ON THE CALCULATION OF PREMIUMS FOR ASSURANCES ON LIVES AND SURVIVORSHIPS BY THE AID OF MR. GOMPERTZ'S HYPOTHESIS.

To the Editor of the Assurance Magazine.

SIR,—Your readers will have observed, with much gratification, the announcement made by Mr. Gompertz (in his letter to Mr. Porter, published in your April Number), that, during the last two or three years, that eminent mathematician has been engaged in adding to his important discoveries in connection with the law of mortality and the methods of computing the values of life contingencies; and they will most sincerely echo the wish that his health may permit him, ere long, to lay the complete results of his very valuable labours before the public, through the medium of the Royal Society.

The discovery of methods for shortening the labour required in computing correctly the values of intricate cases of survivorship assurances must be considered of the highest importance by all engaged in the business of life assurance; and I trust that, notwithstanding the announcement above referred to, which justifies the expectation of a full and comprehensive treatment of this important subject, the following brief description of a general formula (based upon what Mr. Gompertz has already given us), for the solution of the more usual cases of survivorships, may not be altogether devoid of interest.

In a paper headed "On the law of mortality and the construction of annuity tables," published in vol. viii. of this *Magazine*, I ventured to suggest a method which, by means of a slight modification of Mr. Gompertz's formula, appeared to me to possess some utility in facilitating the computation of the values of *annuities* on several lives. The alteration in question consisted simply in introducing into the formula for the probabilities of living an additional constant, in such way that it should combine with the constant representing the interest of money in the corresponding formula for the *values of sums* depending upon those probabilities; and, by this means, preserve an important property of Mr. Gompertz's formula, first observed by Professor De Morgan—viz., the power of substituting an equivalent single age, easily determined, for any combination of joint ages; with this difference, however, that, by the introduction of the additional constant referred to, the substitution consists of an *equal number of lives*, of a certain common age, in lieu of a *single life*.*

The subjoined tables, representing the decrements and the expectation of life at all ages, are constructed upon the principle explained in the paper before referred to. They are based upon the Carlisle Table; but, for the sake of convenience in calculation, I have slightly altered the values of the several constants yielded by that table—considering, for the object in view, a very close adherence to any particular table unnecessary. Nevertheless, upon a comparison of the expectation of life, it will, I think, be found that

* My paper having been drawn up previously to the publication of Professor De Morgan's article in the *Assurance Magazine* for July, 1859 (which is the only one I have seen on the subject), the enunciation of the property in question is given as *new*. In a letter to the Editor, which accompanied my manuscript a few days after the publication of the Number for July, 1859, I referred to Mr. De Morgan's prior discovery, but, by an oversight, omitted to insert a similar reference in the paper itself.

the difference does not exceed the licence usually allowed in the adjustment of mortality tables.

The first table, which I give entire, containing the decrements and expectation of life, needs no explanation, as the values correspond exactly with those formed by the usual methods. The annuity tables, however (of which extracts only are given), are constructed by the following formula—

$$\frac{1}{B_m^\mu}\int_o^\infty \left(B_m^\mu\right)^{qx} \left(\frac{v}{a^\mu}\right)^x . \, dx,^*$$

which denotes the value of an annuity *payable momently* during the joint existence of μ lives each aged m years, and consequently they differ in this respect from annuity tables in ordinary use. Various methods, more or less convenient, may be adopted for calculating the values of the above integral, but it is not my purpose to enter into this subject further than to state that in the tables in question the values are computed to the fourth decimal place. As, however, three decimal places are sufficient for most purposes, that number only is given in the extracts, and it will be observed that each annuity table for any given number of lives to three places could be comprised in the same amount of space as a table for single lives.

Adopting a table of annuities payable *momently* as the basis of calculation, let such an annuity on the joint existence of any given number of lives aged respectively $m, n, r \ldots$ be represented by $A_{m,n,r\ldots}$. To find the value of a similar annuity, payable t times a year, we have the following simple formula—

$$A_{m,n,r\ldots} \pm \frac{1}{2t},$$

the upper sign being taken when the payments are in advance, and the lower sign when in arrear.

By the aid of these *momently* annuities, a convenient general formula may be deduced for the *exact* solution of the following comprehensive problem in survivorship assurances.

Problem.—Required the value of £1 payable at the failure of the joint existence of μ lives aged respectively $m, n \ldots$ and r, provided that event shall happen before the failure of the joint existence of ν other lives aged respectively $u, v \ldots$ and z.

The formula by which this problem is solved is—

$$\frac{S_m}{S_m+S_u}\left\{1+\left(\log_e v - \log_e a \, . \, \frac{\nu S_m - \mu S_u}{S_m}\right) A_{m,n\ldots z}\right\} \quad . \quad [1]$$

where $S_m = q^m + q^n + \ldots + q^r$, $S_u = q^u + q^v + \ldots + q^z$ and $A_{m\ldots z}$ = the value of an annuity (payable momently) during the joint existence of all the lives involved.

In the case of an absolute assurance on the joint lives $m, n \ldots$ and r, the quantities ν and S_u vanish, and the formula becomes—

$$1+\log_e v \, . \, A_{m.n\ldots r} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad [2]$$

For calculating the value of an assurance on a single life (m) against ν other lives (u), (v) . . . (z), the general formula [1] may be put in the following more convenient form—

* The characters correspond with those used in my paper of January, 1860.

$$\frac{q^{m-y}}{q^{m-y}+q^{u-y}+\ldots+q^{z-y}}\left\{1+(\log_e v+C_{u-m}+C_{v-m}+\ldots+C_{z-m})A_{m,u\ldots z}\right\} \quad [3]$$

where $C_n=\log_e a(q^n-1)$, and y is the age of the youngest of all the lives involved.

When the problem refers to one life (m) against another life (u), the formula may be still further simplified, thus—

$$F_{u-m}+G_{u-m}.A_{m,u} \quad \ldots\ldots\ldots \quad [4]$$

where $F_n=(1+q^n)^{-1}$ and $G_n=F_n.(\log_e v+C_n)$.

It should be observed that the preceding formulæ give the value of the reversion payable at the instant of death, from which the value of the same, payable at the expiration of any time, t, after death, may be accurately determined by multiplying by v^t; and the value of the given sum payable at the expiration of the year of death may be considered equivalent to the same payable six months after death, and determined accordingly.

In taking as the basis of calculation the values of annuities payable momently, it would, perhaps, be preferable that the annual premiums should be supposed to be payable in the same manner—*i. e.*, by momently instalments. The general formula for the annual premium payable momently during the joint existence of all the lives would be—

$$\frac{S_m}{S_m+S_u}\left\{\frac{1}{A_{m,n\ldots z}}+\left(\log_e v-\log_e a\frac{\nu S_m-\mu S_u}{S_m}\right)\right\},$$

and the annual premium payable by $\frac{1}{p}$thly instalments (in advance) could be deduced from the value so found by simply deducting therefrom $\frac{1}{2p}$th of a year's interest, provided that the proportion of premium paid in advance for any period beyond the time at which the death takes place be returned to the assured. For instance, if π denote the annual premium payable momently, and t the unexpired fraction of the current year at the date of death, the amount of premium returnable to the assured at the instant of death will be $\pi t\left(1-\frac{rt}{2}\right)$ (r being the yearly interest on £1), for the Office may be supposed to owe the assured the sum πt payable by momently instalments during the time t.

A practice somewhat similar to the above is, I believe, generally followed in India with regard to half-yearly and quarterly premiums; and, independently of the facilities afforded in calculation, it is, perhaps, preferable to the plan adopted in this country. However, the annual, half-yearly, and quarterly premiums, according to the usual practice, may be found from the single premium by the aid of the formula for determining the value of an annuity payable p times a year.

The general formula [1] is the key to the exact solution of all cases of assurances on lives, whether absolute or contingent, treated in Baily's work on annuities and assurances, with the exception of those involving the problem of determining the value of a reversion on a given life, subject to the condition of a second life surviving a third during the lifetime of the first—a satisfactory solution of which has, I believe, never yet been *published.* And the term "exact solution" is to be taken in its fullest sense;

for the hypothesis of constant decrements for single years, or any similar hypothesis, is not resorted to in deducing the formula. The remaining cases of survivorship assurances, involving the problem before referred to, are also capable of exact solution on principles similar to those by which the formula [1] is deduced; but it is not my purpose to enter here upon the subject of these interesting but unusual cases.

A very important point in Mr. Gompertz's letter is the announcement that the method which he is now engaged upon admits of the solution of cases involving combinations of lives subject to different laws of mortality. This problem also admits of solution by tables constructed on the method suggested by me, provided one uniform value of q can be adopted in the construction of the several tables of mortality. By a slight arbitrary modification of the several constants, I have ascertained that this is perfectly practicable in the case of Indian and European lives, and in all probability it would be found equally so in other cases. Mr. Gompertz's promised paper will doubtless treat this subject in a manner worthy of its great practical importance.

I will not add to the length of this communication by inserting the demonstration of the solution of the general problem, which would occupy considerable space, although no particular difficulty is involved in it. I will, therefore, conclude by a single example of the most usual case of survivorships involving three lives.

Example.—Required the value of £1 payable at the death of (28), provided (30) and (33) shall be then both living.

(m) being the youngest life, the formula becomes—

$$\frac{1}{1+q^2+q^5}\left\{1+(\log_e v+C_2+C_5)A_{28,30,33}\right\}.$$

$$\begin{array}{l}
1\cdot \\
1\cdot1860 \\
1\cdot5317 \\
\hline
3\cdot7177 \text{ log.}=\cdot57027^* \\
\quad\text{log. } 3=\cdot47712 \\
\hline
\cdot09315\times\left(27=\dfrac{1}{\log. q}\right) \\
\hline
2\cdot7945 \\
\;\cdot2795 \\
\hline
2\cdot515 \\
28\cdot \\
\hline
30\cdot515=\text{Eq. com. age.}
\end{array}$$

$$\begin{array}{rr}
A_{30.30.30}=12\cdot784 & -167\times\cdot515 \\
 & \overline{84} \\
-86 & 2 \\
\hline
12\cdot698 &
\end{array}$$

$$\begin{array}{r}
\log_e v=\bar{1}\cdot96078 \\
C_2=\ \cdot00132 \\
C_5=\ \cdot00376 \\
\hline
\bar{1}\cdot96586
\end{array}$$

$$\begin{array}{r}
\bar{1}\cdot96586=-\cdot03414 \text{ log.}=\bar{2}\cdot53326 \\
\text{log. } 12\cdot698=1\cdot10374 \\
-\text{log. }(1+q^2+q^5)=\bar{1}\cdot42973^*=\text{log. }\cdot26899 \\
\hline
\bar{1}\cdot06673\ =\text{log. }\cdot11661 \\
\hline
(\textit{Answer})\ \cdot15238
\end{array}$$

I am, Sir,

Your very obedient servant,

5, *Lothbury.* W. M. MAKEHAM.

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Age.	Living.	Decrements.	Expectation of Life.	Age.	Living.	Decrements.	Expectation of Life.
15	10000·000	75·323	44·86	60	5677·788	165·630	14·57
16	9924·677	75·183	44·20	61	5512·158	171·522	13·99
17	9849·494	75·070	43·53	62	5340·636	177·472	13·42
18	9774·424	74·997	42·86	63	5163·164	183·437	12·87
19	9699·427	74·958	42·19	64	4979·727	189·345	12·32
20	9624·469	74·959	41·51	65	4790·382	195·127	11·79
21	9549·510	75·001	40·83	66	4595·255	200·701	11·27
22	9474·509	75·093	40·15	67	4394·554	205·971	10·76
23	9399·416	75·226	39·47	68	4188·583	210·838	10·27
24	9324·190	75·418	38·79	69	3977·745	215·185	9·79
25	9248·772	75·660	38·10	70	3762·560	218·888	9·32
26	9173·112	75·964	37·41	71	3543·672	221·825	8·86
27	9097·148	76·332	36·72	72	3321·847	223·856	8·42
28	9020·816	76·764	36·02	73	3097·991	224·850	7·99
29	8944·052	77·272	35·33	74	2873·141	224·677	7·58
30	8866·780	77·860	34·63	75	2648·464	223·214	7·18
31	8788·920	78·530	33·93	76	2425·250	220·351	6·79
32	8710·390	79·286	33·23	77	2204·899	216·004	6·42
33	8631·104	80·142	32·53	78	1988·895	210·115	6·06
34	8550·962	81·094	31·83	79	1778·780	202·660	5·72
35	8469·868	82·158	31·14	80	1576·120	193·657	5·39
36	8387·710	83·336	30·44	81	1382·463	183·177	5·08
37	8304·374	84·640	29·74	82	1199·286	171·341	4·78
38	8219·734	86·070	29·04	83	1027·945	158·323	4·49
39	8133·664	87·642	28·34	84	869·622	144·355	4·22
40	8046·022	89·360	27·64	85	725·267	129·714	3·95
41	7956·662	91·233	26·95	86	595·553	114·715	3·71
42	7865·429	93·274	26·25	87	480·838	99·703	3·47
43	7772·155	95·483	25·56	88	381·135	85·029	3·25
44	7676·672	97·877	24·87	89	296·106	71·029	3·04
45	7578·795	100·465	24·19	90	225·077	58·013	2·84
46	7478·330	103·248	23·51	91	167·064	46·231	2·65
47	7375·082	106·244	22·83	92	120·833	35·869	2·48
48	7268·838	109·453	22·16	93	84·964	27·029	2·31
49	7159·385	112·887	21·49	94	57·935	19·730	2·15
50	7046·498	116·546	20·82	95	38·205	13·912	2·01
51	6929·952	120·444	20·16	96	24·293	9·448	1·87
52	6809·508	124·578	19·51	97	14·845	6·156	1·74
53	6684·930	128·947	18·87	98	8·689	3·838	1·63
54	6555·983	133·556	18·23	99	4·851	2·277	1·51
55	6422·427	138·396	17·60	100	2·574	1·283	1·41
56	6284·031	143·458	16·97	101	1·291	·681	1·32
57	6140·573	148·735	16·36	102	·610	·341	1·23
58	5991·838	154·204	15·75	103	·269	·158	1·15
59	5837·634	159·846	15·16	104	·111	·069	1·07

Annuities (4 per Cent.).

Ages. (n)	One Life. (A_n)		Two Lives. ($A_{n,n}$)		Three Lives. ($A_{n,n,n}$)		Ages. (n)	One Life. (A_n)		Two Lives. ($A_{n,n}$)		Three Lives. ($A_{n,n,n}$)	
		(−)		(−)		(−)			(−)		(−)		(−)
30	17·492	·164	14·668	·171	12·784	·167	35	16·620	·191	13·758	·199	11·899	·193
31	17·328	·169	14·497	·177	12·617	·171	36	16·429	·197	13·559	·204	11·706	·197
32	17·159	·174	14·320	·182	12·446	·177	37	16·232	·202	13·355	·209	11·509	·203
33	16·985	·180	14·138	·187	12·269	·183	38	16·030	·207	13·146	·215	11·306	·208
34	16·805	·185	13·951	·193	12·086	·187	39	15·823	·214	12·931	·220	11·098	·212

n	q^n	d_n	C_n	C_{-n}	F_n	F_{-n} $(=1-F_n)$	G_n	G_{-n}
1	1·089023	·511	·0006299	$\bar{1}$·9994216	·4786927	·5213073	$\bar{1}$·9815269	$\bar{1}$·9792524
2	1·185971	1·043	·0013158	$\bar{1}$·9988905	·4574626	·5425374	$\bar{1}$·9826599	$\bar{1}$·9781194
3	1·291550	1·596	·0020628	$\bar{1}$·9984028	·4363858	·5636142	$\bar{1}$·9837848	$\bar{1}$·9769944
4	1·406527	2·170	·0028763	$\bar{1}$·9979550	·4155366	·5844634	$\bar{1}$·9848976	$\bar{1}$·9758817
5	1·531740	2·765	·0037623	$\bar{1}$·9975438	·3949853	·6050147	$\bar{1}$·9859945	$\bar{1}$·9747849

ON THE SUPERANNUATION OF EMPLOYÉS IN ASSURANCE OFFICES.

To the Editor of the Assurance Magazine.

Sir,—May I solicit the favour of your allotting a small space in your *Journal* for the insertion of a few remarks upon the subject of superannuation of *employés* in Assurance Offices, in the hope that it may engage the attention that it certainly deserves, but which, I believe, it has not hitherto received.

There may be, and, no doubt, are, systems of superannuation in connexion with some of our public institutions, but they are not, I believe, general; indeed, I am only aware of one instance in which a scheme exists in connexion with a Joint Stock Company for granting retiring pensions after certain periods of service. The Company referred to is the National Provincial Bank of England, and the main features of the scheme are, the option of retirement, after 20 years' service, on one-third of salary; after 30 years, on half salary; or, after 40 years, on two-thirds salary; and in proportion for intermediate periods of service—one of the conditions being, that the age of 60 shall be attained before retiring.

It will be seen that there is here no inducement held out to withdraw from active duties, but there is an option given of doing so, that would be esteemed a boon by very many who yet might never avail themselves of it.

Habit, we all know, has a powerful hold upon men generally—probably upon no class is its influence greater than upon those engaged in official routine—and it may reasonably be supposed that few men would, if in health, readily sacrifice two-thirds, or a half, of their income, merely for the sake of living in idleness.

It may be suggested, that a person has no claim upon the Company by which he has been employed—whatever may have been the length of his services—when incapacitated, by sickness or other infirmity, for further duty; true, he has no *legal* claim, but I am happy to think that boards of directors of Assurance Companies are not usually composed of men who take this view of things.

Assuming, then, a willingness to entertain the question of superannuation, a difficulty may arise respecting the cost; this is, however, rather imaginary than real, as I hope to show.

Waiving the consideration of retiring pensions to heads of departments, I will take the general staff of an Office, and assume that all engagements commence with a salary of £60, rising £10 annually until a maximum of £250 is reached, which would be in 20 years. Supposing, then, an individual who has attained the age of 60, and has been engaged in the service of a Company for 30 years, should feel desirous to retire from the cares and