BULL. AUSTRAL. MATH. SOC. 20H20, 20E07
VOL. 66 (2002) [419–423]

MOST FINITELY GENERATED SUBGROUPS OF INFINITE UNITRIANGULAR MATRICES ARE FREE

WALDEMAR HOŁUBOWSKI

In this note we prove that the group G of infinite dimensional upper unitriangular matrices over a finite field contains an explicit countable subgroup 'full' of free subgroups. We deduce from this fact that, in a suitable sense, almost all $k-$generator subgroups of G are free groups of rank k.

1. INTRODUCTION

Let $G = UT(\infty, p^s)$ be the group of all infinite dimensional (indexed by $\mathbb{N}$) upper unitriangular matrices over the finite field of order p^s (where p is any prime). The set N_m of all matrices a from G such that first m columns of a are the same as those in the unit matrix e is a normal subgroup of G. Clearly, we have $|G : N_m| < \infty$ and G is a profinite group as an inverse limit of $G/N_m \simeq UT(m, p^s)$ [**12, 11**]. The profinite topology induces a metric $d(x, y)$ under which G is a complete metric space. The same is true for $G^k = G \times \ldots \times G$, considering the natural direct product extension of the metric $d(x, y)$. If $x \in G^k$ then $\langle x \rangle$ denotes the subgroup of G generated by the components of x. We put $F = \{x \in G^k \mid \langle x \rangle \text{ is a free group of rank } k\}$.

A subset of a metric space is called *nowhere dense* if its complement contains a dense open subset. The union of a countable family of nowhere dense sets is called a *meagre set* (or of the first category in the sense of Baire). Baire's theorem states that in a complete metric space the complement of a meagre set is dense [**8**]. Thus in a complete metric space a meagre set is very small; for example, the whole space cannot be written as the union of a countable family of meagre subsets.

In [**4**] Epstein showed that almost all $k-$generator subgroups in a connected, non-solvable, finite dimensional Lie group are free groups of rank k, where 'almost all' is interpreted in terms of the natural Haar measure on the group. In [**3**] Dixon proved that almost all $k-$generator subgroups in permutation groups of countably infinite degree are free groups of rank k in the natural permutation group topology. Bhattacharjee obtained similar results in [**1**] for inverse limit of wreath products of non-trivial groups. We prove the following

Received 20th March, 2002

THEOREM 1. *Almost all k-generator subgroups of $G = UT(\infty, p^s)$ are free groups of rank k, in the sense that $G^k \setminus F$ is meagre subset of G^k.*

The theorem above may be contrasted with the fact that the finite dimensional groups $UT(m, p^s)$ are finite and the stable group $UT_\omega(p^s)$, which is the direct limit of $UT(m, p^s)$ under the natural embeddings, is locally finite, so it does not contain free subgroups.

Gartside and Knight proposed in [5] a new approach, using Polish topological groups, which gives many equivalent conditions for the property proved in Theorem 1. The Main Theorem of [5] enables us to strengthen the result above in the following way

COROLLARY 1.

(i) *Almost all countably generated subgroups of $G = UT(\infty, p^s)$ are free groups of countable rank.*

(ii) *$G = UT(\infty, p^s)$ contains a non-discrete free subgroup of rank two.*

Our proof of Theorem 1 is different from those in [**1, 3, 4, 5**], and is extremely explicit. We deduce it from the fact that G has an explicit countable subgroup 'full' of free subgroups. Our main result is

THEOREM 2. *The group $G = UT(\infty, p^s)$ contains a countable subgroup H such that the intersection of H^k with any open ball in G^k contains a free subgroup of rank k, given by explicit generators.*

Another advantage of our approach is the possibility of proving similar results for semigroups. For example, almost all subsemigroups of the multiplicative semigroup of all infinite upper triangular matrices over a finite field are free ([**7**]).

We conclude with some open questions. All free subgroups considered in this note are discrete. So it would be interesting to find explicit generators of non-discrete free subgroups of rank 2, which exist by Corollary 1 (ii). It is known that the group of all permutations of an infinite set contains a free subgroup of rank $2^{\aleph_0}$. Is the same true for $UT(\infty, p^s)$?

2. Proof of the main result

The profinite topology on G induces the following metric.

DEFINITION 1: For $x, y \in G$, if $x = y$, then we put $d(x, y) = 0$, and if $x \neq y$, then we put $d(x, y) = 2^{-m}$, where $m \in \mathbb{N}$ is the least integer such that the m−th column of x and y are different, that is, $xy^{-1} \in N_m$.

The group operations $(x, y) \mapsto xy$ and $x \mapsto x^{-1}$ on G are continuous under this metric and (G, d) is complete (as a metric space). We define a metric d_k on G^k in the natural way:

$$d_k(x, y) = \max\{d(x_i, y_i) \mid i = 1, \ldots, k\},$$

where $x = (x_1, \ldots, x_k)$ and $y = (y_1, \ldots, y_k)$. We denote by $B(x,n)$ the open ball in G^k with radius 2^{-n}. $B(x,n)$ is both open and closed (equal to the closed ball with radius 2^{-n-1}). The ball (G^k, d_k) is complete and totally disconnected.

PROOF OF THEOREM 2: Let $B(x,n)$ be any fixed open ball in G^k. So $x = (x_1, \ldots, x_k) \in G^k$ and $n \in \mathbb{N}$. We want to show first that $B(x,n)$ contains the generators of a free subgroup. Let $x'_1, \ldots, x'_k$ denote the images of $x_1, \ldots, x_k$ under the homomorphism $f : UT(\infty, p^s) \to UT(n, p^s)$ which deletes the rows and columns indexed by $n+1, n+2, \ldots$. If $y_1, \ldots, y_k$ generate a free subgroup of G, then the infinite unitriangular matrices

$$z_1 = \operatorname{diag}(x'_1, y_1), \ \ldots, \ z_k = \operatorname{diag}(x'_k, y_k)$$

generate a free subgroup too. Moreover $z = (z_1, \ldots, z_k) \in B(x,n)$.

Now we give examples of free subgroups of G with explicit generators. The main result of [**10**] shows that $UT(\infty, 2)$ contains a free product of three cyclic groups of order two, and thus, by [**9**], contains a noncyclic free subgroup. In fact, in [**10**] two matrices are given explicitly, which generate a free subgroup. In [**6**] it was shown that the two infinite matrices

$$c = \operatorname{diag}(t_{12}(1), t_{12}(1), \ldots) \ \text{ and } \ d = \operatorname{diag}\bigl(1, t_{12}(1), t_{12}(1), \ldots\bigr),$$

where $t_{12}(1)$ is the elementary transvection $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, generate a free subgroup in $UT(\infty, \mathbb{Z})$. For $p > 2$, they generate in $UT(\infty, p)$ a free product of two cyclic groups of order p, which, by [**9**], also contains a noncyclic free subgroup. For example, $x = c(cd)c^{-1}$, $y = dc(cd)c^{-1}d^{-1}$ generate a free subgroup. Standard considerations show that if x, y generate a free group, then $yxy^{-1}, \ldots, y^k x y^{-k}$ generate a free subgroup of rank k. The above results give also free subgroups in $UT(\infty, p^s)$ for any s, because the finite field of order p^s contains a prime subfield of order p.

The free subgroups in [**6**] and [**10**] belong to an interesting subgroup $UT_a(\infty, p^s)$ of $UT(\infty, p^s)$ connected with finite state automata transformations. An infinite upper unitriangular matrix $A = (a_{ij})$ is called m-banded if $a_{ij} = 0$ for $j > i+m$, and banded if it is banded for some m. By $A_{[n]}$ we denote the submatrix of A which arises by deleting the first n rows and first n columns of A. We say that the sequence $\{A_{[n]}\}$ is almost periodic if it is periodic, that is, $A_{[n+d]} = A_{[n]}$, starting from some fixed n_0. Then $UT_a(\infty, p^s)$ is defined as the subgroup of all matrices A for which both A and A^{-1} are banded and both sequences $\{A_{[n]}\}$ and $\{A^{-1}_{[n]}\}$ are almost periodic. It is clear that $UT_a(\infty, p^s)$ is countable and the matrices

$$z_1 = \operatorname{diag}(x'_1, y_1), \ \ldots, \ z_k = \operatorname{diag}(x'_k, y_k)$$

belong to $UT(n, p^s) \times UT_a(\infty, p^s)$. The group $H = \left[\bigcup_{n=1}^{\infty} UT(n, p^s)\right] \times UT_a(\infty, p^s)$ is countable too and Theorem 2 is proved. ∎

PROOF OF THEOREM 1: If w is a reduced word in the free group of rank k, then we put $F(w) = \{x \in G^k \mid w(x) \neq e\}$. Clearly $F = \bigcap F(w)$ where the intersection is taken over all nontrivial reduced words w. We note that this is a countable intersection. Since

$$G^k \setminus F = G^k \setminus \bigcap F(w) = \bigcup (G^k \setminus F(w))$$

it suffices to prove that $F(w)$ is open and dense. This would show that $G^k \setminus F(w)$ is nowhere dense and $G^k \setminus F$ is meagre. ∎

LEMMA 1. *$F(w)$ is open in G^k.*

PROOF: Since the group operations are continuous, the mapping $\phi : G^k \to G$ defined by $\phi(x) = w(x)$ is continuous too. The set $\{e\}$ is closed in G, so $G \setminus \{e\}$ is open and $F(w) = \phi^{-1}(G \setminus \{e\})$ is open in G^k. ∎

LEMMA 2. *$F(w)$ is dense in G^k.*

PROOF: We show that F is dense in G^k and because $F \subseteq F(w)$ the lemma follows. Let $x = (x_1, \ldots, x_k) \in G^k$ and $n \in \mathbb{N}$. From Theorem 2 it follows that any open ball $B(x, n)$ contains a point of F, so F is dense. ∎

REMARK. The group $UT(\infty, p^s)$ has many interesting non-free subgroups. For example, the Nottingham group $\mathcal{N}$ can be viewed as a subgroup. It is known that every countably-based pro-p-group can be embedded in $\mathcal{N}$ [2], and so in $UT(\infty, p^s)$. In particular, every finitely generated residually finite p-group can be embedded in $UT(\infty, p^s)$.

REFERENCES

[1] M. Bhattacharjee, 'The ubiquity of free subgroups in certain inverse limits of groups', *J. Algebra* **172** (1995), 134–146.

[2] R. Camina, 'Some natural subgroups of the Nottingham group', *J. Algebra* **196** (1997), 101–113.

[3] J.D. Dixon, 'Most finitely generated permutation groups are free', *Bull. London Math. Soc.* **22** (1990), 222–226.

[4] D.B.A. Epstein, 'Almost all subgroups of a Lie group are free', *J. Algebra* **19** (1971), 261–262.

[5] P.M. Gartside and R.W. Knight, 'Ubiquity of free subgroups', (preprint).

[6] W. Hołubowski, 'Free subgroups of the group of infinite unitriangular matrices', *Internat. J. Algebra Comput.* (to appear).

[7] W. Hołubowski, 'The ubiquity of free subsemigroups of infinite triangular matrices', *Semigroup Forum.* (to appear).

[8] J.L. Kelley, *General topology* (Van Nostrand, New York, 1955).

[9] W. Magnus, A. Karras and D. Solitar, *Combinatorial group theory* (John Wiley and Sons, Inc., New York, 1966).

[10] A. Olijnyk and V.S. Sushchansky, 'Free group of infinite unitriangular matrices', *Math. Notes* **67** (2000), 320–324.

https://doi.org/10.1017/S0004972700040272 Published online by Cambridge University Press

[11] L. Ribes and P. Zalesskii, *Profinite groups* (Springer-Verlag, Berlin, Heidelberg, New York, 2000).

[12] J.S. Wilson, *Profinite groups*, London Math. Soc. Monographs New Series **19** (Oxford University Press, Oxford, 1998).

Institute of Mathematics
Silesian Technical University
Kaszubska 23
44-101 Gliwice
Poland
e-mail: wholub@polsl.gliwice.pl