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Nils R. Scheithauer

Abstract

We prove some new structure results for automorphic products of singular weight. First, we give a simple characterisation of the Borcherds function Φ_{12}. Second, we show that holomorphic automorphic products of singular weight on lattices of prime level exist only in small signatures and we derive an explicit bound. Finally, we give a complete classification of reflective automorphic products of singular weight on lattices of prime level.

Contents

1. Introduction

The singular theta correspondence (see [Bor98] and also [Bru02]) is a map from modular forms for the Weil representation of $\mathrm{SL}_2(\mathbb{Z})$ to automorphic forms on orthogonal groups. More precisely, let L be an even lattice of signature $(n,2)$, $n>2$ and even with discriminant form D and F a modular form for the Weil representation of $\mathrm{SL}_2(\mathbb{Z})$ on $\mathbb{C}[D]$ of weight $(2-n)/2$, which is holomorphic on the upper halfplane and has integral principal part. Then Borcherds associates an automorphic form $\Psi(F)$ of weight $c_0(0)/2$ for $O(L)$ to F where $c_0(0)$ denotes the constant coefficient in the Fourier expansion of F_0. The function $\Psi(F)$ has nice product expansions at the rational 0-dimensional cusps and is called the automorphic product associated to L and F. The divisor of $\Psi(F)$ is a linear combination of rational quadratic divisors whose orders are determined by the principal part of F. Bruinier [Bru14] has shown that if L splits two hyperbolic planes, then every automorphic form for $O(L)$ whose divisor is a linear combination of rational quadratic divisors is an automorphic product.

The smallest possible weight of a non-constant holomorphic automorphic form on $O_{n,2}(\mathbb{R})$ is given by $(n-2)/2$. Forms of this so-called singular weight are particularly interesting because their Fourier coefficients are supported only on isotropic vectors. Holomorphic automorphic products

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of singular weight seem to be very rare. The few known examples are all related to infinite-dimensional Lie superalgebras, i.e. given by the denominator functions of generalised Kac–Moody superalgebras. One of the main open problems in the theory of automorphic forms on orthogonal groups is to classify holomorphic automorphic products of singular weight [Bor95]. In this paper, we prove some new results in this direction.

The simplest holomorphic automorphic product of singular weight is the function Φ_{12}. It is the theta lift of the inverse of the Dedekind function Δ on the unimodular lattice $II_{26,2}$. The product expansion of Φ_{12} at a cusp is given by

$$\Phi_{12}(Z) = e((\rho, Z)) \prod_{\alpha \in II_{25,1}^+} (1 - e((\alpha, Z)))^{[1/\Delta](-\alpha^2/2)},$$

where ρ is a primitive norm 0 vector in $II_{25,1}$ corresponding to the Leech lattice. The function Φ_{12} is holomorphic and has zeros of order 1 orthogonal to the roots of $II_{26,2}$. Since Φ_{12} has weight 12, i.e. singular weight, its Fourier coefficients are supported only on norm 0 vectors. This can be used to show that it has the sum expansion

$$e((\rho, Z)) \prod_{\alpha \in II_{25,1}^+} (1 - e((\alpha, Z)))^{[1/\Delta](-\alpha^2/2)} = \sum_{w \in W} \det(w) e((w\rho, Z)) \prod_{n=1}^{\infty} (1 - e((nw\rho, Z)))^{24}.$$

Here W is the reflection group of $II_{25,1}$.

This identity is the denominator identity of an infinite-dimensional Lie algebra describing the physical states of a bosonic string moving on the torus $\mathbb{R}^{25,1}/II_{25,1}$ called the fake monster algebra [Bor90].

The function Φ_{12} also has some nice geometric applications. In [GHS07], the authors show that the moduli space of polarised K3 surfaces of degree d is of general type for $d > 61$ using quasi-pullbacks of Φ_{12}.

The first main result of this paper is the following characterisation (see Theorem 4.5).

The function Φ_{12} is the only holomorphic automorphic product of singular weight on a unimodular lattice.

Next, we consider lattices of prime level. We show that for a given discriminant form D of prime level, the number of lattices with dual quotient isomorphic to D carrying a holomorphic automorphic product of singular weight is finite and we give an explicit bound for the signature. The precise statement is as follows (see Theorems 5.7 and 5.12).

Let $c > 1/\log(\pi e/6) = 2.83309\ldots$. Then there exists a constant d with the following property: let L be an even lattice of signature $(n, 2)$, $n > 2$ and prime level splitting a hyperbolic plane $II_{1,1}$. Let D be the discriminant form of L. Suppose L carries a holomorphic automorphic product of singular weight. Then

$$n \leqslant c \log|D| + d.$$

The constant d does not depend on the level, but only on c. The proof is constructive. We can take, for example, $c = 3.59750\ldots$ and $d = 40.52171\ldots$. Given a discriminant form D of prime level, the theorem allows us to determine all holomorphic automorphic products of singular weight on lattices with dual quotient isomorphic to D by working out the obstruction theory in the possible signatures.

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We sketch the proofs of the first two main results. To obtain a restriction on the signature in the prime level case, we pair the vector valued modular form F associated to the automorphic product Ψ with an Eisenstein series for the dual Weil representation. We obtain a relation between the signature and a sum over the principal part of F. We expand this sum in the degrees of the divisors which are non-negative by the holomorphicity of Ψ. Then we apply the Riemann–Roch theorem to F to derive the bound. In the unimodular case, a similar argument gives the uniqueness.

The expansion of an automorphic form on $O_{n,2}(\mathbb{R})$ at a cusp is sometimes the denominator function of an infinite-dimensional Lie superalgebra. In that case, the divisor of the automorphic form is locally the sum of rational quadratic divisors $\alpha^\perp$ of order 1 where α is a root. An automorphic form on $O_{n,2}(\mathbb{R})$ is called reflective if this condition holds globally (see also [Bor99, GN02]). So far, all known examples of holomorphic automorphic products of singular weight are reflective.

In [Sch06], certain reflective automorphic products of singular weight on lattices of prime level are classified. The assumptions are that the underlying lattice L does not have maximal p-rank and that all roots of a fixed norm give zeros, i.e. the corresponding vector valued modular form is invariant under the orthogonal group of the discriminant form of L. The second condition is quite restrictive. Surprisingly we find only three additional cases when we remove these assumptions. This is the third main result of this paper (see Theorem 6.28).

Let L be a lattice of prime level and signature $(n,2)$ with $n>2$ and Ψ a reflective automorphic product of singular weight on L. Then, as a function on the corresponding Hermitian symmetric domain, Ψ is the theta lift of one of the following modular forms.

p	L	F	Co_0
2	$II_{18,2}(2_{II}^{+10})$	$F_{\eta_{1^{-8}2^{-8}},0}$	$1^8 2^8$
	$II_{10,2}(2_{II}^{+2})$	$F_{16\eta_{1^{-16}2^{8}},0}$	$1^{-8}2^{16}$
	$II_{10,2}(2_{II}^{+10})$	$F_{\eta_{1^{8}2^{-16}},0}$	$1^{-8}2^{16}$
	$II_{6,2}(2_{II}^{-6})$	$F_{\eta_{1^{4}2^{-8}},\gamma}$	$2^{-4}4^8$
3	$II_{14,2}(3^{-8})$	$F_{\eta_{1^{-6}3^{-6}},0}$	$1^6 3^6$
	$II_{8,2}(3^{-3})$	$F_{9\eta_{1^{-9}3^{3}},0}$	$1^{-3}3^9$
	$II_{8,2}(3^{-7})$	$F_{\eta_{1^{3}3^{-9}},0}$	$1^{-3}3^9$
	$II_{6,2}(3^{+6})$	$F_{(1/4)\eta_{(1/3)^{-3}1^{2}3^{-3}},M^+}$	$1^3 3^{-2} 9^3$
	$II_{4,2}(3^{-5})$	$F_{\eta_{1^{1}3^{-3}},\gamma}$	$3^{-1}9^3$
5	$II_{10,2}(5^{+6})$	$F_{\eta_{1^{-4}5^{-4}},0}$	$1^4 5^4$
	$II_{6,2}(5^{+3})$	$F_{5\eta_{1^{-5}5^{1}},0}$	$1^{-1}5^5$
	$II_{6,2}(5^{+5})$	$F_{\eta_{1^{1}5^{-5}},0}$	$1^{-1}5^5$
7	$II_{8,2}(7^{-5})$	$F_{\eta_{1^{-3}7^{-3}},0}$	$1^3 7^3$
11	$II_{6,2}(11^{-4})$	$F_{\eta_{1^{-2}11^{-2}},0}$	$1^2 11^2$
23	$II_{4,2}(23^{-3})$	$F_{\eta_{1^{-1}23^{-1}},0}$	$1^1 23^1$

With three exceptions, all of these functions come from symmetric modular forms. At a suitable cusp Ψ is the twisted denominator function of the fake monster algebra by the indicated element in Conway's group.

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Conversely, all the given modular forms lift to reflective automorphic products of singular weight on the respective lattices.

The cases not coming from symmetric modular forms are those corresponding to the elements of order 4 and 9 in Conway's group.

The sum expansion of the theta lift of $F_{(1/4)\eta_{(1/3)^{-3}1^2 3^{-3}},M^+}$ gives a new infinite product identity (see Proposition 6.23).

The above result can be used to classify generalised Kac–Moody superalgebras whose denominator functions are reflective automorphic products of singular weight on lattices of prime level.

We describe the proof of the theorem. Reflective automorphic products of singular weight associated to symmetric forms can be classified by the Eisenstein condition [Sch06]. It turns out that in the non-symmetric case the Riemann–Roch theorem imposes strong restrictions (see Theorem 6.5). In the remaining cases we work out the obstruction theory and determine the possible reflective modular forms. Many of them lift to the same function leaving us with the above list.

The paper is organised as follows. In § 2, we summarise some results on modular forms for the Weil representation. Then we recall Borcherds' singular theta correspondence and define reflective forms. In § 4, we prove that the only holomorphic automorphic product of singular weight on a unimodular lattice is the theta lift of $1/\Delta$ on $II_{26,2}$. Next, we show that holomorphic automorphic products of singular weight on lattices of prime level exist only in small signatures. Finally, we give a complete classification of reflective automorphic products of singular weight on lattices of prime level.

2. Modular forms for the Weil representation

In this section, we recall some results on modular forms for the Weil representation from [Sch09, Sch15].

Let D be a discriminant form with quadratic form $q : D \to \mathbb{Q}/\mathbb{Z}$ and associated bilinear form $(\ ,\ )$ (see [Sch09, Nik79] and [CS99, ch. 15]). We assume that D has even signature. The level of D is the smallest positive integer N such that $Nq(\gamma) = 0 \bmod 1$ for all $\gamma \in D$. We define a scalar product on the group ring $\mathbb{C}[D]$ which is linear in the first and antilinear in the second variable by $(e^\gamma, e^\beta) = \delta^{\gamma\beta}$. Then there is a unitary action of the group $\Gamma = \mathrm{SL}_2(\mathbb{Z})$ on $\mathbb{C}[D]$ satisfying

$$\rho_D(T)e^\gamma = e(-q(\gamma))e^\gamma,$$
$$\rho_D(S)e^\gamma = \frac{e(\mathrm{sign}(D)/8)}{\sqrt{|D|}} \sum_{\beta \in D} e((\gamma, \beta))e^\beta,$$

where $S = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$ and $T = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ are the standard generators of Γ. This representation is called the Weil representation of Γ on $\mathbb{C}[D]$. It commutes with the orthogonal group $O(D)$ of D. Suppose the level of D divides N and let $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(N)$. Then

$$\rho_D(M)e^\gamma = \left(\frac{a}{|D|}\right) e((a-1)\,\mathrm{oddity}(D)/8)e(-bdq(\gamma))e^{d\gamma}.$$

A general formula for the action of ρ_D is given in [Sch09, Theorem 4.7].

Let

$$F(\tau) = \sum_{\gamma \in D} F_\gamma(\tau)e^\gamma$$

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be a holomorphic function on the complex upper halfplane $\mathcal{H}$ with values in $\mathbb{C}[D]$ and k an integer. Then F is a modular form for ρ_D of weight k if

$$F(M\tau) = (c\tau + d)^k \rho_D(M) F(\tau)$$

for all $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$ and F is meromorphic at ∞. We say that F is symmetric if it is invariant under the action of $O(D)$.

Classical examples of modular forms for the dual Weil representation $\overline{\rho}_D$ are theta functions. Let L be a positive-definite even lattice of even rank $2k$ with discriminant form D. For $\gamma \in D$ define

$$\theta_\gamma(\tau) = \sum_{\alpha \in \gamma + L} q^{\alpha^2/2},$$

where $q^{\alpha^2/2} = e(\tau\alpha^2/2)$. Then

$$\theta = \sum_{\gamma \in D} \theta_\gamma e^\gamma$$

is a modular form for the dual Weil representation $\overline{\rho}_D$ of weight k which is holomorphic at ∞.

Let f be a complex function on $\mathcal{H}$ and k an integer. For $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$ we define the function $f|_{k,M}$ on $\mathcal{H}$ by $f|_{k,M}(\tau) = (c\tau + d)^{-k} f(M\tau)$.

We can easily construct modular forms for the Weil representation by symmetrising scalar-valued modular forms on congruence subgroups (see [Sch15, Theorem 3.1]).

Theorem 2.1. *Let D be a discriminant form of even signature and level dividing N.*

Let f be a scalar-valued modular form on $\Gamma_0(N)$ of weight k and character χ_D and H an isotropic subset of D that is invariant under $(\mathbb{Z}/N\mathbb{Z})^*$. Then

$$F_{\Gamma_0(N),f,H} = \sum_{M \in \Gamma_0(N)\backslash\Gamma} \sum_{\gamma \in H} f|_{k,M} \rho_D(M^{-1}) e^\gamma$$

is a modular form for ρ_D of weight k.

Let $\gamma \in D$ and f a scalar-valued modular form on $\Gamma_1(N)$ of weight k and character χ_γ. Then

$$F_{\Gamma_1(N),f,\gamma} = \sum_{M \in \Gamma_1(N)\backslash\Gamma} f|_{k,M} \rho_D(M^{-1}) e^\gamma$$

is a modular form for ρ_D of weight k.

Let f be a scalar-valued modular form on $\Gamma(N)$ of weight k and $\gamma \in D$. Then

$$F_{\Gamma(N),f,\gamma} = \sum_{M \in \Gamma(N)\backslash\Gamma} f|_{k,M} \rho_D(M^{-1}) e^\gamma$$

is a modular form for ρ_D of weight k.

Every modular form for ρ_D can be written as a linear combination of liftings from $\Gamma_1(N)$ or $\Gamma(N)$.

Explicit formulas for these function are given in [Sch15, § 3].

We also have the following proposition.

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Proposition 2.2. *Let D be a discriminant form of even signature and H an isotropic subgroup of D. Then $D_H = H^\perp/H$ is a discriminant form of the same signature as D.*

Let F_D be a modular form for ρ_D. For $\gamma \in H^\perp$ define

$$F_{D_H,\gamma+H} = \sum_{\beta\in\gamma+H} F_{D,\beta}.$$

Then F_{D_H} is a modular form for ρ_{D_H}.

Conversely, let F_{D_H} be a modular form for the Weil representation of D_H. Define

$$F_{D,\gamma} = F_{D_H,\gamma+H}$$

if $\gamma \in H^\perp$ and $F_{D,\gamma} = 0$ otherwise. Then F_D is a modular form for ρ_D.

We will need the Eisenstein series for the dual Weil representation. They can be constructed as follows. Let D be a discriminant form of even signature and level dividing N. Let $\Gamma_\infty^+ = \{T^n \mid n \in \mathbb{Z}\}$. Then

$$E_k = \frac{1}{2} \sum_{M\in\Gamma_\infty^+\backslash\Gamma_1(N)} 1|_{k,M}$$

is an Eisenstein series for $\Gamma_1(N)$ of weight k. Let $\gamma \in D$ be isotropic. Then

$$E_\gamma = \sum_{M\in\Gamma_1(N)\backslash\Gamma} E_k|_{k,M}\overline{\rho}_D(M^{-1})e^\gamma$$

is an Eisenstein series for the dual Weil representation $\overline{\rho}_D$. It is easy to see that E_γ gives the Eisenstein series defined in [Bru02]. For $\gamma = 0$ we have

$$E_0 = \sum_{M\in\Gamma_0(N)\backslash\Gamma} E_{k,\chi}|_{k,M}\overline{\rho}_D(M^{-1})e^0,$$

where

$$E_{k,\chi} = \sum_{M\in\Gamma_1(N)\backslash\Gamma_0(N)} \chi(M)E_k|_{k,M}$$

is an Eisenstein series for $\Gamma_0(N)$ of weight k and character $\overline{\chi} = \chi = \chi_D$. We will write E for the Eisenstein series E_0.

The dimension of the space of holomorphic modular forms for the Weil representation can be worked out using the Riemann–Roch theorem [Fre12] or the Selberg trace formula [ES95, Bor00].

The residue theorem implies the following result.

Proposition 2.3. *Let D be a discriminant form of even signature and F a modular form for ρ_D of weight $2-k$ with $k \geqslant 3$. Let G be a modular form for $\overline{\rho}_D$ of weight k. Then the constant coefficient of $(F,\overline{G}) = \sum_{\gamma\in D} F_\gamma G_\gamma$ vanishes.*

More generally we have (see [Bor99, Theorem 3.1] and [Bru02, Theorem 1.17]) the following theorem.

Theorem 2.4. *Let* $P = \sum_{\gamma \in D} P_\gamma e^\gamma$, *where*

$$P_\gamma = \sum_{\substack{n \in \mathbb{Z} - q(\gamma) \\ n<0}} c_\gamma(n) q^n$$

is a finite Fourier polynomial with complex coefficients. Then P *is the principal part of a modular form of weight* $2-k$, $k \geqslant 3$, *for* ρ_D *if and only if the linear map*

$$\begin{array}{rccl} \phi_P : & S_{\overline{\rho}_D,k} & \longrightarrow & \mathbb{C} \\ & G & \longmapsto & \text{constant coefficient of } (P, \overline{G}) \end{array}$$

vanishes on $S_{\overline{\rho}_D,k}$.

We will use Theorem 2.1 to work out the obstruction spaces $S_{\overline{\rho}_D,k}$ in several cases in § 6.

3. Automorphic products

We describe some properties of automorphic products [Bor98] and define reflective automorphic products.

Let L be an even lattice of signature $(n, 2)$, $n > 2$ even, $V = L \otimes_{\mathbb{Z}} \mathbb{R}$ and $V(\mathbb{C}) = V \otimes_{\mathbb{R}} \mathbb{C}$. Then

$$\mathcal{K} = \{Z \in V(\mathbb{C}) \mid (Z, Z) = 0, (Z, \overline{Z}) < 0\}$$

is a complex manifold with two connected components that are exchanged by the map $Z \mapsto \overline{Z}$. We choose one of the components and denote it by $\mathcal{H}$. There is a subgroup $O(V)^+$ of index 2 in the orthogonal group $O(V)$, which preserves the two connected components of $\mathcal{K}$. This group acts holomorphically on $\mathcal{H}$.

Let Γ be a finite index subgroup of $O(L)^+$ and $\chi : \Gamma \to \mathbb{C}^*$ a unitary character. Since the abelianisation of Γ is finite, χ has finite order. Let k be an integer. A meromorphic function $\Psi : \mathcal{H} \to \mathbb{C}$ is called an automorphic form of weight k for Γ with character χ if

$$\Psi(MZ) = \chi(M)\Psi(Z),$$
$$\Psi(tZ) = t^{-k}\Psi(Z)$$

for all $M \in \Gamma$ and $t \in \mathbb{C}^*$.

The weight of a holomorphic automorphic form is bounded below (see [Bor95, Corollary 3.3]).

Proposition 3.1. *Let* L *be an even lattice of signature* $(n, 2)$, $n > 2$ *even and rational Witt rank 2. Let* Ψ *be a non-constant holomorphic automorphic form of weight* k *for the discriminant kernel of* $O(L)^+$. *Then,* $k \geqslant (n-2)/2$. *If* Ψ *has weight* $(n-2)/2$, *then the non-vanishing Fourier coefficients correspond to isotropic vectors.*

The weight $(n-2)/2$ is called the singular weight.

Let L be an even lattice of signature $(n, 2)$, $n > 2$ even with discriminant form D. Let F be a modular form for the Weil representation of Γ on $\mathbb{C}[D]$ of weight $1 - n/2$ with integral principal part. We denote the Fourier coefficients of F by $c_\gamma(n)$ and assume that $c_0(0)$ is even. Then Borcherds' singular theta correspondence [Bor98, Theorem 13.3] associates an automorphic form Ψ to F.

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Theorem 3.2. *There is a meromorphic function $\Psi : \mathcal{H} \to \mathbb{C}$ with the following properties.*

(1) *The function Ψ is an automorphic form of weight $c_0(0)/2$ for the group $O(L,F)^+$.*

(2) *The only zeros or poles of Ψ lie on rational quadratic divisors $\gamma^\perp$ where γ is a primitive vector of positive norm in L'. The divisor $\gamma^\perp$ has order*

$$\sum_{m>0} c_{m\gamma}(-m^2\gamma^2/2).$$

(3) *For each primitive isotropic vector z in L and for each Weyl chamber W of $K = (L \cap z^\perp)/\mathbb{Z}z$ the restriction Ψ_z has an infinite product expansion converging in a neighbourhood of the cusp corresponding to z that is up to a constant*

$$e((Z,\rho)) \prod_{\alpha \in K'^+} \prod_{\substack{\gamma \in L'/L \\ \gamma|_{(L\cap z^\perp)} = \alpha}} (1 - e((\gamma, z') + (\alpha, Z)))^{c_\gamma(-\alpha^2/2)}.$$

The function Ψ is called the automorphic product corresponding to F.

Bruinier proved the following converse theorem [Bru14, Theorem 1.2].

Theorem 3.3. *Let L be an even lattice of signature $(n,2)$, $n > 2$ and even and Ψ an automorphic form for the discriminant kernel of $O(L)^+$ whose divisor is a linear combination of rational quadratic divisors. If $L = K \oplus II_{1,1} \oplus II_{1,1}(m)$ for some positive integer m, then up to a constant factor the function Ψ is the theta lift of a modular form for the Weil representation of L.*

Let L and F_L be as above. Suppose $L = K \oplus II_{1,1}(m)$ for some positive integer m. Let M be a finite index sublattice of K. Then $H = K/M \subset K'/M \subset M'/M$ is an isotropic subgroup of the discriminant form of M with orthogonal complement $H^\perp = K'/M$. Note that $H^\perp/H$ is naturally isomorphic to K'/K. The function F_L induces a modular form F_N on $N = M \oplus II_{1,1}(m)$. The embedding $N \to L$ gives an identification of the domains $\mathcal{H}_N$ and $\mathcal{H}_L$.

Proposition 3.4. *Under this identification, the automorphic products $\Psi(F_L)$ and $\Psi(F_N)$ coincide as functions on $\mathcal{H}_L$.*

Proof. We choose a primitive norm 0 vector z in $II_{1,1}(m)$. Then, the product expansion of $\Psi(F_N)$ at the cusp corresponding to z is given by

$$\Psi(F_N)_z(Z) = c_N e((\rho_N, Z)) \prod_{\alpha \in M'^+} \prod_{j \in \mathbb{Z}/m\mathbb{Z}} (1 - e(j/m)e((\alpha, Z)))^{c_{N,\alpha+jz/m}(-\alpha^2/2)}.$$

The components $F_{N,\alpha+jz/m}$ of F_N vanish unless $\alpha \in H^\perp$ and $F_{N,\alpha+jz/m} = F_{L,(\alpha+H)+jz/m}$ in that case. It follows

$$\Psi(F_N)_z(Z) = c_N e((\rho_N, Z)) \prod_{\alpha \in K'^+} \prod_{j \in \mathbb{Z}/m\mathbb{Z}} (1 - e(j/m)e((\alpha, Z)))^{c_{L,\alpha+jz/m}(-\alpha^2/2)}.$$

This implies

$$\Psi(F_N)_z(Z) = \frac{c_N}{c_L} \Psi(F_L)_z(Z).$$

It is not difficult to see that $c_N/c_L = 1$. Hence, $\Psi(F_N)$ and $\Psi(F_L)$ coincide in a neighbourhood of the cusp z and, therefore, coincide on $\mathcal{H}_L$. □

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Let L be an even lattice of signature $(n,2)$, $n>2$ even with discriminant form D. A root of L is a primitive vector α of positive norm in L such that the reflection $\sigma_\alpha(x) = x - 2(x,\alpha)\alpha/\alpha^2$ is in $O(L)$. Let $\gamma \in D$ be of norm $q(\gamma) = 1/k \bmod 1$ for some positive integer k. We say that γ corresponds to roots if *the order of* γ *divides* k and *if there is a vector* $\alpha \in L \cap kL'$ *of norm* $\alpha^2 = 2k$ *with* $\alpha/k = \gamma \bmod L$ *then* α *is a root*. Let F be a modular form for the Weil representation of L. The function F is called reflective if F has weight $1-n/2$ and the only singular terms of F come from components F_γ with γ corresponding to roots of L and are of the form $q^{-1/k}$. An automorphic product Ψ on L is called reflective if it is the theta lift of a reflective modular form F. The divisor of Ψ has a nice geometric description in this case (see [Sch06, § 9]).

PROPOSITION 3.5. *Let* Ψ *be a reflective automorphic product on* L. *Then,* Ψ *is holomorphic and its zeros are zeros of order* 1 *at the rational quadratic divisors* $\alpha^\perp$ *where* α *is a root of* L *with* $\alpha^2 = 2k$ *and* $c_{\alpha/k}(-1/k) = 1$.

4. Singular weight forms on unimodular lattices

In this section, we show that the function Φ_{12} is the only holomorphic automorphic product of singular weight on a unimodular lattice.

Let L be an even unimodular lattice of signature $(n,2)$ with $n>2$ and let $\Psi(F)$ be a holomorphic automorphic product of singular weight on L.

Since L is unimodular, we have that $n = 2 \bmod 8$. By assumption the modular form F has weight $1-n/2$, is holomorphic on $\mathcal{H}$ and has a finite order pole at ∞. We write

$$F(\tau) = \sum_{m\in\mathbb{Z}} c(m)q^m$$

with $c(0) = n-2$ and define $m_\infty = -\nu_\infty(F)$, i.e. m_∞ is the largest integer such that $c(-m_\infty) \neq 0$. The coefficients $c(-m)$, $m>0$ of the principal part of F are integral.

Let

$$E_k(\tau) = 1 - \frac{2k}{B_k}\sum_{m>0}\sigma_{k-1}(m)q^m$$

be the Eisenstein series of weight $k = 1+n/2$ for Γ. Pairing F with E_k (see Proposition 2.3) we obtain the following result.

PROPOSITION 4.1. *The principal part of* F *satisfies*

$$2(k-2) - \frac{2k}{B_k}\sum_{m>0} c(-m)\sigma_{k-1}(m) = 0.$$

This result restricts the possible values of k.

PROPOSITION 4.2. *We have* $k = 2 \bmod 12$.

Proof. The previous proposition implies $(k-2)B_k \in \mathbb{Z}$. The von Staudt–Clausen theorem states that

$$B_k + \sum_{(p-1)|k}\frac{1}{p} \in \mathbb{Z}.$$

Hence, $(k-2)\sum_{(p-1)|k}(1/p) \in \mathbb{Z}$ and $k-2 = 0 \bmod 3$. The assertion now follows from the condition on n. □

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The modular form $F\Delta^{(k-2)/12}$ has weight 0, is holomorphic on H and possibly has a pole at ∞. Hence,

$$m_\infty \geqslant \frac{k-2}{12}.$$

The divisor of $\Psi(F)$ is a linear combination of rational quadratic divisors $\gamma^\perp$ where γ is a primitive vector of positive norm in L. The order of $\gamma^\perp$ is $\sum_{m>0} c(-m^2\gamma^2/2)$. The holomorphicity of $\Psi(F)$ does not imply that the coefficients of the principal part of F are non-negative. However, the function g on the positive integers defined by

$$g(d) = \sum_{m>0} c(-dm^2)$$

is non-negative because the lattice L splits a hyperbolic plane $II_{1,1}$ and therefore contains primitive vectors of arbitrary norm.

Theorem 4.3. *The principal part of F satisfies the inequality*

$$\sum_{m>0} c(-m)\sigma_{k-1}(m) \geqslant m_\infty^{k-1}.$$

Proof. We have

$$\begin{aligned}
\sum_{m>0} c(-m)\sigma_{k-1}(m) &= \sum_{m>0} c(-m) \sum_{d|m} d^{k-1} \\
&= \sum_{d>0} d^{k-1} \sum_{d|m} c(-m) \\
&= \sum_{d>0} d^{k-1} \sum_{t>0} c(-td) \\
&= \sum_{d>0} d^{k-1} \sum_{\substack{m>0 \\ t \text{ squarefree}}} c(-m^2td) \\
&= \sum_{d>0} d^{k-1} \sum_{t \text{ squarefree}} g(td) \\
&= \sum_{m>0} g(m) \sum_{\substack{d|m \\ m/d \text{ squarefree}}} d^{k-1}
\end{aligned}$$

so that

$$\sum_{m>0} c(-m)\sigma_{k-1}(m) \geqslant g(m_\infty)m_\infty^{k-1} = c(-m_\infty)m_\infty^{k-1} \geqslant m_\infty^{k-1}.$$

This proves the theorem. □

We obtain the inequalities

$$\left(\frac{k-2}{12}\right)^{k-1} \leqslant m_\infty^{k-1} \leqslant \frac{k-2}{k}B_k.$$

Note that $k = 2 \bmod 4$ implies that the Bernoulli numbers B_k are positive.

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PROPOSITION 4.4. *The only solution of the inequality*

$$\left(\frac{k-2}{12}\right)^{k-1} \leqslant \frac{k-2}{k} B_k$$

with $k>2$ *and* $k=2 \bmod 12$ *is* $k=14$. *In this case, equality holds.*

Proof. We can write the inequality as

$$1 \leqslant \frac{(k-2)^2}{12k}\left(\frac{12}{k-2}\right)^k B_k.$$

For $k \to \infty$ we have $B_k \sim 2(k!/(2\pi)^k)$ and $k! \sim \sqrt{2\pi k}(k/e)^k$ so that

$$\begin{aligned}\frac{(k-2)^2}{12k}\left(\frac{12}{k-2}\right)^k B_k &\sim 2\sqrt{2\pi k}\frac{(k-2)^2}{12k}\left(\frac{k}{k-2}\right)^k\left(\frac{6}{\pi e}\right)^k \\ &\sim \frac{1}{6}\sqrt{2\pi} e^2 k^{3/2}\left(\frac{6}{\pi e}\right)^k.\end{aligned}$$

Since $\pi e > 6$, the last expression tends to 0 as $k \to \infty$. Hence, the inequality has only finitely many solutions. It is easy to verify that $k=14$ is the only solution. □

Now, the classification result follows.

THEOREM 4.5. *Let* L *be an even unimodular lattice of signature* $(n,2)$ *with* $n>2$ *and let* $\Psi(F)$ *be a holomorphic automorphic product of singular weight on* L. *Then* $n=26$ *and* $F=1/\Delta$. *The expansion of* Ψ *at a cusp is given by*

$$e((\rho,Z)) \prod_{\alpha \in II_{25,1}^+} (1-e((\alpha,Z)))^{[1/\Delta](-\alpha^2/2)} = \sum_{w \in W} \det(w)\Delta((w\rho,Z)).$$

Proof. We have $k=14$ and $m_\infty = 1$. Hence,

$$F(\tau) = q^{-1} + 24 + \cdots$$

by Proposition 4.1. Since F is holomorphic on $\mathcal{H}$ we obtain $F = 1/\Delta$. □

We conclude this section with some examples.

Let

$$j(\tau) = q^{-1} + 744 + 196884q + 21493760q^2 + \cdots$$

be the modular invariant. Then the function

$$\begin{aligned}F(\tau) &= (j(\tau)^3 - 2256j(\tau)^2 + 1105920j(\tau) - 40890369)/\Delta(\tau) \\ &= q^{-4} - q^{-1} + 1610809344 + 11828339932860q + \cdots \\ &= \sum_{m \in \mathbb{Z}} c(m) q^m\end{aligned}$$

is a modular form of weight -12 for Γ, holomorphic on $\mathcal{H}$ with a pole of order 4 at ∞. Note that the coefficient $c(-1) = -1$ of the principal part of F is negative. Let L be an even unimodular lattice of signature $(26,2)$ and $\Psi(F)$ the automorphic product corresponding to F on L. Then $\Psi(F)$ is a holomorphic automorphic form of weight 805404672 whose zeros are zeros of order 1 at the divisors $\gamma^\perp$ where γ is a primitive vector of norm $\gamma^2 = 8$ in L. If γ is a vector of norm $\gamma^2 = 2$ in L, then the divisor $\gamma^\perp$ has order $c(-4) + c(-1) = 0$.

Next, we consider non-holomorphic automorphic products.

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Proposition 4.6. *Let L be an even unimodular lattice of signature $(n,2)$ with*

$$n = 26, 50, 74, 122, 146, 170 \text{ or } 194.$$

Then, L carries infinitely many meromorphic automorphic products of weight 12.

Proof. First, we consider the case $n = 26$. Let $F = (aj + b)/\Delta$ with $a, b \in \mathbb{Z}$. Then

$$F(\tau) = aq^{-2} + (768a + b)q^{-1} + (215064a + 24b) + \cdots .$$

Since $(215064, 24) = 24$ there are infinitely many choices for a and b such that F has constant coefficient 24. This implies that there are infinitely many meromorphic automorphic products of weight 12 on L. In the general case, write $n = 24m + 2$ and let

$$F = (a_m j^m + \cdots + a_1 j + a_0)/\Delta^m.$$

Then there are infinitely many $(a_0, \ldots, a_m) \in \mathbb{Z}^{m+1}$ such that F has constant coefficient 24. □

We explain the exception at $n = 98$. Let

$$F(\tau) = \sum_{m\in\mathbb{Z}} c(m)q^m$$

be a modular form of weight $1 - 98/2 = -48$ for Γ, holomorphic on $\mathcal{H}$ with a pole at ∞. Suppose F has integral principal part. Since the Eisenstein series E_{10} has Fourier expansion

$$E_{10}(\tau) = 1 - 264 \sum_{m>0} \sigma_9(m)q^m$$

the constant coefficient of FE_{10}^5 is given by $c(0) + 264(\cdots)$. This coefficient has to vanish so that $c(0) = 0 \bmod 264$. This implies that the weight of a meromorphic automorphic product on a unimodular lattice of signature $(98, 2)$ is divisible by 132.

Finally, we remark that lifting constants with Gritsenko's additive lift [Gri91] (see also [Bor98, Theorem 14.3]) shows that holomorphic automorphic forms of singular weight exist on any unimodular lattice of signature $(n,2)$ with $n > 2$. By Theorem 3.3, the divisor of such a function is not a linear combination of rational quadratic divisors.

5. The prime level case

Let L be an even lattice of prime level carrying a holomorphic automorphic product of singular weight. We derive an explicit bound for the signature of L.

We consider the cases of even and odd p-ranks separately.

5.1 Even p-rank

Let L be an even lattice of prime level p and genus $II_{n,2}(p^{\epsilon_p n_p})$ with $n > 2$ and n_p even carrying a holomorphic automorphic product $\Psi(F)$ of singular weight.

Let D be the discriminant form of L. The oddity formula (see [CS99, ch. 15, § 7.7])

$$e(\operatorname{sign}(D)/8) = \gamma_p(D)$$

implies

$$e((n-2)/8) = \epsilon_p \left(\frac{-1}{p}\right)^{n_p/2}.$$

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Hence, $n = \pm 2 \bmod 8$ and $k = 1 + n/2$ is an even integer. Note that $k \geqslant 4$. Define

$$\xi = \epsilon_p \left(\frac{-1}{p} \right)^{n_p/2} = -(-1)^{k/2}.$$

Let E be the Eisenstein series of weight k for $\overline{\rho}_D$ corresponding to 0. Write

$$E = \sum_{\gamma \in D} E_\gamma e^\gamma$$

with

$$E_\gamma(\tau) = \sum_{m \in \mathbb{Z} + q(\gamma)} a_\gamma(m) q^m.$$

Define

$$c_{k,p,n_p} = \xi \frac{2k}{B_k} \frac{1}{p^k - 1} \frac{1}{p^{(n_p-2)/2}}.$$

Note that c_{k,p,n_p} is positive. By explicit calculation we can derive the following formulas for the Fourier coefficients $a_\gamma(m)$ (see also [Sch06, Theorem 7.1]).

Proposition 5.1. *Let $\gamma \in D$ and $m \in q(\gamma) + \mathbb{Z}$, $m > 0$.*
If $q(\gamma) \neq 0 \bmod 1$, then

$$a_\gamma(m) = -c_{k,p,n_p} \sigma_{k-1}(pm).$$

Suppose $q(\gamma) = 0 \bmod 1$. Write $m = p^\nu a$ with $(a, p) = 1$. Then

$$a_\gamma(m) = -c_{k,p,n_p} p^{(\nu+1)(k-1)} \sigma_{k-1}(a)$$

if $\gamma \neq 0$ and

$$a_\gamma(m) = -c_{k,p,n_p} p^{(\nu+1)(k-1)} \sigma_{k-1}(a) + \xi c_{k,p,n_p} p^{n_p/2} \sigma_{k-1}(a) - \xi c_{k,p,n_p} p^{(n_p-2)/2} (p-1) \sigma_{k-1}(m)$$

if $\gamma = 0$.

Write

$$F = \sum_{\gamma \in D} F_\gamma e^\gamma$$

with

$$F_\gamma(\tau) = \sum_{m \in \mathbb{Z} - q(\gamma)} c_\gamma(m) q^m.$$

Pairing F with the Eisenstein series E (see Proposition 2.3) we obtain

$$2(k-2) + \sum_{\gamma \in D} \sum_{m > 0} c_\gamma(-m) a_\gamma(m) = 0.$$

In the following, we will often need that L splits a hyperbolic plane $II_{1,1}$. We give a criterion for this.

Proposition 5.2. *The lattice L splits a hyperbolic plane $II_{1,1}$ if and only if*

$$n_p = n \quad \text{and} \quad \xi = +1$$

or

$$n_p \leqslant n - 2.$$

Proof. Suppose L splits $II_{1,1}$, i.e. $II_{n,2}(p^{\epsilon_p n_p}) = II_{n-1,1}(p^{\epsilon_p n_p}) \oplus II_{1,1}$. If $n_p \leqslant n-2$, this gives no restriction on ϵ_p. If $n_p = n$, then the sign rule (see [CS99, ch. 15, § 7.7]) applied to $II_{n-1,1}(p^{\epsilon_p n_p})$ implies $\epsilon_p = (-1/p)$ so that

$$\xi = \epsilon_p \left(\frac{-1}{p}\right)^{n_p/2} = \left(\frac{-1}{p}\right)^{1+n/2} = +1.$$

The converse is now clear. □

Let d be a positive rational number such that pd is integral. We define functions

$$g_\gamma(d) = \sum_{m>0} c_{m\gamma}(-m^2 d),$$

where we assume m to be integral. We have

$$g_\gamma(d) = g_0(p^2 d) + \sum_{(m,p)=1} c_{m\gamma}(-m^2 d).$$

This implies

$$g_\gamma(d) = g_0(p^2 d)$$

if $q(\gamma) \neq d \bmod 1$.

The divisor of $\Psi(F)$ is a linear combination of rational quadratic divisors $\gamma^\perp$, where γ is a primitive vector of positive norm in L. The divisor $\gamma^\perp$ has order $\sum_{m>0} c_{m\gamma}(-m^2\gamma^2/2)$. Since $\Psi(F)$ is holomorphic this is a non-negative integer.

Proposition 5.3. *Suppose L splits a hyperbolic plane $II_{1,1}$. Then*

$$g_\gamma(d) \geqslant 0$$

for all $\gamma \in D$.

Proof. By the above remark we can assume that $d = q(\gamma) \bmod 1$. Write $L = M \oplus II_{1,1}$. Choose a representative of γ in M'. By adding a primitive element of suitable norm in $II_{1,1}$ we obtain a primitive element $\gamma \in L'$ of norm $\gamma^2/2 = d$. The holomorphicity of $\Psi(F)$ implies that

$$g_\gamma(d) = \sum_{m>0} c_{m\gamma}(-m^2 d) = \sum_{m>0} c_{m\gamma}(-m^2\gamma^2/2) \geqslant 0.$$

This proves the proposition. □

We also define the multiplicative function

$$h(m) = \sum_{\substack{d|m \\ m/d \text{ squarefree} \\ (m/d,p)=1}} d^{k-1}.$$

Now we expand the sum $-\sum_{\gamma\in D}\sum_{m>0} c_\gamma(-m) a_\gamma(m)$ in terms of the non-negative divisor degrees g_γ.

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THEOREM 5.4. *Suppose L splits $II_{1,1}$. Let $c_p = 1 - 1/p$. Then*

$$-\sum_{\gamma\in D}\sum_{m>0} c_\gamma(-m)a_\gamma(m) \geqslant c_{k,p,n_p} \sum_{\substack{\gamma\in D \\ q(\gamma)\neq 0 \bmod 1}} \sum_{m/p=q(\gamma) \bmod 1} g_\gamma(m/p)h(m) + c_{k,p,n_p}p^{k-1} \sum_{\substack{\gamma\in D\backslash\{0\} \\ q(\gamma)=0 \bmod 1}} \sum_{m>0} g_\gamma(m)h(m) + c_p c_{k,p,n_p}p^{k-1}\sum_{m>0} g_0(m)h(m).$$

Proof. Let $\gamma \in D$ with $q(\gamma) \neq 0 \bmod 1$. Then

$$\begin{aligned}
-\sum_{j=1}^{p-1}\sum_{m>0} c_{j\gamma}(-m)a_{j\gamma}(m) &= c_{k,p,n_p}\sum_{j=1}^{p-1}\sum_{(m,p)=1} c_{j\gamma}(-m/p)\sum_{d|m} d^{k-1} \\
&= c_{k,p,n_p}\sum_{(d,p)=1} d^{k-1}\sum_{j=1}^{p-1}\sum_{(t,p)=1} c_{j\gamma}(-td/p) \\
&= c_{k,p,n_p}\sum_{(d,p)=1} d^{k-1}\sum_{j=1}^{p-1}\sum_{\substack{t \text{ squarefree} \\ (t,p)=1}}\sum_{(m,p)=1} c_{j\gamma}(-m^2td/p) \\
&= c_{k,p,n_p}\sum_{(d,p)=1} d^{k-1}\sum_{j=1}^{p-1}\sum_{l=1}^{p-1}\sum_{\substack{t \text{ squarefree} \\ (t,p)=1}}\sum_{m=l \bmod p} c_{j\gamma}(-m^2td/p) \\
&= c_{k,p,n_p}\sum_{(d,p)=1} d^{k-1}\sum_{j=1}^{p-1}\sum_{l=1}^{p-1}\sum_{\substack{t \text{ squarefree} \\ (t,p)=1}}\sum_{m=l \bmod p} c_{lj\gamma}(-m^2td/p) \\
&= c_{k,p,n_p}\sum_{(d,p)=1} d^{k-1}\sum_{j=1}^{p-1}\sum_{\substack{t \text{ squarefree} \\ (t,p)=1}} (g_{j\gamma}(td/p) - g_0(tdp)) \\
&= c_{k,p,n_p}\sum_{j=1}^{p-1}\sum_{(m,p)=1} (g_{j\gamma}(m/p) - g_0(mp))\sum_{\substack{d|m \\ m/d \text{ squarefree}}} d^{k-1} \\
&= c_{k,p,n_p}\sum_{j=1}^{p-1}\sum_{m/p=q(j\gamma) \bmod 1} (g_{j\gamma}(m/p) - g_0(mp))h(m).
\end{aligned}$$

For $\gamma \in D\backslash\{0\}$ with $q(\gamma) = 0 \bmod 1$ we find analogously

$$-\sum_{j=1}^{p-1}\sum_{m>0} c_{j\gamma}(-m)a_{j\gamma}(m) = c_{k,p,n_p}p^{k-1}\sum_{j=1}^{p-1}\sum_{m>0}(g_{j\gamma}(m) - g_0(mp^2))h(m).$$

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For $\gamma = 0$, we have

$$\begin{aligned}
-\sum_{m>0} c_\gamma(-m)a_\gamma(m) &= c_{k,p,n_p}p^{k-1}\sum_{d>0} d^{k-1} \sum_{(t,p)=1} c_0(-td) \\
&\quad - \xi c_{k,p,n_p}p^{n_p/2} \sum_{(d,p)=1} d^{k-1} \sum_{t>0} c_0(-td) \\
&\quad + \xi c_{k,p,n_p}p^{(n_p-2)/2}(p-1)\sum_{d>0} d^{k-1}\sum_{t>0} c_0(-td) \\
&= c_{k,p,n_p}p^{k-1}\sum_{d>0} d^{k-1} \sum_{\substack{t \text{ squarefree} \\ (t,p)=1}} (g_0(td) - g_0(tdp^2)) \\
&\quad - \xi c_{k,p,n_p}p^{n_p/2} \sum_{(d,p)=1} d^{k-1} \sum_{\substack{t \text{ squarefree} \\ (t,p)=1}} (g_0(td) + g_0(tdp)) \\
&\quad + \xi c_{k,p,n_p}p^{(n_p-2)/2}(p-1)\sum_{d>0} d^{k-1} \sum_{\substack{t \text{ squarefree} \\ (t,p)=1}} (g_0(td) + g_0(tdp)) \\
&= c_{k,p,n_p}p^{k-1}\sum_{m>0}(g_0(m) - g_0(mp^2))h(m) \\
&\quad - \xi c_{k,p,n_p}p^{n_p/2} \sum_{(m,p)=1} (g_0(m) + g_0(mp))h(m) \\
&\quad + \xi c_{k,p,n_p}p^{(n_p-2)/2}(p-1)\sum_{m>0}(g_0(m) + g_0(mp))h(m).
\end{aligned}$$

Using

$$\begin{aligned}
\sum_{m>0} g_0(m)h(m) &= \sum_{(m,p)=1} g_0(m)h(m) + p^{k-1}\sum_{(m,p)=1} g_0(mp)h(m) \\
&\quad + p^{2(k-1)}\sum_{m>0} g_0(mp^2)h(m)
\end{aligned}$$

and

$$\sum_{m>0} g_0(mp)h(m) = \sum_{(m,p)=1} g_0(mp)h(m) + p^{k-1}\sum_{m>0} g_0(mp^2)h(m),$$

we find

$$\begin{aligned}
-\sum_{\gamma\in D}\sum_{m>0} c_\gamma(-m)a_\gamma(m) &= c_{k,p,n_p}\sum_{\substack{\gamma\in D \\ q(\gamma)\neq 0 \bmod 1}} \sum_{m/p=q(\gamma) \bmod 1} g_\gamma(m/p)h(m) \\
&\quad + c_{k,p,n_p}p^{k-1}\sum_{\substack{\gamma\in D\setminus\{0\} \\ q(\gamma)=0 \bmod 1}} \sum_{m>0} g_\gamma(m)h(m) \\
&\quad + c_{k,p,n_p}c^0_{k,p,n_p}\sum_{(m,p)=1} g_0(m)h(m)
\end{aligned}$$

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$$+ c_{k,p,n_p} \sum_{j=1}^{p-1} c^1_{k,n_p,j} \sum_{m=j \bmod p} g_0(mp)h(m)$$
$$+ c_{k,p,n_p} c^{\geqslant 2}_{k,p,n_p} \sum_{m>0} g_0(mp^2)h(m)$$

with

$$\begin{aligned} c^0_{k,p,n_p} &= p^{k-1} - \xi p^{(n_p-2)/2}, \\ c^1_{k,p,n_p,j} &= p^n - a_{k,p,n_p,j} + \xi p^{(n_p-2)/2}((p-1)p^{k-1} - 1), \\ c^{\geqslant 2}_{k,p,n_p} &= p^{k-1}(p^n - a_{k,p,n_p,0} + \xi p^{(n_p-2)/2}(p-1)(p^{k-1}+1)), \end{aligned}$$

where $a_{k,p,n_p,j}$ denotes the number of elements $\gamma \in D$ of norm $q(\gamma) = j/p \bmod 1$. For $j \neq 0 \bmod p$ we have

$$a_{k,p,n_p,j} = p^{n_p-1} - \xi p^{(n_p-2)/2}$$

(see [Sch06, Proposition 3.2]) so that

$$\begin{aligned} c^1_{k,p,n_p,j} &= p^n - p^{n_p-1} + \xi p^{(n+n_p-2)/2}(p-1), \\ c^{\geqslant 2}_{k,p,n_p} &= p^{k-1}(p^n - p^{n_p-1} + \xi p^{(n+n_p-2)/2}(p-1)). \end{aligned}$$

Since L splits $II_{1,1}$, we obtain the following bounds

$$\begin{aligned} c^0_{k,p,n_p} &\geqslant (1-1/p)p^{k-1}, \\ c^1_{k,p,n_p,j} &\geqslant (1-1/p)p^{2(k-1)}, \\ c^{\geqslant 2}_{k,p,n_p} &\geqslant (1-1/p)p^{3(k-1)}. \end{aligned}$$

Applying the above formula for $\sum_{m>0} g_0(m)h(m)$ once more, we obtain

$$\begin{aligned} -\sum_{\gamma \in D} \sum_{m>0} c_\gamma(-m)a_\gamma(m) \geqslant{}& c_{k,p,n_p} \sum_{\substack{\gamma \in D \\ q(\gamma) \neq 0 \bmod 1}} \sum_{m/p = q(\gamma) \bmod 1} g_\gamma(m/p)h(m) \\ &+ c_{k,p,n_p} p^{k-1} \sum_{\substack{\gamma \in D \setminus \{0\} \\ q(\gamma) = 0 \bmod 1}} \sum_{m>0} g_\gamma(m)h(m) \\ &+ c_p c_{k,p,n_p} p^{k-1} \sum_{m>0} g_0(m)h(m). \end{aligned}$$

This proves the theorem. □

Define $m_\infty = \max_{\gamma \in D}(-\nu_\infty(F_\gamma))$. Note that $m_\infty > 0$.

Proposition 5.5. *Suppose L splits $II_{1,1}$. Then*

$$m_\infty \geqslant \frac{k-2}{12}.$$

Let $\gamma \in D$ such that $\nu_\infty(F_\gamma) = -m_\infty$. Then $c_\gamma(-m_\infty)$ is a positive integer.

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Proof. The function F_0 is a non-zero modular form for $\Gamma_0(p)$ of weight $2-k$. Applying the Riemann–Roch theorem to F_0 we obtain

$$p\nu_0(F_0)+\nu_\infty(F_0)\leqslant -\frac{k-2}{12}(p+1)$$

(see [HBJ94, Theorem 4.1]). The formula for the S-transformation (see § 2) implies

$$\nu_0(F_0)=\nu_\infty\bigg(\sum_{\gamma\in D}F_\gamma\bigg).$$

Let $\gamma\in D$ such that $\nu_\infty(F_\gamma)$ is minimal. Since L splits $II_{1,1}$, there is a primitive vector μ in L' with $\mu=\gamma \bmod L$ and $\mu^2/2=m_\infty$. Then, the divisor $\mu^\perp$ has order $c_\gamma(-m_\infty)$ which is a positive integer by the holomorphicity of $\Psi(F)$. Hence,

$$\nu_\infty\bigg(\sum_{\gamma\in D}F_\gamma\bigg)=\min_{\gamma\in D}\nu_\infty(F_\gamma).$$

It follows

$$p\min_{\gamma\in D}\nu_\infty(F_\gamma)+\min_{\gamma\in D}\nu_\infty(F_\gamma)\leqslant -\frac{k-2}{12}(p+1).$$

This completes the proof. □

We obtain the following inequalities.

Proposition 5.6. *Suppose L splits $II_{1,1}$. Then*

$$\bigg(\frac{k-2}{12}\bigg)^{k-1}\leqslant m_\infty^{k-1}\leqslant \xi\frac{p^{n_p/2}}{c_p}\frac{k-2}{k}B_k.$$

Proof. Suppose $\nu_\infty(F_0)<\nu_\infty(F_\gamma)$ for all $\gamma\in D\backslash\{0\}$. Then the Eisenstein condition and the estimate in Theorem 5.4 give

$$\begin{aligned}2(k-2)&=-\sum_{\gamma\in D}\sum_{m>0}c_\gamma(-m)a_\gamma(m)\\&\geqslant c_pc_{k,p,n_p}p^{k-1}g_0(m_\infty)h(m_\infty)\\&\geqslant c_pc_{k,p,n_p}p^{k-1}m_\infty^{k-1}\end{aligned}$$

so that

$$m_\infty^{k-1}\leqslant \xi\frac{p^{n_p/2}}{c_p}\frac{k-2}{k}B_k.$$

The assertion now follows from Proposition 5.5. Suppose $\nu_\infty(F_\gamma)\leqslant\nu_\infty(F_0)$ for some $\gamma\in D\backslash\{0\}$. Choose $\gamma\neq 0$ such that $-\nu_\infty(F_\gamma)=m_\infty$. Then,

$$-\sum_{\gamma\in D}\sum_{m>0}c_\gamma(-m)a_\gamma(m)\geqslant c_{k,p,n_p}p^{k-1}m_\infty^{k-1}$$

and the statement follows analogously. □

We remark that the first inequality in the proposition is a consequence of the Riemann–Roch theorem and the second is a consequence of the Eisenstein condition.

THEOREM 5.7. *Let L be an even lattice of level p and genus $II_{n,2}(p^{\epsilon_p n_p})$ with $n > 2$ and n_p even splitting a hyperbolic plane $II_{1,1}$. Suppose L carries a holomorphic automorphic product of singular weight. Then for each $c > 1/\log(\pi e/6)$ there exists a constant d depending only on c such that*

$$n \leqslant c n_p \log(p) + d.$$

Proof. Recall that $k \geqslant 4$. Using $2\zeta(k) = \xi((2\pi)^k/k!)B_k$ and $k! \leqslant e\sqrt{k}(k/e)^k$ we derive from Proposition 5.6 the inequality

$$1 \leqslant e^2 p^{n_p/2} k^{3/2} \left(\frac{6}{\pi e}\right)^k$$

respectively

$$0 \leqslant 2 + \frac{n_p}{2}\log(p) + \frac{3}{2}\log(k) - k\log\left(\frac{\pi e}{6}\right).$$

If t is a tangent of the real logarithm then $\log(x) \leqslant t(x)$ for all $x > 0$. Thus, $\log(k) \leqslant (k-x)/x + \log(x)$ for all $x > 0$. It follows

$$0 \leqslant -\left(\log\left(\frac{\pi e}{6}\right) - \frac{3}{2x}\right)k + \frac{n_p}{2}\log(p) + \frac{3}{2}(\log(x) - 1) + 2$$

for all $x > 0$. If $x > 3/2\log(\pi e/6) = 4.24964\ldots$ this gives an upper bound on k and on n, i.e.

$$n \leqslant c(x) n_p \log(p) + d(x)$$

with

$$c(x) = \frac{2}{2\log(\pi e/6) - 3/x}$$
$$d(x) = (3\log(x) + 1)c(x) - 2$$

in this case. □

Note that the proof is constructive. For example, taking $x = 20$ gives the bounds $c = 3.59750\ldots$ and $d = 33.92899\ldots$.

5.2 Odd p-rank

Now let L be an even lattice of prime level p and genus $II_{n,2}(p^{\epsilon_p n_p})$ with $n > 2$ and n_p odd. Suppose $\Psi(F)$ is a holomorphic automorphic product of singular weight on L.

Since n_p is odd, it follows that p is odd as well.

The oddity formula implies

$$e((n-2)/8) = \begin{cases} \epsilon_p \left(\dfrac{2}{p}\right) & \text{if } p = 1 \bmod 4, \\ \epsilon_p \left(\dfrac{2}{p}\right)(-1)^{(n_p-1)/2} e(1/4) & \text{if } p = 3 \bmod 4, \end{cases}$$

so that

$$n = \begin{cases} \pm 2 \bmod 8 & \text{if } p = 1 \bmod 4, \\ 0 \bmod 4 & \text{if } p = 3 \bmod 4. \end{cases}$$

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Define $k = 1 + n/2$ and

$$\xi = \epsilon_p \left(\frac{2}{p}\right)\left(\frac{-1}{p}\right)^{(n_p-1)/2}.$$

Then

$$\xi = \begin{cases} -(-1)^{k/2} & \text{if } p = 1 \bmod 4, \\ -(-1)^{(k-1)/2} & \text{if } p = 3 \bmod 4. \end{cases}$$

Let $\chi(j) = (j/p)$. Define the twisted divisor sum

$$\sigma_{l,\chi}(m) = \sum_{d|m} \chi(m/d) d^l$$

and the generalised Bernoulli numbers $B_{m,\chi}$ by

$$\sum_{j=1}^{p} \frac{\chi(j) x e^{jx}}{e^{px} - 1} = \sum_{m \geqslant 0} B_{m,\chi} \frac{x^m}{m!}$$

(see [Iwa72]). Let

$$c_{k,p,n_p} = \xi \frac{2k}{B_{k,\chi}} \frac{1}{p^{(n_p-1)/2}}.$$

The positivity of $L(k, \chi)$ implies that c_{k,p,n_p} is positive. We describe the Fourier coefficients $a_\gamma(m)$ of the Eisenstein series E.

Proposition 5.8. *Let $\gamma \in D$ and $m \in q(\gamma) + \mathbb{Z}$, $m > 0$.*

If $q(\gamma) \neq 0 \bmod 1$, then

$$a_\gamma(m) = -c_{k,p,n_p} \sigma_{k-1,\chi}(pm).$$

Suppose $q(\gamma) = 0 \bmod 1$. Write $m = p^\nu a$ with $(a, p) = 1$. Then

$$a_\gamma(m) = -c_{k,p,n_p} p^{(\nu+1)(k-1)} \sigma_{k-1,\chi}(a)$$

if $\gamma \neq 0$ and

$$a_\gamma(m) = -c_{k,p,n_p} p^{(\nu+1)(k-1)} \sigma_{k-1,\chi}(a) - \xi c_{k,p,n_p} p^{(n_p-1)/2} \chi(a) \sigma_{k-1,\chi}(a)$$

if $\gamma = 0$.

We have the following result.

Proposition 5.9. *The lattice L splits a hyperbolic plane $II_{1,1}$ if and only if*

$$n_p \leqslant n - 1.$$

As above, we denote the Fourier coefficients of F by c_γ and define the functions g_γ. We also define

$$h_\chi(m) = \sum_{\substack{d|m \\ m/d \text{ squarefree}}} \chi(m/d) d^{k-1}.$$

The function h_χ is bounded below by $h_\chi(m) \geqslant (2 - \zeta(2)) m^{k-1} \geqslant m^{k-1}/3$.

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Theorem 5.10. *Suppose L splits $II_{1,1}$. Let $c_p = 1 - 1/p$. Then*

$$-\sum_{\gamma \in D} \sum_{m>0} c_\gamma(-m) a_\gamma(m) \geqslant c_{k,p,n_p} \sum_{\substack{\gamma \in D \\ q(\gamma) \neq 0 \bmod 1}} \sum_{m/p = q(\gamma) \bmod 1} g_\gamma(m/p) h_\chi(m)$$
$$+ c_{k,p,n_p} p^{k-1} \sum_{\substack{\gamma \in D \backslash \{0\} \\ q(\gamma) = 0 \bmod 1}} \sum_{m>0} g_\gamma(m) h_\chi(m)$$
$$+ c_p c_{k,p,n_p} p^{k-1} \sum_{m>0} g_0(m) h_\chi(m).$$

Proof. The argument is analogous to the proof of Theorem 5.4. We describe the necessary modifications.

Let $\gamma \in D$ with $q(\gamma) \neq 0 \bmod 1$. Then

$$-\sum_{j=1}^{p-1} \sum_{m>0} c_{j\gamma}(-m) a_{j\gamma}(m) = c_{k,p,n_p} \sum_{j=1}^{p-1} \sum_{m/p = q(j\gamma) \bmod 1} (g_{j\gamma}(m/p) - g_0(mp)) h_\chi(m).$$

For $\gamma \in D\backslash\{0\}$ with $q(\gamma) = 0 \bmod 1$ we find

$$-\sum_{j=1}^{p-1} \sum_{m>0} c_{j\gamma}(-m) a_{j\gamma}(m) = c_{k,p,n_p} p^{k-1} \sum_{j=1}^{p-1} \sum_{m>0} (g_{j\gamma}(m) - g_0(mp^2)) h_\chi(m).$$

If $\gamma = 0$, then

$$-\sum_{m>0} c_\gamma(-m) a_\gamma(m) = c_{k,p,n_p} p^{k-1} \sum_{m>0} (g_0(m) - g_0(mp^2)) h_\chi(m)$$
$$+ \xi c_{k,p,n_p} p^{(n_p-1)/2} \sum_{(m,p)=1} (g_0(m) + g_0(mp)) \chi(m) h_\chi(m).$$

Using

$$\sum_{m>0} g_0(m) h_\chi(m) = \sum_{(m,p)=1} g_0(m) h_\chi(m) + p^{k-1} \sum_{(m,p)=1} g_0(mp) h_\chi(m)$$
$$+ p^{2(k-1)} \sum_{m>0} g_0(mp^2) h_\chi(m),$$

we obtain

$$-\sum_{\gamma \in D} \sum_{m>0} c_\gamma(-m) a_\gamma(m) = c_{k,p,n_p} \sum_{\substack{\gamma \in D \\ q(\gamma) \neq 0 \bmod 1}} \sum_{m/p = q(\gamma) \bmod 1} g_\gamma(m/p) h_\chi(m)$$
$$+ c_{k,p,n_p} p^{k-1} \sum_{\substack{\gamma \in D \backslash \{0\} \\ q(\gamma) = 0 \bmod 1}} \sum_{m>0} g_\gamma(m) h_\chi(m)$$
$$+ c_{k,p,n_p} \sum_{j=1}^{p-1} \sum_{m = j \bmod p} c^0_{k,p,n_p,j} g_0(m) h_\chi(m)$$

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$$+ c_{k,p,n_p} \sum_{j=1}^{p-1} \sum_{m=j \bmod p} c^1_{k,p,n_p,j} g_0(mp) h_\chi(m)$$
$$+ c_{k,p,n_p} c^{\geqslant 2}_{k,p,n_p} \sum_{m>0} g_0(mp^2) h_\chi(m)$$

with

$$\begin{aligned} c^0_{k,p,n_p,j} &= p^{k-1} + \xi\chi(j) p^{(n_p-1)/2}, \\ c^1_{k,p,n_p,j} &= p^{2(k-1)} - a_{k,p,n_p,j} + \xi\chi(j) p^{(n_p-1)/2}, \\ c^{\geqslant 2}_{k,p,n_p} &= p^{3(k-1)} - p^{k-1} a_{k,p,n_p,0}, \end{aligned}$$

where $a_{k,p,n_p,j}$ denotes the number of elements $\gamma \in D$ of norm $q(\gamma) = j/p \bmod 1$. Since L splits $II_{1,1}$ we have

$$\begin{aligned} c^0_{k,p,n_p,j} &\geqslant (1 - 1/p) p^{k-1}, \\ c^1_{k,p,n_p,j} &\geqslant (1 - 1/p^2) p^{2(k-1)}, \\ c^{\geqslant 2}_{k,p,n_p,m} &\geqslant (1 - 1/p^2) p^{3(k-1)}. \end{aligned}$$

This implies the assertion. □

Pairing F with the Eisenstein series E and applying the Riemann–Roch theorem to F_0, we obtain the following result.

Proposition 5.11. *Suppose L splits $II_{1,1}$. Then*

$$\left(\frac{k-2}{12}\right)^{k-1} \leqslant m_\infty^{k-1} \leqslant 3\xi \frac{p^{(n_p+1)/2}}{c_p} \frac{k-2}{k} \frac{B_{k,\chi}}{p^k}.$$

We can now derive a bound on n.

Theorem 5.12. *Let L be an even lattice of level p and genus $II_{n,2}(p^{\epsilon_p n_p})$ with $n > 2$ and n_p odd splitting a hyperbolic plane $II_{1,1}$. Suppose L carries a holomorphic automorphic product of singular weight. Then for each $c > 1/\log(\pi e/6)$ there exists a constant d depending only on c such that*

$$n \leqslant c n_p \log(p) + d.$$

Proof. Here we use $2L(k,\chi) = \xi\sqrt{p}((2\pi)^k/k!)(B_{k,\chi}/p^k)$ and $L(k,\chi) \leqslant \zeta(3)$ to obtain

$$1 \leqslant \frac{5}{2} e^2 p^{n_p/2} k^{3/2} \left(\frac{6}{\pi e}\right)^k.$$

As above, this implies

$$n \leqslant c(x) n_p \log(p) + d(x)$$

with

$$\begin{aligned} c(x) &= \frac{2}{2\log(\pi e/6) - 3/x}, \\ d(x) &= (3\log(x) + 1 + 2\log(5/2)) c(x) - 2 \end{aligned}$$

for $x > 3/2 \log(\pi e/6)$. □

Note that the constant d is slightly larger here than in Theorem 5.7. Taking $x = 20$ we obtain the bounds $c = 3.59750\ldots$ and $d = 40.52171\ldots$.

5.3 An example

Let L be a lattice of genus $II_{n,2}(2_{II}^{+n_2})$ with $n > 2$ and $n_2 = 2,\ 4$ or 6 carrying a holomorphic automorphic product of singular weight. Then $n \leqslant 34,\ 42$ respectively 42 and

$$\frac{k-2}{12} \leqslant m_\infty \leqslant \left(2^{(n_2+2)/2}\frac{k-2}{k}B_k\right)^{1/(k-1)}$$

by Theorem 5.7 and Proposition 5.6. The values of the bounds are given in the following table.

n	k	$(k-2)/12$	2_{II}^{+2}	2_{II}^{+4}	2_{II}^{+6}
10	6	$0.33333\ldots$	$0.57616\ldots$	$0.66183\ldots$	$0.76024\ldots$
18	10	$0.66666\ldots$	$0.85431\ldots$	$0.92271\ldots$	$0.99658\ldots$
26	14	1	$1.11253\ldots$	$1.17346\ldots$	$1.23772\ldots$
34	18	$1.33333\ldots$	$1.36385\ldots$	$1.42060\ldots$	$1.47973\ldots$
42	22	$1.66666\ldots$	$1.61161\ldots$	$1.66570\ldots$	$1.72159\ldots$
50	26	2	$1.85716\ldots$	$1.90937\ldots$	$1.96305\ldots$

Since m_∞ is half-integral we obtain the following theorem.

Theorem 5.13. *Let L be a lattice of genus $II_{n,2}(2_{II}^{+n_2})$ with $n > 2$ and $n_2 = 2, 4$ or 6 carrying a holomorphic automorphic product of singular weight. Then $n = 10$ or 26.*

6. Reflective forms

In this section, we remove the hypotheses made in [Sch06] and give a complete classification of reflective automorphic products of singular weight on lattices of prime level.

6.1 General results

We derive some general bounds and formulate the Eisenstein condition for reflective modular forms.

Let L be an even lattice of prime level p and genus $II_{n,2}(p^{\epsilon_p n_p})$ with $n > 2$. Let $F = \sum_{\gamma\in D} F_\gamma e^\gamma$ be a non-zero reflective modular form on L (see § 3). Then F has weight $1 - n/2$,

$$F_0(\tau) = c_0(-1)q^{-1} + \sum_{\substack{m\in\mathbb{Z}\\ m\geqslant 0}} c_0(m)q^m$$

with $c_0(-1) = 0$ or 1,

$$F_\gamma(\tau) = c_\gamma(-1/p)q^{-1/p} + \sum_{\substack{m\in\mathbb{Z}-1/p\\ m>0}} c_\gamma(m)q^m$$

with $c_\gamma(-1/p) = 0$ or 1 if $q(\gamma) = 1/p \bmod 1$ and the other components F_γ of F are holomorphic at ∞. We define integers $c_1 = c_0(-1)$ and $c_p = |\{\gamma \in D \mid q(\gamma) = 1/p \bmod 1 \text{ and } F_\gamma \text{ singular}\}|$.

Proposition 6.1. *We have $n < 26$. If $c_1 = 0$, then $n \leqslant 2 + 24/(p+1)$.*

Proof. The conditions imply $F_0 \neq 0$. Since F is reflective, the product $F_0\Delta$ is a modular form for $\Gamma_0(p)$ of weight $13 - n/2$ which is holomorphic on the upper halfplane and at the cusps. Hence, $n \leqslant 26$. If $n = 26$ the function F_0 must be $1/\Delta$. However, as a result, F does not transform

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correctly under S. This proves the first statement. If $c_1 = 0$ the Riemann–Roch theorem applied to F_0 gives

$$-1 \leqslant p\nu_0(F_0) + \nu_\infty(F_0) \leqslant \frac{m}{12}(p+1),$$

where $m = 1 - n/2$ is the weight of F_0. This implies the second statement. □

Pairing F with the Eisenstein series E of weight $k = 1 + n/2$ we obtain (see Propositions 5.1 and 5.8) the following result.

Proposition 6.2. *Suppose F_0 has constant coefficient $n-2$. Then*

$$\frac{k-2}{k} B_k(p^k - 1) = \xi_{\text{even}}(p^{k-n_p/2}c_1 + p^{1-n_p/2}c_p) - c_1$$

with $\xi_{\text{even}} = -(-1)^{k/2}$ if n_p is even and

$$\frac{k-2}{k} B_{k,\chi} = \xi_{\text{odd}}(p^{k-(n_p+1)/2}c_1 + p^{(1-n_p)/2}c_p) + c_1$$

with

$$\xi_{\text{odd}} = \begin{cases} -(-1)^{k/2} & \text{if } p = 1 \bmod 4, \\ -(-1)^{(k-1)/2} & \text{if } p = 3 \bmod 4, \end{cases}$$

if n_p is odd.

We will also need the following result.

Proposition 6.3. *If $n_p = n+2$, then $n - 2 = 0 \bmod 8$ and L is a rescaling of $II_{n,2}$ by p.*

Proof. Since $\gamma_p(D)$ is a fourth root of unity the oddity formula $e(\text{sign}(D)/8) = \gamma_p(D)$ implies that n is even. Then n_p is also even and

$$\gamma_p(D) = \epsilon_p \left(\frac{-1}{p}\right)^{n_p/2}.$$

Hence, $n-2 = 0$ or $4 \bmod 8$ and $\gamma_p(D) = \epsilon_p$. The lattice L has determinant p^{n+2} so that $\epsilon_1\epsilon_p = 1$ by the sign rule. Now $\epsilon_1 = +1$ because L has maximal p-rank and therefore $\epsilon_p = +1$. Applying the oddity formula again, we obtain $n - 2 = 0 \bmod 8$. The second statement follows from the fact that there is only one class in the genus $II_{n,2}(p^{\epsilon_p n_p})$ under the given conditions. □

6.2 Symmetric forms

Here we classify reflective modular forms that are invariant under $O(D)$.

Let L be an even lattice of prime level p and genus $II_{n,2}(p^{\epsilon_p n_p})$ with $n > 2$. Then the number of elements γ in D of order p and norm $q(\gamma) = 1/p \bmod 1$ is given by

$$p^{n_p-1} - \xi_{\text{even}}p^{(n_p-2)/2}$$

if n_p is even and by

$$p^{n_p-1} + \xi_{\text{odd}}p^{(n_p-1)/2}$$

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if n_p is odd (see [Sch06, Proposition 3.2]). Suppose L carries a symmetric reflective modular form F with $[F_0](0) = n-2$. Then the Eisenstein condition takes the form

$$\frac{k-2}{k}B_k(p^k-1) = \xi_{\mathrm{even}}(p^{k-n_p/2}d_1 + p^{n_p/2}d_p) - d_1 - d_p$$

if n_p is even and

$$\frac{k-2}{k}B_{k,\chi} = \xi_{\mathrm{odd}}(p^{k-(n_p+1)/2}d_1 + p^{(n_p-1)/2}d_p) + d_1 + d_p$$

if n_p is odd. Here d_1 and d_p can be 0 or 1. In the case $n_p < n+2$, the solutions of these equations have been determined in [Sch06].

Theorem 6.4. *Let L be an even lattice of prime level p and genus $II_{n,2}(p^{\epsilon_p n_p})$ with $n > 2$ carrying a symmetric reflective modular form F. Suppose F_0 has constant coefficient $n-2$. Then L and F are given in the following table.*

p	L	F
2	$II_{18,2}(2_{II}^{+10})$	$\eta_{1^{-8}2^{-8}}$
	$II_{10,2}(2_{II}^{+2})$, $II_{10,2}(2_{II}^{+10})$	$16\eta_{1^{-16}2^{8}}$, $\eta_{1^{8}2^{-16}}$
3	$II_{14,2}(3^{-8})$	$\eta_{1^{-6}3^{-6}}$
	$II_{8,2}(3^{-3})$, $II_{8,2}(3^{-7})$	$9\eta_{1^{-9}3^{3}}$, $\eta_{1^{3}3^{-9}}$
5	$II_{10,2}(5^{+6})$	$\eta_{1^{-4}5^{-4}}$
	$II_{6,2}(5^{+3})$, $II_{6,2}(5^{+5})$	$5\eta_{1^{-5}5^{1}}$, $\eta_{1^{1}5^{-5}}$
7	$II_{8,2}(7^{-5})$	$\eta_{1^{-3}7^{-3}}$
11	$II_{6,2}(11^{-4})$	$\eta_{1^{-2}11^{-2}}$
23	$II_{4,2}(23^{-3})$	$\eta_{1^{-1}23^{-1}}$

The η-product in the last column is a modular form for $\Gamma_0(p)$ whose lift on 0 gives F.

Conversely, each of these functions is a reflective modular form on L with the above stated properties.

Proof. We only have to consider the case $n_p = n+2$. Then $n = 10$ or 18 and $\xi_{\mathrm{even}} = +1$ by Propositions 6.3 and 6.1. The Eisenstein condition simplifies to

$$\frac{k-2}{k}B_k = d_p.$$

Now the left-hand side is $1/63$ for $k = 6$ and $2/33$ for $k = 10$. Hence, there are no reflective forms if $n_p = n+2$. □

6.3 Bounds in the non-symmetric case

In this section, we derive bounds on the signature for reflective modular forms which are not invariant under $O(D)$.

First, we recall the Riemann–Roch theorem for $\Gamma_1(p)$.

Let p be prime. For $p \geqslant 3$, the group $\Gamma_1(p)$ has $p-1$ classes of cusps which can be represented by $1/c$ with $c = 1, \ldots, (p-1)/2$ of width p and a/p with $a = 1, \ldots, (p-1)/2$ of width 1. The cusps of $\Gamma_1(2)$ can be represented by $1/2$ of width 1 and $1/1$ of width 2. Let $f \neq 0$ be a meromorphic

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modular form on $\Gamma_1(p)$ of weight m and finite-order character. For $p \geqslant 5$, there are no torsion points and the Riemann–Roch theorem states

$$\sum_{c=1}^{(p-1)/2} p\nu_{1/c}(f) + \sum_{a=1}^{(p-1)/2} \nu_{a/p}(f) + \sum_{\tau\in\Gamma_1(p)\backslash H} \nu_\tau(f) = \frac{m}{24}(p^2-1).$$

For $p = 3$, we have

$$3\nu_{1/1}(f) + \nu_{1/3}(f) + \frac{1}{3}\nu_{e_3}(f) + \sum_{\substack{\tau\in\Gamma_1(3)\backslash H \\ \tau\neq e_3 \bmod \Gamma_1(3)}} \nu_\tau(f) = \frac{m}{3}$$

with $e_3 = (3+i\sqrt{3})/6$ and

$$2\nu_{1/1}(f) + \nu_{1/2}(f) + \frac{1}{2}\nu_{e_2}(f) + \sum_{\substack{\tau\in\Gamma_1(2)\backslash H \\ \tau\neq e_2 \bmod \Gamma_1(2)}} \nu_\tau(f) = \frac{m}{4}$$

with $e_2 = (1+i)/2$ if $p = 2$.

Theorem 6.5. *Let L be an even lattice of prime level p and signature $(n,2)$ with $n > 2$ carrying a non-symmetric reflective modular form F. Then $p \leqslant 11$ and $n \leqslant 2 + 24/p$.*

Proof. Since F is non-symmetric there are $\gamma_1, \gamma_2 \in D\backslash\{0\}$ of the same norm such that

$$f = F_{\gamma_1} - F_{\gamma_2} \neq 0.$$

The function f is a modular form on $\Gamma_1(p)$ of weight $m = 1 - n/2$ and finite-order character.

Let $\gamma \in D$ and $M = \left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right) \in \Gamma$. Then

$$F_\gamma|_{m,M} = \left(\frac{a}{|D|}\right) e(-abq(\gamma)) F_{a\gamma}$$

if $c = 0 \bmod p$ and

$$F_\gamma|_{m,M} = \frac{e(-\mathrm{sign}(D)/8)}{\sqrt{|D|}} \left(\frac{-c}{|D|}\right) \sum_{\mu\in D} e(-c^{-1}dq(\mu)) e(-b(\mu,\gamma)) e(-abq(\gamma)) F_{a\gamma+\mu}$$

if $c \neq 0 \bmod p$. The coefficient at F_0 in this sum is

$$e(-c^{-1}dq(\mu)) e(-b(\mu,\gamma)) e(-abq(\gamma)) = e(-c^{-1}aq(\gamma)),$$

i.e. only depends on the norm of γ.

This implies that for all $M \in \Gamma$, the function $f|_{m,M}$ is a linear combination of functions F_γ with $\gamma \neq 0$. Hence,

$$\nu_s(f) \geqslant -1/p$$

for all cusps s of $\Gamma_1(p)$. It follows that

$$-\frac{p-1}{2}\left(1 + \frac{1}{p}\right) \leqslant \frac{m}{24}(p^2-1).$$

This proves the theorem. □

Note that the bounds do not hold in the symmetric case.

Using Theorem 6.5, we can determine the non-symmetric forms on lattices of prime level by analysing the obstructions in a finite number of cases. For $p = 3$, which is the most complicated case, we describe this explicitly in the next section. The other cases are analogous.

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6.4 Level 3

In this section we determine the reflective forms on lattices of level 3 and signature $(n,2)$ where $n = 4,\ 6,\ 8$ or 10.

Let L be a lattice of genus $II_{10,2}(3^{\epsilon_3 n_3})$ and F a reflective form on L. Suppose F_0 has constant coefficient $[F_0](0) = 8$. Then $c_1 = 1$ (see Proposition 6.1) and the Eisenstein condition gives the following value for c_3 (see Proposition 6.2).

	$II_{10,2}(3^{-2})$	$II_{10,2}(3^{+4})$	$II_{10,2}(3^{-6})$	$II_{10,2}(3^{+8})$	$II_{10,2}(3^{-10})$	$II_{10,2}(3^{+12})$
$c_1 = 1$	$-2074/9$	$-616/3$	-130	96	774	2808

Since c_3 should be a non-negative integer, this already excludes the first three cases.

The space $S_6(\Gamma(3))$ has dimension 3 and is spanned by the functions $\eta_{1^8}\theta_{A_2}^2$, $\eta_{1^8}\theta_{A_2}\theta_{\nu+A_2}$ and $\eta_{1^6 3^6}$. The liftings of these functions generate the obstruction space $S_{\overline{\rho}_D,6}$.

Pairing F with the lift $F_{\eta_{1^6 3^6},0}$ of the η-product $\eta_{1^6 3^6}(\tau) = \eta(\tau)^6\eta(3\tau)^6$ we obtain

$$1 - \frac{1}{3^{n_3/2}} - \frac{c_3}{3^{(n_3+4)/2}} = 0.$$

Proposition 6.6. *There are no reflective modular forms with constant coefficient 8 on lattices of genus $II_{10,2}(3^{\epsilon_3 n_3})$.*

Next, we consider the case $n = 8$. Let L be a lattice of genus $II_{8,2}(3^{\epsilon_3 n_3})$ and F a reflective modular form on L with $[F_0](0) = 6$. Then we obtain the following for c_3.

	$II_{8,2}(3^{+1})$	$II_{8,2}(3^{-3})$	$II_{8,2}(3^{+5})$	$II_{8,2}(3^{-7})$	$II_{8,2}(3^{+9})$
$c_1 = 0$	2	6	18	54	162
$c_1 = 1$	-78	-72	-54	0	162

The discriminant form of type 3^{+1} contains no elements γ of norm $q(\gamma) = 1/3 \bmod 1$. Hence, this case can be excluded.

The space $S_5(\Gamma(3))$ has dimension 2 and is spanned by the functions $\eta_{1^8}\theta_{A_2}$ and $\eta_{1^8}\theta_{\nu+A_2}$. The liftings of these functions generate the obstruction space $S_{\overline{\rho}_D,5}$.

The lattice A_2 has genus $II_{2,0}(3^{-1})$ and is isomorphic to its rescaled dual $A_2'(3)$. The theta functions of A_2 can be written as

$$\theta_{A_2} = \frac{\eta_{1^3} + 9\eta_{9^3}}{\eta_{3^1}},$$
$$\theta_{\nu+A_2} = \tfrac{1}{2}(\theta_{A_2'} - \theta_{A_2}).$$

They transform under $S = \left(\begin{smallmatrix}0 & -1\\ 1 & 0\end{smallmatrix}\right)$ as

$$\theta_{A_2}|_{1,S} = \frac{e(-1/4)}{\sqrt{3}}(\theta_{A_2} + 2\theta_{\nu+A_2}) = \frac{e(-1/4)}{\sqrt{3}}\theta_{A_2'},$$
$$\theta_{\nu+A_2}|_{1,S} = \frac{e(-1/4)}{\sqrt{3}}(\theta_{A_2} - \theta_{\nu+A_2}).$$

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Let $\gamma \in D$ be of norm $q(\gamma) = 1/3 \bmod 1$. Then the lift of $\eta_{1^8}\theta_{A_2}$ with respect to the dual Weil representation $\overline{\rho}_D$ on γ is given by

$$F_{\eta_{1^8}\theta_{A_2},\gamma} = F_{1/3} + F_{1/1}$$

with

$$F_{1/3} = \eta_{1^8}\theta_{A_2}(e^{\gamma} + e^{-\gamma})$$

and

$$F_{1/1} = -\frac{1}{3^{(n_3-1)/2}} \sum_{\mu \in D} e((\gamma,\mu)) g_{j_\mu}(e^{\mu} + e^{-\mu}),$$

where

$$\eta_{1^8}(\theta_{A_2} + 2\theta_{\nu + A_2}) = g_0 + g_1 + g_2$$

and $g_j|_{5,T} = e(j/3)g_j$. Note that $g_0 = 0$. We obtain an analogous result for the lift of $\eta_{1^8}\theta_{\nu+A_2}$ with respect to $\overline{\rho}_D$ on an element $\gamma \in D$ of norm $q(\gamma) = 2/3 \bmod 1$.

Let

$$M = \{\gamma \in D \mid q(\gamma) = 1/3 \bmod 1 \text{ and } F_\gamma \text{ singular}\}.$$

We assume now that M is non-empty. Then $|M| = c_3 = 2 \cdot 3^{(n_3-1)/2}$ and $M = -M$ because $F_\gamma = F_{-\gamma}$. The crucial result to determine the structure of M is the following proposition.

Proposition 6.7. *Let $\gamma \in D$ be of norm $q(\gamma) \neq 0 \bmod 1$. Then*

$$|M \cap \gamma^\perp| = \begin{cases} 2|M|/3 & \text{if } \gamma \in M, \\ |M|/3 & \text{otherwise.} \end{cases}$$

Proof. Let $\gamma \in D$ be of norm $q(\gamma) = 1/3 \bmod 1$. Suppose $\gamma \in M$. Then pairing F with $F_{\eta_{1^8}\theta_{A_2},\gamma}$ gives

$$2 - \frac{1}{3^{(n_3-1)/2}} \sum_{\mu \in M} (e((\gamma,\mu)) + e(-(\gamma,\mu))) = 0$$

so that

$$\sum_{\mu \in M} (e((\gamma,\mu)) + e(-(\gamma,\mu))) = |M|.$$

This implies

$$|M \cap \gamma^\perp| = 2|M|/3.$$

If $\gamma \notin M$ the same argument shows $|M \cap \gamma^\perp| = |M|/3$. In case $q(\gamma) = 2/3 \bmod 1$ the statement follows from pairing F with $F_{\eta_{1^8}\theta_{\nu+A_2},\gamma}$. ☐

The proposition implies that $M^\perp$ is an isotropic subgroup of D. Let $\gamma \in M$ and $\mu \in M^\perp$. Then $M \cap \gamma^\perp = M \cap (\gamma+\mu)^\perp$. Hence, the group $M^\perp$ acts on M by translations.

Proposition 6.8. *Let $\gamma, \mu \in M$ such that $(\gamma,\mu) = 2/3 \bmod 1$. Then $\gamma + \mu \in M$.*

Proof. The sets $M \cap \gamma^\perp$ and $M \cap \mu^\perp$ are both subsets of $M\backslash\{\pm\gamma\}$. Hence,

$$|(M \cap \gamma^\perp) \cap (M \cap \mu^\perp)| \geqslant 4|M|/3 - (|M| - 2) = |M|/3 + 2.$$

Since $(M \cap \gamma^\perp) \cap (M \cap \mu^\perp) \subset (M \cap (\gamma+\mu)^\perp)$ this implies $|M \cap (\gamma+\mu)^\perp| = 2|M|/3$ and $\gamma + \mu \in M$. ☐

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Proposition 6.9. *Let* $\gamma, \mu \in M$ *such that* $(\gamma, \mu) = 0 \bmod 1$. *Then*

$$(M \cap \gamma^{\perp}) \cap (M \cap \mu^{\perp}) = M \cap (\gamma + \mu)^{\perp}.$$

Proof. We have $|M \cap \gamma^{\perp}| = |M \cap \mu^{\perp}| = 2|M|/3$ so that

$$|(M \cap \gamma^{\perp}) \cap (M \cap \mu^{\perp})| \geqslant 4|M|/3 - |M| = |M|/3.$$

On the other hand, $(M \cap \gamma^{\perp}) \cap (M \cap \mu^{\perp}) \subset (M \cap (\gamma + \mu)^{\perp})$ and $|M \cap (\gamma + \mu)^{\perp}| = |M|/3$ because $q(\gamma + \mu) = 2/3 \bmod 1$. This implies the statement. □

Proposition 6.10. *Let* $\gamma, \mu, \nu \in M$ *such that*

$$(\gamma, \mu) = (\mu, \nu) = 2/3 \mod 1.$$

Then

$$(\gamma, \nu) = 2/3 \mod 1.$$

Proof. First suppose $(\gamma, \nu) = 0 \bmod 1$. Define $\sigma = \gamma - \nu$. Then $(\sigma, \mu) = 0$. However, this contradicts $(M \cap \gamma^{\perp}) \cap (M \cap \nu^{\perp}) = M \cap \sigma^{\perp}$. Next we assume $(\gamma, \nu) = 1/3 \bmod 1$. Here we define $\sigma = \gamma + \mu + \nu$. Note that $\gamma + \mu$ is in M and $(\gamma + \mu, \nu) = 0 \bmod 1$. Then $(\sigma, \mu) = 0 \bmod 1$. This contradicts $(M \cap (\gamma + \mu)^{\perp}) \cap (M \cap \nu^{\perp}) = M \cap \sigma^{\perp}$. Hence, $(\gamma, \nu) = 2/3 \bmod 1$. □

A consequence of this result is the following proposition.

Proposition 6.11. *Let* $\gamma, \mu \in M$ *such that* $(\gamma, \mu) \neq 0 \bmod 1$. *Then*

$$M \cap \gamma^{\perp} = M \cap \mu^{\perp}.$$

Proposition 6.12. *Let* $\gamma, \mu \in M$ *such that* $(\gamma, \mu) = 2/3 \bmod 1$. *Then* $\gamma - \mu \in M^{\perp}$.

Proof. Define $\sigma = \gamma - \mu$. Then $M \cap \gamma^{\perp} = M \cap \mu^{\perp}$ implies $(M \cap \gamma^{\perp}) \subset (M \cap \sigma^{\perp})$. Let $\nu \in M$ such that $(\gamma, \nu) = 2/3 \bmod 1$. Then $(\gamma, \mu) = (\mu, \nu) = (\gamma, \nu) = 2/3 \bmod 1$ by the above transitivity result. Hence, $(\sigma, \nu) = 0 \bmod 1$. Similarly, if $\nu \in M$ such that $(\gamma, \nu) = 1/3 \bmod 1$, then $(\sigma, \nu) = 0 \bmod 1$. Hence, all elements in M are orthogonal to σ. □

Proposition 6.13. *The group* $M^{\perp}$ *is an isotropic subgroup of* D *order* $3^{(n_3-3)/2}$.

Proof. Let $\gamma \in M$. Then the elements $\mu \in M$ with $(\gamma, \mu) \neq 0 \bmod 1$ are in $\pm\gamma + M^{\perp}$. Hence, M decomposes as

$$M = (\gamma + M^{\perp}) \cup (-\gamma + M^{\perp}) \cup (M \cap \gamma^{\perp})$$

so that

$$|M| = 2|M^{\perp}| + 2|M|/3.$$

This implies the statement. □

Proposition 6.14. *The set* M *is of the form*

$$M = \bigcup_{i=1}^{3} (\gamma_i + M^{\perp}) \cup \bigcup_{i=1}^{3} (-\gamma_i + M^{\perp})$$

with $\gamma_i \in M$ *and* $(\gamma_i, \gamma_j) = 0 \bmod 1$ *for* $i \neq j$.

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Let H be an isotropic subgroup of D of order $|H| = 3^{(n_3-3)/2}$. Then the lift of $9\eta_{1^{-9}3^3}$, with respect to ρ_D on H, is given by

$$F_{9\eta_{1^{-9}3^3},H} = F_{1/3} + F_{1/1}$$

with

$$F_{1/3} = \sum_{\gamma \in H} 9\eta_{1^{-9}3^3} e^{\gamma}$$

and

$$F_{1/1} = \sum_{\gamma \in H^{\perp}} g_{j_\gamma} e^{\gamma}$$

where $\eta_{1^3 3^{-9}}(\tau/3) = g_0(\tau) + g_1(\tau) + g_2(\tau)$ and $g_j|_{-3,T} = e(j/3) g_j$. Note that

$$\begin{aligned} g_0 &= -3\eta_{1^{-9}3^3}, \\ g_1 &= 0. \end{aligned}$$

The function $F_{9\eta_{1^{-9}3^3},H}$ has 0-component $F_0 = 6\eta_{1^{-9}3^3}$ and is reflective. The singular components are the F_γ with $\gamma \in H^{\perp}$ and $q(\gamma) = 1/3 \mod 1$. The discriminant form $H^{\perp}/H$ is of type 3^{-3}. It is generated by elements $\{\gamma_1, \gamma_2, \gamma_3\}$ with $q(\gamma_i) = 1/3 \mod 1$ and $(\gamma_i, \gamma_j) = 0 \mod 1$ for $i \neq j$. We obtain the following result (see Theorem 2.1 and Proposition 2.2).

Proposition 6.15. *Let L be a lattice of genus $II_{8,2}(3^{\epsilon_3 n_3})$ carrying a reflective modular form. Suppose F_0 is holomorphic at ∞ and has constant coefficient 6. Then $n_3 \geqslant 3$ and $F = F_{9\eta_{1^{-9}3^3},H}$ for some isotropic subgroup H of D of order $|H| = 3^{(n_3-3)/2}$. In this case, the overlattice L_H of L corresponding to H has genus $II_{8,2}(3^{-3})$ and the function F can also be induced from the symmetric form $F_{9\eta_{1^{-9}3^3},0}$ on L_H.*

We can decompose $L = K \oplus II_{1,1}(3)$, where K has genus $II_{7,1}(3^{-\epsilon_3(n_3-2)})$ and assume that H is a maximal isotropic subgroup of the discriminant form of K. Then the embedding $K \subset K_H$ gives an embedding $L \subset L_H$ and identifies the corresponding domains $\mathcal{H}_L$ and $\mathcal{H}_{L_H}$. Proposition 3.4 implies the following proposition.

Proposition 6.16. *The theta lifts of F and $F_{9\eta_{1^{-9}3^3},0}$ coincide as functions under this identification.*

We calculate the product expansions of the automorphic product Ψ corresponding to $F_{9\eta_{1^{-9}3^3},0}$ on L of genus $II_{8,2}(3^{-3})$.

First, we decompose $L = K \oplus II_{1,1}(3)$. Then $K = E_6 \oplus II_{1,1}$. We choose a primitive norm 0 vector z in $II_{1,1}(3)$.

Proposition 6.17. *The expansion of Ψ at the cusp corresponding to z is given by*

$$\begin{aligned} &\prod_{\alpha \in K^+} (1 - e((\alpha, Z)))^{[9\eta_{1^{-9}3^3}](-\alpha^2/2)} \prod_{\alpha \in (3K')^+} (1 - e((\alpha, Z)))^{[-3\eta_{1^{-9}3^3}](-\alpha^2/6)} \\ &\quad = 1 + \sum c(\lambda) e((\lambda, Z)) \end{aligned}$$

where $c(\lambda)$ is the coefficient at q^n in $\eta_{1^9 3^{-3}}$ if λ is n times a primitive norm 0 vector in K^+ and 0 otherwise.

Proof. The product expansion of Ψ at the cusp corresponding to z is

$$\prod_{\alpha\in K'^+}(1-e((\alpha,Z)))^{[F_\alpha](-\alpha^2/2)}(1-e(1/3)e((\alpha,Z)))^{[F_{\alpha+z/3}](-\alpha^2/2)} \times(1-e(2/3)e((\alpha,Z)))^{[F_{\alpha+2z/3}](-\alpha^2/2)}.$$

By the above formulas for the components of $F_{9\eta_{1^{-9}3^3},0}$, this product is equal to

$$\prod_{\alpha\in K^+}(1-e((\alpha,Z)))^{[9\eta_{1^{-9}3^3}](-\alpha^2/2)}\prod_{\alpha\in(3K')^+}(1-e((\alpha,Z)))^{[-3\eta_{1^{-9}3^3}](-\alpha^2/6)}.$$

Since Ψ has singular weight, the Fourier expansion of Ψ_z is supported only on norm 0 vectors of K'. Hence, Ψ_z has the stated sum expansion. □

This is the twisted denominator identity of the fake monster superalgebra [Sch00] corresponding to an element of class $3A$ in $O(E_8)$ (see [Sch01, Proposition 6.1]).

Now, we decompose $L=K\oplus II_{1,1}$ with $K=E_6\oplus II_{1,1}(3)$ and choose a primitive norm 0 vector z in $II_{1,1}$. Then we have the following result.

Proposition 6.18. *The expansion of Ψ at the cusp corresponding to z is given by*

$$e((\rho,Z))\prod_{\alpha\in K'^+}(1-e((\alpha,Z)))^{[\eta_{1^3 3^{-9}}](-3\alpha^2/2)}\prod_{\alpha\in K^+}(1-e((\alpha,Z)))^{[9\eta_{1^{-9}3^3}](-\alpha^2/2)} = \sum_{w\in W}\det(w)\eta_{1^{-3}3^9}((w\rho,Z)),$$

where W is the reflection group of K' generated by the roots of norm $\alpha^2=2/3$.

This is the twisted denominator identity of the fake monster algebra corresponding to an element of class $3C$ in Co_0 (see [Sch04, Proposition 10.7]).

Again, let L be a lattice of genus $II_{8,2}(3^{\epsilon_3 n_3})$ carrying a reflective modular form F.

The lift of $\eta_{1^3 3^{-9}}(\tau)=\eta(\tau)^3\eta(3\tau)^{-9}$ with respect to ρ_D on 0 is given by

$$F_{\eta_{1^3 3^{-9}},0}=F_{1/3}+F_{1/1}$$

with

$$F_{1/3}=\eta_{1^3 3^{-9}}e^0$$

and

$$F_{1/1}=3^{(11-n_3)/2}\sum_{\gamma\in D}g_{j_\gamma}e^\gamma,$$

where $\eta_{1^{-9}3^3}(\tau/3)=g_0(\tau)+g_1(\tau)+g_2(\tau)$ and $g_j|_{-3,T}=e(j/3)g_j$. The modular form $F_{\eta_{1^3 3^{-9}},0}$ is reflective and has 0-component

$$F_0(\tau)=q^{-1}+(3^{(11-n_3)/2}-3)+\cdots.$$

Proposition 6.19. *Let L be a lattice of genus $II_{8,2}(3^{\epsilon_3 n_3})$ and let F be a reflective modular form on L with $c_1=1$ and $[F_0](0)=6$. Then, $n_3=7$ and $F=F_{\eta_{1^3 3^{-9}},0}$ or $n_3=9$ and $F=F_{\eta_{1^3 3^{-9}},0}+F_{9\eta_{1^{-9}3^3},H}$ for some isotropic subgroup H of order 27.*

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Suppose L has genus $II_{8,2}(3^{-7})$. Then the level 1 expansion of the theta lift of $F_{\eta_{1^3 3^{-9}},0}$ on L is the twisted denominator identity of the fake monster superalgebra corresponding to an element in $O(E_8)$ of class $3A$ and the level 3 expansion gives the twisted denominator identity of the fake monster algebra corresponding to an element in Co_0 of class $3C$.

The case $n_3 = 9$ has already been described above because we have the following result.

Proposition 6.20. *Let L be of genus $II_{8,2}(3^{+9})$. Then the theta lift of $F_{\eta_{1^3 3^{-9}},0}$ on L is constant.*

Proof. We decompose $L = K \oplus II_{1,1}(3)$, where K has genus $II_{7,1}(3^{-7})$ and choose a primitive norm 0 vector z in $II_{1,1}(3)$. Then the product expansion of the theta lift Ψ of $F_{\eta_{1^3 3^{-9}},0}$ at the cusp corresponding to z is given by

$$e((\rho, Z)) \prod_{\alpha \in K^+} (1 - e((\alpha, Z)))^{[\eta_{1^3 3^{-9}}](-\alpha^2/2)} \prod_{\alpha \in (3K')^+} (1 - e((\alpha, Z)))^{[3\eta_{1^{-9} 3^3}](-\alpha^2/6)}.$$

The Fourier coefficients $[\eta_{1^3 3^{-9}}](n)$ vanish for $n = 1 \bmod 3$ and $K = E_6'(3) \oplus II_{1,1}(3)$ contains no elements α of norm $-\alpha^2/2 = 2 \bmod 3$. This implies that the first product extends only over the elements $\alpha \in K$ satisfying $\alpha^2/2 = 0 \bmod 3$, i.e., $\alpha \in 3K'$. Now, $[\eta_{1^3 3^{-9}}](3n) = -[3\eta_{1^{-9} 3^3}](n)$ so that the product is constant. This finishes the proof. □

Now we consider the case $n = 6$. Let L be a lattice of genus $II_{6,2}(3^{\epsilon_3 n_3})$ and F a reflective form on L with $[F_0](0) = 4$. We find the following value for c_3.

	$II_{6,2}(3^{+2})$	$II_{6,2}(3^{-4})$	$II_{6,2}(3^{+6})$
$c_1 = 0$	$4/3$	4	12
$c_1 = 1$	$-80/3$	-26	-24

Hence, we can assume that F_0 is holomorphic at ∞ and $n_3 = 4$ or 6.

The space $S_4(\Gamma(3))$ has dimension 1 and is spanned by the function η_{1^8}. The liftings of this function generate the obstruction space $S_{\overline{\rho}_D,4}$.

Let $\gamma \in D$ be of norm $q(\gamma) = 1/3 \bmod 1$. Then the lift of $\eta_{1^8}(\tau) = \eta(\tau)^8$ with respect to the dual Weil representation $\overline{\rho}_D$ on γ is given by

$$F_{\eta_{1^8},\gamma} = F_{1/3} + F_{1/1}$$

with

$$F_{1/3} = \eta_{1^8}(e^{\gamma} + e^{-\gamma})$$

and

$$F_{1/1} = -\frac{1}{3^{(n_3-2)/2}} \sum_{\substack{\mu \in D \\ q(\mu) = 1/3 \bmod 1}} e((\mu, \gamma)) \eta_{1^8}(e^{\mu} + e^{-\mu}).$$

As above, we define

$$M = \{\gamma \in D \mid q(\gamma) = 1/3 \bmod 1 \text{ and } F_\gamma \text{ singular}\}.$$

Then $|M| = 4 \cdot 3^{(n_3-4)/2}$ and $M = -M$. Pairing F with $F_{\eta_{1^8},\gamma}$ we obtain the following result.

Proposition 6.21. *Let $\gamma \in D$ be of norm $q(\gamma) = 1/3 \bmod 1$. Then*

$$|M \cap \gamma^\perp| = \begin{cases} 5|M|/6 & \text{if } \gamma \in M, \\ |M|/3 & \text{otherwise.} \end{cases}$$

This excludes the case $n_3 = 4$. We assume now that $n_3 = 6$. Then the proposition shows that M must be of the form

$$M = \{\pm\gamma_1, \ldots, \pm\gamma_6\}$$

with $(\gamma_i, \gamma_j) = 0 \bmod 1$ for $i \neq j$. In particular, $M^+ = \{\gamma_1, \ldots, \gamma_6\}$ is a basis of D. Let $\gamma \in D$ be of norm $q(\gamma) = 1/3 \bmod 1$ with $\gamma \notin M$. Then γ is a linear combination of four of the γ_i so that $|M \cap \gamma^\perp| = 4$. Hence, the principal part of F satisfies all obstructions coming from $S_{\overline{\rho}_D,4}$. This implies that a reflective modular form with constant coefficient 4 on $II_{6,2}(3^{+6})$ exists.

We give an explicit construction. Let $\gamma \in D$ be of norm $q(\gamma) = 1/3 \bmod 1$. Then the lift of $\theta_{A_2}^2/\eta_{1^8}$ on γ with respect to ρ_D is given by

$$F_{\theta_{A_2}^2/\eta_{1^8},\gamma} = F_{1/3} + F_{1/1}$$

with

$$F_{1/3} = \frac{\theta_{A_2}^2}{\eta_{1^8}}(e^\gamma + e^{-\gamma})$$

and

$$F_{1/1} = \frac{1}{3^3}\sum_{\mu \in D} e(-(\gamma,\mu))g_{j_\mu}(e^\mu + e^{-\mu})$$

where $\theta_{A_2'}^2/\eta_{1^8} = g_0 + g_1 + g_2$ and $g_j|_{-2,T} = e(j/3)g_j$. Note that $g_2 = \theta_{A_2}^2/\eta_{1^8}$.

The function $\eta_{(1/3)^{-3}1^2 3^{-3}}(\tau) = \eta_{1^{-3}3^2 9^{-3}}(\tau/3)$ is a modular form for $\Gamma(3)$ of weight -2. If we decompose $\eta_{(1/3)^{-3}1^2 3^{-3}} = h_0 + h_1 + h_2$ with $h_j|_{-2,T} = e(j/3)h_j$, then $g_2 = h_2$, $g_1 = 4h_1$ and $g_0 = 4h_0$. It follows

$$F_{\theta_{A_2}^2/\eta_{1^8},\gamma} = \tfrac{1}{3}F_{\eta_{(1/3)^{-3}1^2 3^{-3}},\gamma}.$$

Now let $M^+ = \{\gamma_1, \ldots, \gamma_6\} \subset D$ such that $q(\gamma_i) = 1/3 \bmod 1$, $(\gamma_i, \gamma_j) = 0 \bmod 1$ for $i \neq j$ and $M = M^+ \cup (-M^+)$. Define

$$F_{3\theta_{A_2}^2/4\eta_{1^8},M^+} = \frac{3}{4}\sum_{i=1}^{6} F_{\theta_{A_2}^2/\eta_{1^8},\gamma_i}.$$

The components of $F_{3\theta_{A_2}^2/4\eta_{1^8},M^+}$ can be described as follows. Write $\mu \in D$ as $\mu = \sum_{i=1}^6 c_i\gamma_i$ and let $\mathrm{wt}(\mu)$ denote the number of non-zero c_i. Then

$$F_\mu(\tau) = g_2(\tau) = q^{-1/3} + 20q^{2/3} + 176q^{5/3} + 1020q^{8/3} + 4794q^{11/3} + \cdots$$

if $\mu \in M$ and

$$F_\mu = \tfrac{1}{12}(4 - \mathrm{wt}(\mu))g_{j_\mu}$$

with $j_\mu/3 = -q(\mu) \bmod 1$ otherwise. In particular,

$$F_0(\tau) = \tfrac{1}{3}g_0(\tau) = 4 + 60q + 432q^2 + 2328q^3 + 10320q^4 + 40068q^5 + \cdots$$

and $F_\mu = 0$ if $q(\mu) = 1/3 \bmod 1$ and $\mu \notin M$. Hence, F is reflective. Conversely, we have the following proposition.

PROPOSITION 6.22. *Let L be a lattice of genus $II_{6,2}(3^{\epsilon_3 n_3})$ and F a reflective form on L with $[F_0](0) = 4$. Then $n_3 = 6$ and $F = F_{3\theta_{A_2}^2/4\eta_{1^8},M^+}$ for some $M^+ \subset D$ as above.*

Let L be a lattice of genus $II_{6,2}(3^{+6})$. We can decompose L as $L = K \oplus II_{1,1}(3)$ with $K = A_2 \oplus A_2 \oplus II_{1,1}(3)$. Then K has genus $II_{5,1}(3^{-4})$. We choose an orthogonal basis $\{\gamma_1, \gamma_2, \gamma_3, \mu_4\}$ of the discriminant form of K satisfying $q(\gamma_1) = q(\gamma_2) = q(\gamma_3) = -q(\mu_4) = 1/3 \bmod 1$ and an orthogonal basis $\{\mu_5, \gamma_6\}$ of the discriminant form of $II_{1,1}(3)$ satisfying $-q(\mu_5) = q(\gamma_6) = 1/3 \bmod 1$. We define $\gamma_4 = \mu_4 + \mu_5$, $\gamma_5 = \mu_4 - \mu_5$ and $M^+ = \{\gamma_1, \dots, \gamma_6\}$. Let Ψ be the theta lift of $F = F_{3\theta_{A_2}^2/4\eta_{1^8},M^+}$ on L. We choose a primitive norm 0 vector z in $II_{1,1}(3)$. Then z has level 3 and $\mathrm{wt}(z/3) = 3$.

Proposition 6.23. *The expansion of Ψ at the cusp corresponding to z is given by*

$$\begin{aligned}
& e((\rho, Z)) \prod_{\substack{\alpha \in K'^+ \\ \mathrm{wt}(\alpha)=0}} (1 - e((\alpha, Z)))^{[g_0/4](-\alpha^2/2)} (1 - e((3\alpha, Z)))^{[g_0/12](-\alpha^2/2)} \\
& \times \prod_{\substack{\alpha \in K'^+ \\ \mathrm{wt}(\alpha)=3 \\ \mathrm{wt}(\alpha \pm z/3)=3}} (1 - e((3\alpha, Z)))^{[g_0/12](-\alpha^2/2)} \\
& \times \prod_{\substack{\alpha \in K'^+ \\ \mathrm{wt}(\alpha)=3 \\ \mathrm{wt}(\alpha \pm z/3)=6}} (1 - e((\alpha, Z)))^{[g_0/4](-\alpha^2/2)} (1 - e((3\alpha, Z)))^{[-g_0/6](-\alpha^2/2)} \\
& \times \prod_{\substack{\alpha \in K'^+ \\ \mathrm{wt}(\alpha)=2 \\ \mathrm{wt}(\alpha \pm z/3)=2}} (1 - e((3\alpha, Z)))^{[g_1/6](-\alpha^2/2)} \\
& \times \prod_{\substack{\alpha \in K'^+ \\ \mathrm{wt}(\alpha)=2 \\ \mathrm{wt}(\alpha \pm z/3)=5}} (1 - e((\alpha, Z)))^{[g_1/4](-\alpha^2/2)} (1 - e((3\alpha, Z)))^{[-g_1/12](-\alpha^2/2)} \\
& \times \prod_{\substack{\alpha \in K'^+ \\ \mathrm{wt}(\alpha)=5}} (1 - e((\alpha, Z)))^{[-g_1/12](-\alpha^2/2)} \\
& \times \prod_{\substack{\alpha \in K'^+ \\ \mathrm{wt}(\alpha)=1}} (1 - e((\alpha, Z)))^{[g_2](-\alpha^2/2)} \\
& = \sum_{w \in W} \det(w) \eta_{1^3 3^{-2} 9^3}((w\rho, Z)),
\end{aligned}$$

where ρ is a primitive norm 0 vector in K' with $\mathrm{wt}(\rho) = 3$ and $\mathrm{wt}(\rho \pm z/3) = 6$ and W is the reflection group of K' generated by the roots $\alpha \in K'$ of norm $\alpha^2 = 2/3$ and weight $\mathrm{wt}(\alpha) = 1$.

This identity is a new infinite product identity. One can show that it can also be obtained by twisting the denominator identity of the fake monster algebra by an element of class $9C$ in Co_0.

Finally, we consider the case $n = 4$. Let L be a lattice of genus $II_{4,2}(3^{\epsilon_3 n_3})$ and F a reflective form on L with $[F_0](0) = 2$. Then, the Eisenstein condition gives the following value for c_3.

	$II_{4,2}(3^{-1})$	$II_{4,2}(3^{+3})$	$II_{4,2}(3^{-5})$
$c_1 = 0$	$2/9$	$2/3$	2
$c_1 = 1$	$-88/9$	$-34/3$	-16

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Since $S_3(\Gamma(3))$ is trivial, the obstruction space $S_{\overline{\rho}_D,3}$ vanishes. Hence, L carries a reflective form with constant coefficient 2 if and only if it has genus $II_{4,2}(3^{-5})$.

Let D be a discriminant form of type 3^{-5} and $\gamma \in D$ of norm $q(\gamma) = 1/3 \bmod 1$. Then, the lift of $\eta_{1^1 3^{-3}}$ on γ with respect to the Weil representation ρ_D is given by

$$F_{\eta_{1^1 3^{-3}},\gamma} = F_{1/3} + F_{1/1}$$

with

$$F_{1/3} = \eta_{1^1 3^{-3}}(e^{\gamma} + e^{-\gamma})$$

and

$$F_{1/1} = \sum_{\mu \in D} e(-(\gamma,\mu)) g_{j_\mu}(e^{\mu} + e^{-\mu}),$$

where $\eta_{1^{-3}3^1}(\tau/3) = g_0(\tau) + g_1(\tau) + g_2(\tau)$ and $g_j|_{-1,T} = e(j/3)g_j$. Note that $F_{\eta_{1^1 3^{-3}},\gamma}$ is reflective and F_0 has constant coefficient 2.

PROPOSITION 6.24. *Let L be a lattice of genus $II_{4,2}(3^{\epsilon_3 n_3})$ and F a reflective form on L with $[F_0](0) = 2$. Then L has genus $II_{4,2}(3^{-5})$ and $F = F_{\eta_{1^1 3^{-3}},\gamma}$ for some element $\gamma \in D$ of norm $q(\gamma) = 1/3 \bmod 1$.*

Let L be a lattice of genus $II_{4,2}(3^{-5})$. We choose an element $\gamma \in D$ of norm $q(\gamma) = 1/3 \bmod 1$. Let Ψ be the automorphic product corresponding to $F_{\eta_{1^1 3^{-3}},\gamma}$ on L.

We decompose $L = K \oplus II_{1,1}(3)$ such that γ is in the discriminant form of $II_{1,1}(3)$ and choose a primitive norm 0 vector z in $II_{1,1}(3)$. Then $(\gamma, z/3) \neq 0 \bmod 1$. Note that $K = A_2 \oplus II_{1,1}(3)$.

PROPOSITION 6.25. *The expansion of Ψ at the cusp corresponding to z is given by*

$$\prod_{\alpha \in K'^+} (1 - e((\alpha, Z)))^{[3\eta_{1^{-3}3^1}](-3\alpha^2/2)} (1 - e((3\alpha, Z)))^{[-\eta_{1^{-3}3^1}](-3\alpha^2/2)} = 1 + \sum c(\lambda) e((\lambda, Z))$$

where $c(\lambda)$ is the coefficient at q^n in $\eta_{1^3 3^{-1}}$ if λ is n times a primitive norm 0 vector in K'^+ and 0 otherwise.

This is the twisted denominator identity of the fake monster superalgebra corresponding to an element of class $9A$.

We can also decompose $L = K \oplus II_{1,1}(3)$ such that γ is in the discriminant form of K. Again we choose a primitive norm 0 vector z in $II_{1,1}(3)$. Then we have the following result.

PROPOSITION 6.26. *The expansion of Ψ at the cusp corresponding to z is given by*

$$\begin{aligned} e((\rho, Z)) &\prod_{\alpha \in K'^+} (1 - e((3\alpha, Z)))^{[(e((\gamma,\alpha)) + e(-(\gamma,\alpha)))\eta_{1^{-3}3^1}](-3\alpha^2/2)} \\ &\times \prod_{\substack{\alpha \in K'^+ \\ \alpha = \pm\gamma \bmod K}} (1 - e((\alpha, Z)))^{[\eta_{1^1 3^{-3}}](-\alpha^2/2)} \\ &= \sum_{w \in W} \det(w) \eta_{3^{-1}9^3}((w\rho, Z)), \end{aligned}$$

where W is the reflection group of K' generated by the vectors $\alpha \in K'$ of norm $\alpha^2 = 2/3$ satisfying $\alpha = \pm\gamma \bmod K$.

This is the twisted denominator identity of the fake monster algebra corresponding to an element in Co_0 of class $9B$.

The automorphic product Ψ was first described in [DHS15].

6.5 Classification

In this section, we formulate the classification theorems for reflective forms.

First, we list the reflective modular forms on lattices of prime level.

THEOREM 6.27. *Let L be a lattice of prime level and signature $(n,2)$ with $n>2$ carrying a reflective modular form F. Suppose F_0 has constant coefficient $n-2$. Then L and F are given in the following table.*

p	L	F	*Remarks*
2	$II_{18,2}(2_{II}^{+10})$	$F_{\eta_{1^{-8}2^{-8}},0}$	*symmetric*
	$II_{10,2}(2_{II}^{+2})$	$F_{16\eta_{1^{-16}2^{8}},0}$	*symmetric*
	$II_{10,2}(2_{II}^{+n_2})$, $n_2=4,6,\dots,12$	$F_{16\eta_{1^{-16}2^{8}},H}$	$\lvert H\rvert=2^{(n_2-2)/2}$
	$II_{10,2}(2_{II}^{+10})$	$F_{\eta_{1^{8}2^{-16}},0}$	*symmetric*
	$II_{10,2}(2_{II}^{+12})$	$F_{\eta_{1^{8}2^{-16}},0}+F_{16\eta_{1^{-16}2^{8}},H}$	$\lvert H\rvert=2^5$
	$II_{6,2}(2_{II}^{-6})$	$F_{\eta_{1^{4}2^{-8}},\gamma}$	
3	$II_{14,2}(3^{-8})$	$F_{\eta_{1^{-6}3^{-6}},0}$	*symmetric*
	$II_{8,2}(3^{-3})$	$F_{9\eta_{1^{-9}3^{3}},0}$	*symmetric*
	$II_{8,2}(3^{\epsilon_3 n_3})$, $n_3=5,7,9$	$F_{9\eta_{1^{-9}3^{3}},H}$	$\lvert H\rvert=3^{(n_3-3)/2}$
	$II_{8,2}(3^{-7})$	$F_{\eta_{1^{3}3^{-9}},0}$	*symmetric*
	$II_{8,2}(3^{+9})$	$F_{\eta_{1^{3}3^{-9}},0}+F_{9\eta_{1^{-9}3^{3}},H}$	$\lvert H\rvert=3^3$
	$II_{6,2}(3^{+6})$	$F_{(1/4)\eta_{(1/3)^{-3}1^{2}3^{-3}},M^+}$	$M^+=\{\gamma_1,\dots,\gamma_6\}$, $(\gamma_i,\gamma_j)=0 \bmod 1$
	$II_{4,2}(3^{-5})$	$F_{\eta_{1^{1}3^{-3}},\gamma}$	
5	$II_{10,2}(5^{+6})$	$F_{\eta_{1^{-4}5^{-4}},0}$	*symmetric*
	$II_{6,2}(5^{+3})$	$F_{5\eta_{1^{-5}5^{1}},0}$	*symmetric*
	$II_{6,2}(5^{+n_5})$, $n_5=5,7$	$F_{5\eta_{1^{-5}5^{1}},H}$	$\lvert H\rvert=5^{(n_5-3)/2}$
	$II_{6,2}(5^{+5})$	$F_{\eta_{1^{1}5^{-5}},0}$	*symmetric*
	$II_{6,2}(5^{+7})$	$F_{\eta_{1^{1}5^{-5}},0}+F_{5\eta_{1^{-5}5^{1}},H}$	$\lvert H\rvert=5^2$
7	$II_{8,2}(7^{-5})$	$F_{\eta_{1^{-3}7^{-3}},0}$	*symmetric*
11	$II_{6,2}(11^{-4})$	$F_{\eta_{1^{-2}11^{-2}},0}$	*symmetric*
23	$II_{4,2}(23^{-3})$	$F_{\eta_{1^{-1}23^{-1}},0}$	*symmetric*

Conversely each of the functions F is a reflective modular form on L with constant coefficient $[F_0](0)=n-2$.

We have seen that many of these forms give the same function under the singular theta correspondence.

THEOREM 6.28. *Let L be a lattice of prime level and signature $(n,2)$ with $n>2$ and let Ψ be a reflective automorphic product of singular weight on L. Then, as a function on the corresponding Hermitian symmetric domain, the automorphic product Ψ is the theta lift of one of the following modular forms.*

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p	L	F	Co_0
2	$II_{18,2}(2_{II}^{+10})$	$F_{\eta_{1^{-8}2^{-8}},0}$	$1^8 2^8$
	$II_{10,2}(2_{II}^{+2})$	$F_{16\eta_{1^{-16}2^8},0}$	$1^{-8}2^{16}$
	$II_{10,2}(2_{II}^{+10})$	$F_{\eta_{1^8 2^{-16}},0}$	$1^{-8}2^{16}$
	$II_{6,2}(2_{II}^{-6})$	$F_{\eta_{1^4 2^{-8}},\gamma}$	$2^{-4}4^8$
3	$II_{14,2}(3^{-8})$	$F_{\eta_{1^{-6}3^{-6}},0}$	$1^6 3^6$
	$II_{8,2}(3^{-3})$	$F_{9\eta_{1^{-9}3^3},0}$	$1^{-3}3^9$
	$II_{8,2}(3^{-7})$	$F_{\eta_{1^3 3^{-9}},0}$	$1^{-3}3^9$
	$II_{6,2}(3^{+6})$	$F_{(1/4)\eta_{(1/3)^{-3}1^2 3^{-3}},M^+}$	$1^3 3^{-2}9^3$
	$II_{4,2}(3^{-5})$	$F_{\eta_{1^1 3^{-3}},\gamma}$	$3^{-1}9^3$
5	$II_{10,2}(5^{+6})$	$F_{\eta_{1^{-4}5^{-4}},0}$	$1^4 5^4$
	$II_{6,2}(5^{+3})$	$F_{5\eta_{1^{-5}5^1},0}$	$1^{-1}5^5$
	$II_{6,2}(5^{+5})$	$F_{\eta_{1^1 5^{-5}},0}$	$1^{-1}5^5$
7	$II_{8,2}(7^{-5})$	$F_{\eta_{1^{-3}7^{-3}},0}$	$1^3 7^3$
11	$II_{6,2}(11^{-4})$	$F_{\eta_{1^{-2}11^{-2}},0}$	$1^2 11^2$
23	$II_{4,2}(23^{-3})$	$F_{\eta_{1^{-1}23^{-1}},0}$	$1^1 23^1$

Hence, with three exceptions, all these functions come from symmetric modular forms. Moreover, at a suitable cusp Ψ is the twisted denominator identity of the fake monster algebra by the indicated element in Conway's group.

Conversely all the given modular forms lift to reflective automorphic products of singular weight on the corresponding lattices.

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