(\delta + 1)}\|_\infty).$

The result follows by summing over $i$, and using $nh < \sigma$.

3.2. $\Pi_n$ is an interpolatory projection

Let $f$ be a function of $C^{l+1}(a, b)$, such that $f(w_j) = 0, j = 1, \ldots, r + 1$, where the $\{w_j\}_{j=1}^{r+1}$ are $r + 1$ distinct points in $(a, b)$. The $(r + 1)$th divided difference...
of $f$ on the points $w_1, \ldots, w_{r+1}$ is denoted by $\delta[w_1, w_2, \ldots, w_{r+1}, \cdot]f$. Then

$$f(s) = (s - w_1) \cdots (s - w_{r+1}) \delta[w_1, w_2, \ldots, w_{r+1}, s]f, \quad \text{for } s \notin \{w_j\}_{r+1}^1.$$

We set, for $s \in [a, b]$,

$$g^*(s) = \begin{cases} 
\frac{f(s)}{v(s)} & \text{if } s \notin \{w_j\}_{r+1}^1, \\
\lim_{t \to w_j} \left( \frac{f(s)}{v(s)} \right) & \text{if } t = w_j, j = 1, \ldots, r + 1,
\end{cases}$$

where $v(s) = (s - w_1) \cdots (s - w_{r+1})$.

**Lemma 4.** If $f \in C^{l+1}(a, b)$, then $g^* \in C^l(a, b)$.

There is only a need to prove that $g^*$ is $C^l$ in the neighborhood of any $w_j, j = 1, \ldots, r + 1$ (see Lebbar [10]). If $f \in C^{l+1}(a, b)$, the divided difference $\delta[w_1, \ldots, w_{r+1}, \cdot]f$ may therefore be prolonged by continuity on $[a, b]$, up to the order $l$.

We shall apply this lemma on each $\Delta_i$, with the $\{w_j\}_{r+1}^1$ being the Gauss points $\{\tau_j\}_{j+1}^r$. For $f \in C^{l+1}_{\Delta_i}$, $\Pi f_i$ is the polynomial of degree $\leq r$ on $\Delta_i$ which interpolates $f_i$ at the Gauss points $\{\tau_j\}_{j+1}^r$. Hence $(1 - \Pi)f_i(\tau_j) = 0$ for $j = 1, \ldots, r + 1$. We consider the divided difference $\delta[\tau_1, \ldots, \tau_{r+1}, \cdot](1 - \Pi)f_i$, and set $q_{it} := l_{it}\delta[\tau_1, \ldots, \tau_{r+1}, \cdot](1 - \Pi)f_i$. $q_{it} \in C^{l+1}_{\Delta_i}$ (resp. $C^r_{\Delta_i}$), implies that $l_{it} \in C^{l+1}_{\Delta_i}$ (resp. $C^r_{\Delta_i}$) and $q_{it} \in C^l_{\Delta_i}$ (resp. $C^r_{\Delta_i}$) for $\delta < l$.

**Theorem 5.** With the above definitions, then for $f \in C^{l+1}_{\Delta_i}$ and $z$ in $\rho(T)$

i) if $k$ is a smooth kernel of order $l$, then for $t \in [0, 1]$, $\|<l_t, (1 - \Pi_n)f>\| < Ch^{l+1+\alpha}\|q_{it}^{(0)}\|_{\Delta_i}$ and globally $\|T(1 - \Pi_n)f\|_{\infty} < M_{\Delta} h^{l+1+\alpha}$,

ii) if $k$ is a Green's kernel, of order $l$ and continuity $\delta$, $0 < \delta < l$, then for $t_i \in \Delta$,

$$\|<l_{t_i}, (1 - \Pi_n)f>\| < Ch^{l+1+\alpha}\|q_{t_i}^{(\delta+1)}\|_{\Delta_i}, \quad i = 0, \ldots, n,$$

for $t \notin \Delta$,

$$\|<l_t, (1 - \Pi_n)f>\| < Ch^{l+1+\alpha}\max(\|q_{1t}^{(\delta+1)}\|_{\infty}, \|q_{2t}^{(\delta+1)}\|_{\infty}),$$

and globally $\|T(1 - \Pi_n)f\|_{\infty} < M_{\Delta} h^{l+r+1+\alpha}$.

If $f \in C^{l+r+1}_{\Delta_i}$, then $M_{\Delta} < C$.

**Proof.** It is adapted from de Boor-Swartz [1].

$$\int_0^1 l_t(s)(1 - \Pi_n)f(s) \, ds = \sum_{i=1}^n \int_{\Delta_i} l_{it}(s)(1 - \Pi)f_i(s) \, ds$$

$$= \sum_{i=1}^n \int_{\Delta_i} l_{it}(s)\delta[\tau_1, \ldots, \tau_{r+1}, s](1 - \Pi)f_i(s) \left( s - \tau_i \right) \cdots \left( s - \tau_{r+1}^i \right) \, ds.$$
When \( q_{it} \in C^l_{(\Delta)}, q_{it}(s) = q_{it}(t_{i-1}) + \cdots + ((s - t_{i-1})^{l}/l!)q_{it}^{(l)}(\theta), \ t_{i-1} < \theta < s. \) Making use of \( \int_{\Delta} v(s)p(s) \, ds = 0 \) for all polynomial \( p \) of degree \( \leq r \) on \( \Delta_i \), we get \( |\int_{\Delta_i} q_{it}(s)v(s) \, ds| \leq Ch^{r+2+\alpha} \|q_{it}^{(a)}\|_{\infty}, \) which gives, for \( l_i \in C^l_{\Delta}, \)

\[
\langle l_i, (1 - \Pi_n)f \rangle \leq Ch^{r+1+\alpha} \|q_{it}^{(a)}\|_{\Delta}.
\]

When \( l_{it} \in C^\delta_{(\Delta)}, \)

\[
\left| \int_{\Delta_i} l_{it}(s)(1 - \Pi)f_i(s) \, ds \right| \leq Ch^{r+2+\min(r+1,\delta+2)}
\]

and \( |\langle l_i, (1 - \Pi_n)f \rangle| \leq Ch^{r+1+\alpha_*}, \) by summing over \( i. \)

Theorems 3 and 5 play a central role to derive the convergence rates, as we shall see in the next section.

### 4. Convergence rates

We recall that \( \alpha = \min(l, r + 1) \) and \( \alpha^* = \min(l, r + 1, \delta + 2) \). In practice \( \delta + 2 \leq r + 1 \leq l \), so that \( \alpha = r + 1 \) and \( \alpha^* = \delta + 2 \). We assume throughout this section that the kernel \( k \) is of order \( l \) for the Galerkin method \( (f \in C^l_{\Delta} \Rightarrow \tilde{x}_n \in C^l_{\Delta}, \tilde{\phi}_n \in C^l_{\Delta}) \) and of order \( l + r + 1 \) for the collocation method \( (f \in C^{l+r+1}_{\Delta} \Rightarrow \tilde{x}_n \in C^{l+r+1}_{\Delta} \) and \( \tilde{\phi}_n \in C^{l+r+1}_{\Delta}). \)

#### 4.1. Convergence rate for the eigenvalues

The definitions are those of Section 2.2.

**Theorem 6.** For both types of kernel \( k \)

\[
\lambda - \hat{\lambda}_n = O(\varepsilon_n), \quad \max_i |\lambda - \lambda_{in}| = O(\varepsilon_n^{1/\mu}), \quad \min_i |\lambda - \lambda_{in}| = O(\varepsilon_n^{8/\mu})
\]

where (a) \( \varepsilon_n = h^{2\alpha} \) for the Galerkin method, and (b) \( \varepsilon_n = h^{r+1+\alpha^*} \) for the collocation method.

**Proof.** It is adapted from de Boor-Swartz [2] where it is noticed that \( \lambda \) (resp. \( \lambda_{in} \)) are the eigenvalues of two \( m \times m \) matrices such that the \((i,j)\)th coefficient of the difference is \( \langle x_i^*, (1 - \Pi_n)(P_{i_{\Delta^*}})^{-1}x_j \rangle \). Theorem 3 applies where \( l_i \) is replaced by \( x_i^* \in C^l_{\Delta} \) and \( T(P_{i_{\Delta^*}})^{-1}x_j \in C^l_{\Delta} \) if the kernel is of order \( l \). Similarly, Theorem 5 applies if \( k \) is of order \( l + r + 1 \). And the results follow from classical theorems in matrix theory (see Wilkinson [18], pp. 80–81).
4.2. Convergence rate for the solutions and the eigenvectors

(a) The Galerkin method. We suppose that \( f \in C^l_\Delta \) and \( k \) is of order \( l \). Then \( \tilde{x}_n, \tilde{\phi}_n \in C^l_\Delta \). \( \Delta \) is quasi-uniform.

**Theorem 7.** With a smooth kernel, \( \|x - \tilde{x}_n\|_\infty \) and \( \text{dist}(\tilde{\phi}_n, M) \) are of the order \( h^{2\alpha} \), \( \text{dist}(\phi_n, E) = O(h^{2\alpha}/\mu) \). With a Green’s kernel, then: at \( t_i \in \Delta, |x(t_i) - \tilde{x}_n(t_i)| \) and \( |\tilde{\phi}_n(t_i) - (P\tilde{\phi}_n)(t_i)| \) are of the order \( h^{2\alpha} \), \( i = 0, \ldots, n \), whereas globally \( \|x - \tilde{x}_n\|_\infty \) and \( \text{dist}(\tilde{\phi}_n, M) \) are of the order \( h^{\alpha+a} \), \( \text{dist}(\phi_n, E) = O(h^{2\alpha}/\mu) \), for \( \mu > 1 \).

**Proof.** We apply Theorem 3 to \( (\tilde{x}_n - x)(t) = \langle l_t, (1 - \Pi_n)\tilde{x}_n \rangle \), \( \langle \tilde{\phi}_n - P\tilde{\phi}_n \rangle(t) = \langle l'_t, (1 - \Pi_n)\tilde{\phi}_n \rangle \), and Theorem 6 to

\[
\text{dist}(\tilde{\phi}_n, E) \leq C(\|T(1 - \Pi_n)\tilde{\phi}_n\|_\infty + |\lambda_n - \lambda|).
\]

(b) The collocation method. We suppose that \( f \in C^{l+r+1}_\Delta \) and \( k \) is of order \( l + r + 1 \). Then \( \tilde{x}_n, \tilde{\phi}_n \in C^{l+r+1}_\Delta \). We get, as Theorem 8, the analog of Theorem 7, where \( h^{2\alpha} \) (resp. \( h^{\alpha+a} \)) is replaced by \( h^{r+1+\alpha} \) (resp. \( h^{r+1+a} \)). The convergence rates in Theorems 7 and 8 are the best we could hope from the known results. It should be noticed that the computation of \( \tilde{x}_n \) (resp. \( \tilde{\phi}_n \)) from \( x_n \) (resp. \( \lambda_n, \phi_n \)) does not require much extra work: let \( \dim X_n = n \) (say), let \( \{e_i^n\}_1^n \) be a basis of \( X_n \); if \( x_n = \sum_{i=1}^n \xi_i^n e_i^n \), then \( TX_n = \sum_{i=1}^n \xi_i^n T e_i^n \) where the \( \{T e_i^n\}_1^n \) have already been computed to get the coefficients of the matrix associated with the projection method.

5. Numerical Example

We end this paper with a numerical example illustrating the behavior of the iterated collocation solution for the Fredholm equation

\[
\int_0^1 k(t, s) x(s) \, ds - \frac{1}{4} x(t) = -\cosh(1), \quad 0 < t < 1,
\]

with

\[
k(t, s) = \begin{cases} 
-t(1 - s) & \text{if } s > t, \\
-s(1 - t) & \text{if } s < t.
\end{cases}
\]

The exact solution is \( x(t) = \cosh(2t - 1) \).

We choose the partition \( \Delta = \{i/5\}_0^5 \), \( h = \frac{1}{5} \), and on each interval \( \Delta_i \), the \( r + 1 = 4 \) Gauss points. We display in Table 1 the values of \( x - x_n \) and \( x - \tilde{x}_n \).
at the partition points \( t_i \), \( i = 1, 2, 3, 4 \). The kernel \( k \) is of “Green’s function” type with \( \delta = 0 \).

### Table 1

Error values at the partition points

<table>
<thead>
<tr>
<th>( i )</th>
<th>((x - x_n)(t_i))</th>
<th>((x - x_n)(t_i^+))</th>
<th>((x - \bar{x}_n)(t_i))</th>
</tr>
</thead>
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<tr>
<td>1</td>
<td>8.10^{-1}</td>
<td>7.10^{-5}</td>
<td>-5.10^{-12}</td>
</tr>
<tr>
<td>2</td>
<td>6.10^{-5}</td>
<td>6.10^{-5}</td>
<td>-7.10^{-12}</td>
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<td>6.10^{-5}</td>
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</tr>
<tr>
<td>4</td>
<td>7.10^{-5}</td>
<td>8.10^{-5}</td>
<td>-5.10^{-12}</td>
</tr>
</tbody>
</table>

### References


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