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# THE CENTERS OF SEMI-SIMPLE ALGEBRAS OVER A COMMUTATIVE RING, II

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***Dedicated to Professor Keizo Asano on his 60th birthday***

In this note we assume that all rings have identities and denote by $R$ a commutative ring. All $R$-algebras considered are assumed to be finitely generated faithful $R$-modules. An $R$-algebra $\Lambda$ is said to be semi-simple ([5]), if any finitely generated $\Lambda$-module is $(\Lambda, R)$-projective. Further $\Lambda$ is said to be weakly semi-simple, if for any maximal ideal $\mathfrak{m}$ of $R$, $\Lambda/\mathfrak{m}\Lambda$ is semi-simple over $R/\mathfrak{m}$. Especially a (weakly) semi-simple $R$-algebra is called (weakly) simple if it is indecomposable as a ring. It was shown in [5] that any semi-simple $R$-algebra is weakly semi-simple. Formal properties of (weakly) semi-simple $R$-algebras were studied in [5], [6], [3], etc. The purpose of this note is, as a continuation to [3], to solve negatively the following basic problems on semi-simple $R$-algebras:

(I) *Is any central semi-simple $R$-algebra, which is a projective $R$-module, separable?*

(II) *Let $\Lambda$ be a semi-simple $R$-algebra which is a projective $R$-module. Is the center of $\Lambda$ also semi-simple over $R$?*

(III) *Is any weakly semi-simple $R$-algebra, which is a projective $R$-module, semi-simple over $R$?*

**1.** We have proved in [3], (2. 1) that the answer to (I) is affirmative for any Artinian ring $R$. Now we give a negative answer to (I) in case $R$ is a discrete (rank-one) valuation ring in the following

THEOREM 1. *Let $R$ be a discrete (rank-one) valuation ring with a maximal ideal $\mathfrak{p}$. We assume that the characteristic of $R/\mathfrak{p}$ is $p>0$. If the characteristic of $R$ is zero, we further assume that $R$ contains the primitive $p$-th root $\zeta_p$ of 1. Then the following conditions are equivalent for $R$:*

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(1) $[R/\mathfrak{p} : (R/\mathfrak{p})^p] \geqq p^2$.

(2) *There exists a central simple, non-separable R-algebra which is a free R-module.*

In [2], (8. 1) we gave the affirmative answer to [III] for any Dedekind domain $R$. Hence Theorem 1 is an immediate consequence of the following more general

THEOREM 2. *Let $R$ be a local integral domain with a maximal ideal $\mathfrak{m} \neq 0$ and $K$ the quotient field of $R$. We assume that the characteristic of $R/\mathfrak{m}$ is $p > 0$. If the characteristic of $R$ is 0, we further assume that $R$ contains the primitive $p$-th root $\zeta_p$ of 1. Then the following conditions are equivalent for $R$:*

(1) $[R/\mathfrak{m} : (R/\mathfrak{m})^p] \geqq p^2$.

(2) *There exists a central weakly semi-simple, non-separable $R$-algebra $\Lambda$, which is a free $R$-module, such that $K \underset{R}{\otimes} \Lambda$ is separable over $K$.*

We can observe in the proof of [3], (2. 3) that, if the answer to (II) is affirmative for any Artinian ring $R$, then the answer to (I) is also affirmative for any Noetherian ring $R$. Therefore, by Theorem 1, the answer to (II) is negative for an Artinian ring $R$.

Hence we have only to give the proof of Theorem 2 and a counter-example to (III).

**2.** We shall give here the proof of Theorem 2. The implication (1) $\Longrightarrow$ (2). Suppose that $[R/\mathfrak{m} : (R/\mathfrak{m})^p] \geqq p^2$. Then there exist elements $\bar{\alpha}, \bar{\beta} \in (R/\mathfrak{m})^{1/p}$ such that $[R/\mathfrak{m}(\bar{\alpha}, \bar{\beta}) : R/\mathfrak{m}] = p^2$. We put $\bar{a} = \bar{\alpha}^p$ and $\bar{b} = \bar{\beta}^p$ and denote by $a$, $b$ the representatives of $\bar{a}$, $\bar{b}$ in $R$, respectively. Now it suffices to construct a central weakly simple, non-separable $R$-algebra $\Lambda$, which is a free $R$-module, such that $K \underset{R}{\otimes} \Lambda$ is separable over $K$, in each of the following cases:

(i) $R$ is of characteristic $p$.

(ii) $R$ is of characteristic 0 and contains the primitive $p$-th root, $\zeta_p$, of 1.

Let $X$, $Y$ be two indeterminates and $u$ a non-zero element in $\mathfrak{m}$. We put $F(X) = X^p - u^{p-1}X - a$ in Case (i) and $F(X) = X^p - a$ in Case (ii). Then we have $F(X)K[X] \cap R[X] = F(X)R[X]$, since $F(X)$ is monic, and therefore, putting $L = K[X]/F(X)K[X]$ and $S = R[X]/F(X)R[X]$, $L$ is the total quotient ring of $S$. Let $\alpha$ be the residue of $X$ in $L$. As is well known, $L$ is a Galois extension of $K$ whose Galois group, $G$, is a cyclic group of order $p$ which is generated by $\sigma$ such that $\sigma(\alpha)=\alpha+u$ in Case (i) and $\sigma(\alpha)=\alpha\zeta_p$ in Case (ii). Obviously $\sigma$ operates on $S$ as an automorphism over $R$ and the subring of $S$ consisting of all elements in $S$ fixed under $G$ coincides with $R$. Let $S[Y]$ be the non-commutative polynomial ring $S$ such that $s^{\sigma}Y=Ys$ for any $s\in S$ and put $\Lambda=S[Y]/(Y^p-b)S[Y]$ and $\sum=K\otimes_R\Lambda$. Then $\Lambda$ is a free $R$-module, and $\sum$ is a central separable $K$-algebra because it is a crossed product. Since $S/\mathfrak{m}S\cong R[\alpha]/\mathfrak{m}R[\alpha]\cong R/\mathfrak{m}[X]/(X^p-\bar{a})R/\mathfrak{m}[X]\cong R/\mathfrak{m}[\alpha]$, $\sigma$ induces the identity on $S/\mathfrak{m}S$, so that $\Lambda/\mathfrak{m}\Lambda$ is commutative. Hence we observe $\Lambda/\mathfrak{m}\Lambda\cong(R/\mathfrak{m}[\bar{\alpha}])[Y]/(Y^p-\bar{b})(R/\mathfrak{m}[\alpha])[Y]$. Because, by the assumption, $Y^p-\bar{b}$ is irreducible in $(R/\mathfrak{m}[\bar{\alpha}])[Y]$, we have $\Lambda/\mathfrak{m}\Lambda=R/\mathfrak{m}[\bar{\alpha},\bar{\beta}]$. Thus $\Lambda$ is a central weakly simple, non-separable $R$-algebra as required.

The implication $(2)\Longrightarrow(1)$. If $R/\mathfrak{m}$ is perfect, then by [4], (1. 1), any weakly semi-simple $R$-algebra is separable. Therefore it suffices to prove, under the assumption that $[R/\mathfrak{m}:(R/\mathfrak{m})^p]=p$, that any central weakly simple $R$-algebra $\Lambda$, which is a free $R$-module, such that $K\otimes_R\Lambda$ is separable over $K$ is separable over $R$. Let $\Lambda$ be a central weakly simple $R$-algebra, which is a free $R$-module, such that $K\otimes_R\Lambda$ is separable over $K$. By using the Henselization of $R$, we may suppose that $\Lambda/\mathfrak{m}\Lambda$ is a division $R/\mathfrak{m}$-algebra. Let $\bar{C}$ be a maximal commutative subfield of $\Lambda/\mathfrak{m}\Lambda$, and put $n^2=\dim_{R/\mathfrak{m}}\Lambda/\mathfrak{m}\Lambda=\operatorname{rank}_R\Lambda$ and $m=\dim_{R/\mathfrak{m}}\bar{C}$. Then it is well known that $n\leqq m$. However, since $[R/\mathfrak{m}:(R/\mathfrak{m})^p]=p$, we have $\bar{C}=R/\mathfrak{m}[\bar{\alpha}]$ for some $\bar{\alpha}\in\bar{C}$. If we denote by $\alpha$ a representative of $\bar{\alpha}$ in $\Lambda$, then $\{1,\alpha,\alpha^2,\cdots,\alpha^{n-1}\}$ is a subset of a free $R$-basis of $\Lambda$. But, since the degree of the reduced characteristic polynomial of $\alpha$ in $K\otimes_R\Lambda$ is equal to $n$ by [4], § 3, we find that $\alpha^n\in K+K\alpha+\cdots+K\alpha^{n-1}$. From this it follows that $m\leqq n$, i.e., $m=n$. Thus the center of $\Lambda/\mathfrak{m}\Lambda$ coincides with $R/\mathfrak{m}$, so that $\Lambda/\mathfrak{m}\Lambda$ is a central simple $R/\mathfrak{m}$-algebra. Again by [4], (1. 1), we know that $\Lambda$ is separable over $R$. This completes the proof of the theorem.

We note that, in Theorem 2, the hypothesis that $R$ is a local integral domain with a maximal ideal $\mathfrak{m}\neq 0$ can be replaced by the weaker one that $R$ is a local ring whose maximal ideal, $\mathfrak{m}$, contains a non-zero divisor in $R$ and that the implication $(2)\Longrightarrow(1)$ in Theorem 2 was proved without assuming that $R$ contains the primitive $p$-th root of 1 in case it is of characteristic 0.

Also it should be noted that there exists a discrete valuation ring $R$ which satisfies the condition (1) in Theorem 1 (or 2). In fact let $T_1$, $T_2$ and $U$ be three indeterminates and $p$ a prime integer. Now we put $k = Z/pZ(T_1, T_2)$. Then the formal power series ring $k[[U]]$ over $k$ with $U$ is a discrete valuation ring of equi-characteristic $p > 0$ which satisfies the condition (2) in Theorem 1.

**3.** We shall give a negative answer to (III) by exhibiting an example of a commutative weakly simple, non-simple algebra over a regular local ring $R$ with Krull dimension 2 which is a free $R$-module.

Let $R_0$ be a regular local ring of characteristic 3 with Krull dimension 2 and $\mathfrak{m}_0 = uR_0 + vR_0$ the unique maximal ideal of $R_0$. Let $T$, $X$ be indeterminates. If we put $R = R_0[T]_{\mathfrak{m}_0 R_0[T]}$ and $\mathfrak{m} = \mathfrak{m}_0 R$, then $R$ is also a regular local ring with a maximal ideal $\mathfrak{m}$. Further put $F(X) = X^3 - uX^2 + 2vX - T \in R[X]$ and $S = R[X]/F(X)R[X]$. Then we have $S/\mathfrak{m}S \cong R/\mathfrak{m}[X]/(X^3 - T)\, R/\mathfrak{m}[X] \cong R/\mathfrak{m}[T^{1/3}]$, and therefore $S$ is weakly simple over $R$. An element $v^3 + u^2v^2 - u^3T$ of $R$ is prime, because it is prime in $R_0[T]$ and contained in $\mathfrak{m}$. Therefore, putting $\mathfrak{p} = (v^3 + u^2v^2 - u^3T)R$, $\mathfrak{p}$ is a prime ideal of height 1 in $R$. Now we denote by $\bar{u}$, $v$ the residues of $u$, $v$ in $R_\mathfrak{p}/\mathfrak{p}R_\mathfrak{p}$ and by $\bar{F}(X)$ the residue of $F(X)$ in $R_\mathfrak{p}/\mathfrak{p}R_\mathfrak{p}[X]$. Then we have $d\bar{F}(X)/dX = -2(\bar{u}X - v)$ and $\bar{F}(v/\bar{u}) = \frac{1}{\bar{u}^3}(v^3 + \bar{u}^2v^2 - \bar{u}^3T) = 0$ and so $\bar{F}(X)$ has a multiple root $v/\bar{u}$. Hence, since $S_\mathfrak{p}/\mathfrak{p}S_\mathfrak{p} \cong R_\mathfrak{p}/\mathfrak{p}R_\mathfrak{p}[X]/\bar{F}(X)R_\mathfrak{p}/\mathfrak{p}R_\mathfrak{p}[X]$, $S_\mathfrak{p}/\mathfrak{p}S_\mathfrak{p}$ is not simple over $R_\mathfrak{p}/\mathfrak{p}R_\mathfrak{p}$. This implies that $S_\mathfrak{p}$ is not simple over $R_\mathfrak{p}$. Accordingly $S$ is not a simple $R$-algebra. Thus $S$ is as required.

Let $R$ be a Noetherian integrally closed integral domain with quotient field $K$ and $\Lambda$ an $R$-order, which is a projective $R$-module, in a separable $K$-algebra. For such $\Lambda$ we consider the following statements:

(1) $\Lambda$ is weakly semi-simple over $R$.

(2) For any prime ideal $\mathfrak{p}$ of height 1 in $R$, $\Lambda_\mathfrak{p}$ is semi-simple over $R_\mathfrak{p}$.

The above example means that (1) $\Longrightarrow$ (2) is not always valid. We suppose that $\Lambda$ satisfies one of the following conditions:

(i) $R$ is a regular domain and $\Lambda$ is commutative.

(ii) $\Lambda$ has $R$ as its center.

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Then it is well known (cf. [7], (41. 1) and [1], (4. 6)) that $\Lambda$ is separable over $R$ whenever, for any prime ideal $\mathfrak{p}$ of height 1 in $R$, $\Lambda_{\mathfrak{p}}$ is separable over $R_{\mathfrak{p}}$. Finally we shall show that (2) does not imply (1) generally even in case $\Lambda$ satisfies (i) or (ii).

Let $k$ be a field of characteristic 2 and $T$, $U$, $V$, $X$, $Y$ be indeterminates. Now we put $R = k[T, U, V]_{Uk[T,U,V]+Vk[T,U,V]}$ and $\mathfrak{m} = UR + VR$. Then $R$ is a regular local ring with a maximal ideal $\mathfrak{m}$ and with Krull dimension 2, and we have $R/\mathfrak{m} \cong k(T)$. Further put $F(X) = X^2 - UX - V$ and $L = K[X]/F(X)K[X]$. Since $F(X)$ is a monic polynomial of the Artin-Schreier type whose coefficients are contained in $R$, $L$ is a Galois extension of $K$ and $F(X)K[X] \cap R[X] = F(X)R[X]$. Therefore, putting $S = R[X]/F(X)R[X]$, $L$ can be considered as the quotient field of $S$. Obviously $S$ is a regular local ring with a maximal ideal $\mathfrak{m} = XS + US$ and $S/\mathfrak{m} \cong R/\mathfrak{m} \cong k(T)$. It can be easily seen that $S/\mathfrak{m}S \cong R/\mathfrak{m}[X]/X^2R/\mathfrak{m}[X]$, and so $S$ is not weakly semi-simple over $R$. However, for a prime ideal $\mathfrak{p} = UR$ of $R$, we have $S_{\mathfrak{p}}/\mathfrak{p}S_{\mathfrak{p}} \cong R_{\mathfrak{p}}/\mathfrak{p}R_{\mathfrak{p}}[X]/(X^2 - V)R_{\mathfrak{p}}/\mathfrak{p}R_{\mathfrak{p}}[X] = k(T, V^{1/2})$, so that $S_{\mathfrak{p}}$ is simple over $R_{\mathfrak{p}}$. On the other hand, for any prime ideal $\mathfrak{q} \not\ni U$ of height 1 in $R$, the residue $\bar{F}(X)$ of $F(X)$ in $R_{\mathfrak{q}}/\mathfrak{q}R_{\mathfrak{q}}[X]$ is a separable polynomial and $S_{\mathfrak{q}}/\mathfrak{q}S_{\mathfrak{q}} \cong R_{\mathfrak{q}}/\mathfrak{q}R_{\mathfrak{q}}[X]/\bar{F}(X)R_{\mathfrak{q}}/\mathfrak{q}R_{\mathfrak{q}}[X]$, so that $S_{\mathfrak{q}}$ is separable over $R_{\mathfrak{q}}$. Thus a commutative $R$-algebra $S$ satisfies (2) but does not satisfy (1). If we denote the residue of $X$ in $L$ by $x$, then the Galois group of a Galois extension $L$ of $K$ is generated by $\sigma$ such that $\sigma(x) = x + 1$. Hence $\sigma$ operates on $S$ as an automorphism over $R$ and the subring of $S$ consisting of all elements of $S$ fixed under $\sigma$ coincides with $R$. Let $S[Y]$ be the non-commutative polynomial ring over $S$ with $Y$ such that $s^{\sigma}Y = Ys$ for any $s \in S$ and put $\Lambda = S[Y]/(Y^2 - T)S[Y]$. Then $\Lambda$ is a central $R$-algebra which is a free $R$-module and $K \otimes_R \Lambda$ is separable over $K$. It can easily be shown that $\Lambda$ is not weakly semi-simple over $R$ but, for any prime ideal $\mathfrak{p}$ of height 1 in $R$, $\Lambda_{\mathfrak{p}}$ is simple over $R_{\mathfrak{p}}$.

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