Bull. Aust. Math. Soc. **99** (2019), 413–420
doi:10.1017/S0004972718001132

ON σ-QUASINORMAL SUBGROUPS OF FINITE GROUPS

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(Received 30 August 2018; accepted 21 September 2018; first published online 12 November 2018)

Abstract

Let G be a finite group and $\sigma = \{\sigma_i \mid i \in I\}$ some partition of the set of all primes $\mathbb{P}$, that is, $\mathbb{P} = \bigcup_{i\in I} \sigma_i$ and $\sigma_i \cap \sigma_j = \emptyset$ for all $i \neq j$. We say that G is *σ-primary* if G is a σ_i-group for some i. A subgroup A of G is said to be: *σ-subnormal in G* if there is a subgroup chain $A = A_0 \leq A_1 \leq \cdots \leq A_n = G$ such that either $A_{i-1} \trianglelefteq A_i$ or $A_i/(A_{i-1})_{A_i}$ is σ-primary for all $i = 1, \ldots, n$; *modular in G* if the following conditions hold: (i) $\langle X, A \cap Z\rangle = \langle X, A\rangle \cap Z$ for all $X \leq G, Z \leq G$ such that $X \leq Z$ and (ii) $\langle A, Y \cap Z\rangle = \langle A, Y\rangle \cap Z$ for all $Y \leq G, Z \leq G$ such that $A \leq Z$; and *σ-quasinormal in G* if A is modular and σ-subnormal in G. We study σ-quasinormal subgroups of G. In particular, we prove that if a subgroup H of G is σ-quasinormal in G, then every chief factor H/K of G between H^G and H_G is *σ-central in G*, that is, the semidirect product $(H/K) \rtimes (G/C_G(H/K))$ is σ-primary.

2010 *Mathematics subject classification*: primary 20D10; secondary 20D15.

Keywords and phrases: finite group, σ-nilpotent group, σ-subnormal subgroup, modular subgroup, σ-quasinormal subgroup.

1. Introduction

Throughout this paper, all groups are finite and G always denotes a finite group. Moreover, $\mathbb{P}$ is the set of all primes, $\pi = \{p_1, \ldots, p_n\} \subseteq \mathbb{P}$ and $\pi' = \mathbb{P} \setminus \pi$. If n is an integer, the symbol $\pi(n)$ denotes the set of all primes dividing n; as usual, $\pi(G) = \pi(|G|)$, the set of all primes dividing the order of G.

A subgroup A of G is said to be *modular in G* if it is a modular element (in the sense of Kurosh [9, page 43]) of the lattice of all subgroups of G, that is, the following conditions hold:

(i) $\langle X, A \cap Z\rangle = \langle X, A\rangle \cap Z$ for all $X \leq G, Z \leq G$ such that $X \leq Z$; and
(ii) $\langle A, Y \cap Z\rangle = \langle A, Y\rangle \cap Z$ for all $Y \leq G, Z \leq G$ such that $A \leq Z$.

In what follows, σ is some partition of $\mathbb{P}$, that is, $\sigma = \{\sigma_i \mid i \in I\}$, where $\mathbb{P} = \bigcup_{i\in I} \sigma_i$ and $\sigma_i \cap \sigma_j = \emptyset$ for all $i \neq j$. By analogy with the notation $\pi(n)$, we write $\sigma(n)$ to denote the set $\{\sigma_i \mid \sigma_i \cap \pi(n) \neq \emptyset\}$ and $\sigma(G) = \sigma(|G|)$. The group G is said to be *σ-primary* [10] if $|\sigma(G)| \leq 1$, that is, G is a σ_i-group for some i.

Research is supported by an NNSF grant of China (Grant No. 11401264) and a TAPP of Jiangsu Higher Education Institutions (PPZY 2015A013).

If $K \leq H$ are normal subgroups of G and $C \leq C_G(H/K)$, then we can form the semidirect product $(H/K) \rtimes (G/C)$ putting $(hK)^{gC} = g^{-1}hgK$ for all $hK \in H/K$ and $gC \in G/C$. A chief factor H/K of G is said to be *σ-central in G* (as defined in [10]) if $(H/K) \rtimes (G/C_G(H/K))$ is σ-primary; G is called *σ-nilpotent* [10] if every chief factor of G is σ-central. In view of [3, Proposition 2.7], G is σ-nilpotent if and only if $G = G_1 \times \cdots \times G_t$ for some σ-primary groups $G_1, \ldots, G_t$. We use $\mathfrak{N}_\sigma$ to denote the class of all σ-nilpotent groups.

A subgroup A of G is said to be *σ-subnormal in G* [10] if it is *$\mathfrak{N}_\sigma$-subnormal in G* in the sense of Kegel [6], that is, there is a subgroup chain $A = A_0 \leq A_1 \leq \cdots \leq A_n = G$ such that either $A_{i-1} \trianglelefteq A_i$ or $A_i/(A_{i-1})_{A_i}$ is σ-nilpotent for $i = 1, \ldots, n$, and it is *σ-seminormal in G* (J. C. Beidleman) if $x \in N_G(A)$ for all $x \in G$ such that

$$\sigma(|x|) \cap \sigma(A) = \emptyset.$$

Finally, recall that a subgroup A of G is said to be *quasinormal* [8] or *permutable* [1] in G if A permutes with every subgroup L of G, that is, $AL = LA$.

The quasinormal subgroups have many interesting properties. For instance, if A is quasinormal in G, then A is subnormal in G (see Ore [8]), A/A_G is nilpotent (see Ito and Szép [5]) and, in general, A/A_G is not necessarily abelian (see Thompson [12]). Every quasinormal subgroup A of G is modular in G [9]. Moreover, the following properties of quasinormal subgroups are well known.

Theorem 1.1 (see [9, Theorem 5.1.1]). *A subgroup A of G is quasinormal in G if and only if A is modular and subnormal in G.*

Theorem 1.2. *If A is a quasinormal subgroup of G, then:*

(i) *A^G/A_G is nilpotent (this follows from the above-mentioned results in [5, 8]);*
(ii) *every chief factor H/K of G between A^G and A_G is central in G, that is, $C_G(H/K) = G$ (see Maier and Schmid [7]).*

Since every subnormal subgroup of G is σ-subnormal in G, Theorems 1.1 and 1.2 make it natural to ask: *What can we say about the quotient A^G/A_G provided the subgroup A is σ-quasinormal in G in the sense of the following definition?*

Definition 1.3. Let A be a subgroup of G. Then we say that A is *σ-quasinormal* in G if A is modular and σ-subnormal in G.

In this note we give the answer to this question. But before continuing, we consider the following example.

Example 1.4. Let $p > q, r, t$ be distinct primes, where t divides $r - 1$. Let Q be a simple $\mathbb{F}_q C_p$-module which is faithful for C_p, let $C_r \rtimes C_t$ be a nonabelian group of order rt and let $A = C_t$. Finally, let $G = (Q \rtimes C_p) \times (C_r \rtimes C_t)$ and B be a subgroup of order q in Q. Then $B < Q$ since $p > q$. It is not difficult to show that A is modular in G (see [9, Lemma 5.1.8]). On the other hand, A is σ-subnormal in G, where $\sigma = \{\{q, r, t\}, \{q, r, t\}'\}$. Hence, A is σ-quasinormal in G. It is clear also that A is not subnormal in G, so A is not quasinormal in G. Finally, note that B is not modular in G by Lemma 2.2 below.

https://doi.org/10.1017/S0004972718001132 Published online by Cambridge University Press

Our main goal here is to prove the following theorem.

THEOREM 1.5. *Let A be a σ-quasinormal subgroup of G.*

(i) *If G possesses a Hall σ_i-subgroup, then A permutes with each Hall σ_i-subgroup of G.*
(ii) *The quotients A^G/A_G and $G/C_G(A^G/A_G)$ are σ-nilpotent.*
(iii) *Every chief factor of G between A^G and A_G is σ-central in G.*
(iv) *$\sigma_i \in \sigma(A^G/A_G)$ for every i such that $\sigma_i \in \sigma(G/C_G(A^G/A_G))$.*
(v) *A is σ-seminormal in G.*

The subgroup A of G is subnormal in G if and only if it is σ-subnormal in G, where $\sigma = \sigma^1 = \{\{2\}, \{3\}, \ldots\}$ (using the notation in [11]). It is clear also that G is nilpotent if and only if G is σ^1-nilpotent, and a chief factor H/K of G is central in G if and only if H/K is σ^1-central in G. Therefore, Theorem 1.2 is a special case of Theorem 1.5 when $\sigma = \sigma^1$.

In the other classical case when $\sigma = \sigma^\pi = \{\pi, \pi'\}$, G is σ^π-nilpotent if and only if G is *π-decomposable*, that is, $G = O_\pi(G) \times O_{\pi'}(G)$, and a subgroup A of G is σ^π-subnormal in G if and only if there is a subgroup chain

$$A = A_0 \leq A_1 \leq \cdots \leq A_n = G$$

such that either $A_{i-1} \trianglelefteq A_i$ or $A_i/(A_{i-1})_{A_i}$ is a π_0-group, where $\pi_0 \in \{\pi, \pi'\}$, for all $i = 1, \ldots, n$. Thus, in this case Theorem 1.5 gives the following corollary.

COROLLARY 1.6. *Suppose that A is a modular subgroup of G and there is a subgroup chain*

$$A = A_0 \leq A_1 \leq \cdots \leq A_n = G$$

such that either $A_{i-1} \trianglelefteq A_i$ or $A_i/(A_{i-1})_{A_i}$ is a π_0-group, where $\pi_0 \in \{\pi, \pi'\}$, for all $i = 1, \ldots, n$. Then the following statements hold.

(i) *If G possesses a Hall π_0-subgroup, where $\pi_0 \in \{\pi, \pi'\}$, then A permutes with each Hall π_0-subgroup of G.*
(ii) *The quotients A^G/A_G and $G/C_G(A^G/A_G)$ are π-decomposable.*
(iii) *For every chief factor H/K of G between A^G and A_G, the semidirect product $(H/K) \rtimes (G/C_G(H/K))$ is either a π-group or a π'-group.*

In fact, in the theory of π-soluble groups ($\pi = \{p_1, \ldots, p_n\}$), we deal with the partition $\sigma = \sigma^{1\pi} = \{\{p_1\}, \ldots, \{p_n\}, \pi'\}$ of $\mathbb{P}$. Note that G is $\sigma^{1\pi}$-nilpotent if and only if G is *π-special* [2], that is, $G = O_{p_1}(G) \times \cdots \times O_{p_n}(G) \times O_{\pi'}(G)$. A subgroup A of G is $\sigma^{1\pi}$-subnormal in G if and only if it is $\mathfrak{F}$-subnormal in G in the sense of Kegel [6], where $\mathfrak{F}$ is the class of all π'-groups. Therefore, in this case Theorem 1.5 gives the following corollary.

COROLLARY 1.7. *Suppose that A is a modular subgroup of G and there is a subgroup chain*

$$A = A_0 \leq A_1 \leq \cdots \leq A_n = G$$

https://doi.org/10.1017/S0004972718001132 Published online by Cambridge University Press

such that either $A_{i-1} \trianglelefteq A_i$ or $A_i/(A_{i-1})_{A_i}$ is a π'-group. Then the following statements hold.

(i) *A permutes with every Sylow p-subgroup of G for all $p \in \pi$ and, if G possesses a Hall π'-subgroup, then A permutes with each Hall π'-subgroup of G.*
(ii) *The quotients A^G/A_G and $G/C_G(A^G/A_G)$ are π-special.*
(iii) *Every element of G induces a π'-automorphism on every noncentral chief factor of G between A^G and A_G.*

2. Proof of Theorem 1.5

Recall that G is a nonabelian P-group if $G = A \rtimes \langle t\rangle$, where A is an elementary abelian p-group and an element t of prime order $q \neq p$ induces a nontrivial power automorphism on A (see [9, page 49]). In this case we say that G is a *P-group of type* (p, q).

LEMMA 2.1 (see [9, Lemma 2.2.2(d)]). *If $G = A \rtimes \langle t\rangle$ is a P-group of type (p, q), then $\langle t\rangle^G = G$.*

The following remarkable result of Schmidt plays a key role in the proof of Theorem 1.5.

LEMMA 2.2 (see [9, Theorem 5.1.14]). *Let M be a modular subgroup of G with $M_G = 1$. Then $G = S_1 \times \cdots \times S_r \times K$, where $0 \le r \in \mathbb{Z}$ and, for all $i, j \in \{1, \ldots, r\}$:*

(a) *S_i is a nonabelian P-group;*
(b) *$(|S_i|, |S_j|) = 1 = (|S_i|, |K|)$ for all $i \neq j$;*
(c) *$M = Q_1 \times \cdots \times Q_r \times (M \cap K)$ and Q_i is a nonnormal Sylow subgroup of S_i;*
(d) *$M \cap K$ is quasinormal in G.*

The following lemma is a corollary of general properties of modular subgroups [9, page 201] and σ-subnormal subgroups [10, Lemma 2.6].

LEMMA 2.3. *Let A, B and N be subgroups of G, where A is σ-quasinormal and N is normal in G.*

(1) *The subgroup $A \cap B$ is σ-quasinormal in B.*
(2) *The subgroup AN/N is σ-quasinormal in G/N.*
(3) *If $N \le B$ and B/N is σ-quasinormal in G/N, then B is σ-quasinormal in G.*

PROOF OF THEOREM 1.5. Suppose that this theorem is false and let G be a counterexample of minimal order. Then $1 < A < G$. We can assume without loss of generality that $\sigma(A) = \{\sigma_1, \ldots, \sigma_m\}$.

CLAIM 1. *Statement (i) holds for G.*

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Suppose that this is false. Then, for some Hall σ_i-subgroup V of G, we have $AV \neq VA$. First note that $\langle A, V\rangle = G$. Indeed, A is σ-quasinormal in $\langle A, V\rangle$ by Lemma 2.3(1), so in the case when $\langle A, V\rangle < G$ the choice of G implies that $AV = VA$, contrary to our assumption about V.

Since A is σ-subnormal in G, there is a subgroup chain $A = A_0 \leq A_1 \leq \cdots \leq A_n = G$ such that either $A_{i-1} \trianglelefteq A_i$ or $A_i/(A_{i-1})_{A_i}$ is σ-primary for $i = 1, \ldots, n$. We can assume without loss of generality that $M = A_{n-1} < G$. Then A permutes with every Hall σ_i-subgroup of M by the choice of G. Moreover, the modularity of A in G implies that

$$M = M \cap \langle A, V\rangle = \langle A, M \cap V\rangle$$

since $\langle A, V\rangle = G$. On the other hand, $M \cap V$ is a Hall σ_i-subgroup of M by [10, Lemma 2.6(7)]. Hence, $M = A(M \cap V) = (M \cap V)A$.

If $V \leq M_G$, then $A(M \cap V) = AV = VA$ and so $V \nleq M_G$. Now note that $VM = MV$. This is clear if M is normal in G. Otherwise, G/M_G is σ-primary and so $G = MV = VM$ since $V \nleq M_G$ and V is a Hall σ_i-subgroup of G. Therefore,

$$VA = V(M \cap V)A = VM = MV = A(M \cap V)V = AV.$$

This contradiction completes the proof of Claim 1. □

Claim 2. *We have* $A_G = 1$.

Suppose that $A_G \neq 1$ and let R be a minimal normal subgroup of G contained in A_G. Then the hypothesis holds for $(G/R, A/R)$ by Lemma 2.3(2). Therefore, the choice of G implies that Statements (ii)–(v) hold for $(G/R, A/R)$. Hence,

$$(A/R)^{G/R}/(A/R)_{G/R} = (A^G/R)/(A_G/R) \simeq A^G/A_G$$

and

$$(G/R)/C_{G/R}((A/R)^{G/R}/(A/R)_{G/R}) = (G/R)/(C_G(A^G/A_G)/R) \simeq G/C_G(A^G/A_G)$$

are σ-nilpotent, so Statement (ii) holds for G.

Now let T/L be any chief factor of G between A^G and A_G. Then $(T/R)/(L/R)$ is a chief factor of G/R between $(A/R)^{G/R}$ and $(A/R)_{G/R}$ and so $(T/R)/(L/R)$ is σ-central in G/R, that is,

$$((T/R)/(L/R)) \rtimes ((G/R)/C_{(G/R)}((T/R)/(L/R)))$$

is σ-primary. Since the factors $(T/R)/(L/R)$ and T/L are G-isomorphic, it follows that $(T/L) \rtimes (G/C_G(T/L))$ is σ-primary too. Hence, T/L is σ-central in G. Thus, Statement (iii) holds for G.

If i is such that

$$\sigma_i \cap \pi(G/C_G(A^G/A_G)) = \sigma_i \cap \pi((G/R)/C_{G/R}((A/R)^{G/R}/(A/R)_{G/R})) \neq \emptyset,$$

then

$$\sigma_i \cap \pi(A^G/A_G) = \sigma_i \cap \pi((A/R)^{G/R}/(A/R)_{G/R}) \neq \emptyset$$

and so Statement (iv) holds for G too.

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Finally, if $x \in G$ and $\sigma(A) \cap \sigma(|x|) = \emptyset$, then $\sigma(A/R) \cap \sigma(|xR|) = \emptyset$, so $xR \in N_{G/R}(A/R) = N_G(A)/R$ and hence Statement (v) holds for G. Therefore, in view of Claim 1, the conclusion of the theorem holds for G, which contradicts the choice of G. Hence, $A_G = 1$. □

Claim 3. *The inclusion $O_{\sigma_i}(A) \leq O_{\sigma_i}(G)$ holds for all i.*

It is enough to show that $A \leq O_{\sigma_i}(G)$ for any σ-subnormal σ_i-subgroup A of G. Assume that this is false and let G be a counterexample of minimal order. Then $1 < A < G$. Let R be a minimal normal subgroup of G and let $D = O_{\sigma_i}(G)$ and $O/R = O_{\sigma_i}(G/R)$. Then the choice of G and Lemma 2.3(2) imply that $AR/R \leq O/R$. Therefore, $R \nleq D$, so $D = 1$ and $A \cap R < R$. It is clear also that $O^{\sigma_i}(R) = R$. Suppose that $L = A \cap R \neq 1$. Lemma 2.3(1) implies that L is σ-subnormal in R. If $R < G$, the choice of G implies that $L \leq O_{\sigma_i}(R) \leq D$ since $O_{\sigma_i}(R)$ is a characteristic subgroup of R. But then $D \neq 1$, which is a contradiction. Hence, $R = G$ is a simple group, which is also impossible since $1 < A < G$. Therefore, $R \cap A = 1$. If $O < G$, the choice of G implies that $A \leq O_{\sigma_i}(O) \leq D = 1$. Therefore, $G/R = O/R$ is a σ_i-group. Hence, R is the unique minimal normal subgroup of G. It is clear also that $R \nleq \Phi(G)$, so $C_G(R) \leq R$ since $C_G(R)$ is normal in G.

Now we show that $G = RA$. Indeed, if $RA < G$, then the choice of G and Lemma 2.3(1) imply that $A \leq O_{\sigma_i}(RA)$ and so $A = O_{\sigma_i}(RA)$ since $O_{\sigma_i}(R) = 1$, which implies that $RA = R \times A$. But then $A \leq C_G(R) \leq R$ and so $A = 1$ since $A \cap R = 1$. This contradiction shows that $G = RA$.

Since A is σ-subnormal in G, there is a subgroup M such that $A \leq M < G$ and either $M \trianglelefteq G$ or G/M_G is σ-primary. Since R is the unique minimal normal subgroup of G and $A \leq M < G = RA$, it follows that $R \nleq M$ and $G/M_G = G/1$ is a σ_i-group. Therefore, $A \leq O_{\sigma_i}(G) = G$. This contradiction completes the proof of Claim 3. □

Claim 4. *We have $A \leq O_{\sigma_1}(G) \times \cdots \times O_{\sigma_m}(G)$. Hence, $A^G = O_{\sigma_1}(A^G) \times \cdots \times O_{\sigma_m}(A^G)$.*

By Claim 2, Theorem 1.2(i) and Lemma 2.2(c) and (d), $A = A_1 \times \cdots \times A_m$, where A_i is a Hall σ_i-subgroup of A for $i = 1, \ldots, m$. On the other hand, since A is σ-subnormal in G, we have $A_i \leq O_{\sigma_i}(G)$ by Claims 3 and 4. □

Claim 5. *Statements (ii), (iii) and (iv) hold for G.*

If $A^G = G$, this follows from Claim 4. Now assume that $A^G \neq G$. By Lemma 2.2, $G = S_1 \times \cdots \times S_r \times K$, where, for $i, j \in \{1, \ldots, r\}$, the following hold:

(a) S_i is a nonabelian P-group;
(b) $(|S_i|, |S_j|) = 1 = (|S_i|, |K|)$ for $i \neq j$;
(c) $A = Q_1 \times \cdots \times Q_r \times (A \cap K)$ and Q_i is a nonnormal Sylow subgroup of S_i;
(d) $A \cap K$ is quasinormal in G.

In view of Lemma 2.1 and Claim 4,

$$\begin{aligned} A^G &= Q_1^G \times \cdots \times Q_r^G \times (A \cap K)^G = S_1 \times \cdots \times S_r \times (A \cap K)^G \\ &= O_{\sigma_1}(A^G) \times \cdots \times O_{\sigma_m}(A^G), \end{aligned}$$

where $(A \cap K)^G \leq Z_\infty(G)$ by Theorem 1.2(ii) since $(A \cap K)_G \leq A_G = 1$ by Claim 2.

Now note that for all i, j, either $S_i \leq O_{\sigma_j}(A^G)$ or $S_i \cap O_{\sigma_j}(A^G) = 1$. Indeed, assume that $S_i \cap O_{\sigma_j}(A^G) \neq 1$. It is clear that $Q_i \leq O_{\sigma_t}(A^G)$ for some t. Then $Q_i^G = S_i \leq O_{\sigma_t}(A^G)$ by Lemma 2.1. Hence, $j = t$ since $O_{\sigma_j}(A^G) \cap O_{\sigma_t}(A^G) = 1$ for $j \neq t$. Therefore, all S_i are σ-primary. Moreover, if S_i is a σ_l-group, then $G/C_G(S_i)$ is a σ_l-group since $G = S_1 \times \cdots \times S_r \times K$.

From Claim 4, $A^G = O_{\sigma_1}(A^G) \times \cdots \times O_{\sigma_m}(A^G)$. Consequently,

$$C_G(A^G) = C_G(O_{\sigma_1}(A^G)) \cap \cdots \cap C_G(O_{\sigma_m}(A^G)).$$

On the other hand, $G/C_G(O_{\sigma_i}(A^G))$ is a σ_i-group for $i = 1, \ldots, m$. Therefore, in view of [4, Ch. I, Lemma 9.6],

$$\begin{aligned} G/C_G(A^G) &= G/(C_G(O_{\sigma_1}(A^G)) \cap \cdots \cap C_G(O_{\sigma_m}(A^G))) \\ &\simeq V \leq (G/C_G(O_{\sigma_1}(A^G))) \times \cdots \times (G/C_G(O_{\sigma_m}(A^G))) \end{aligned}$$

is σ-nilpotent and $\sigma_i \in \sigma(A^G)$ for every i such that $\sigma_i \in \sigma(G/C_G(A^G))$. Hence, Statements (ii) and (iv) hold for G.

Finally, let T/L be any chief factor of G below A^G. Suppose that T/L is not σ-central in G. By Theorem 1.2(ii), T is not contained in $(A \cap K)^G$. Therefore, in view of the Jordan–Hölder theorem for the chief series, we can assume without loss of generality that $T \leq S_k$ for some k. But then $G/C_G(S_k)$ is a σ_l-group, where S_k is a σ_l-group and so from $C_G(S_k) \leq C_G(T/L)$ we deduce that T/L is σ-central in G, which is a contradiction. This proves Claim 5. □

Claim 6. *Statement (v) holds for G.*

Suppose that $x \in G$ is such that $\sigma(A) \cap \sigma(|x|) = \emptyset$. Then the modularity of A and Claim 4 imply that $A = O_{\sigma_1}(A^G) \times \cdots \times O_{\sigma_m}(A^G) \cap \langle A, \langle x \rangle \rangle$ is normal in $\langle A, \langle x \rangle \rangle$, so $x \in N_G(A)$. This proves Claim 6. □

From Claims 1, 5 and 6, it follows that the conclusion of the theorem holds for G, which contradicts the choice of G. The theorem is proved. □

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