A mapping $f$ of a ring $R$ into itself is called skew-commuting on a subset $S$ of $R$ if $f(s)a + sf(s) = 0$ for all $s \in S$. We prove two theorems which show that under rather mild assumptions a nonzero additive mapping cannot have this property. The first theorem asserts that if $R$ is a prime ring of characteristic not 2, and $f: R \to R$ is an additive mapping which is skew-commuting on an ideal $I$ of $R$, then $f(I) = 0$. The second theorem states that zero is the only additive mapping which is skew-commuting on a 2-torsion free semiprime ring.

Let $S$ be a subset of a ring $R$. A mapping $f$ of $R$ into itself is said to be skew-commuting on $S$ if $f(s)s + sf(s) = 0$ for all $s \in S$. For results on skew-commuting mappings and their generalisations (such as semi-commuting, skew-centralising, semi-centralising mappings) we refer the reader to [4, 6, 7, 8]. In these papers the authors have showed that nonzero derivations and ring endomorphisms cannot be skew-commuting (semi-commuting,...) on certain subsets of prime rings (for example, ideals). In the present paper we prove theorems of this kind for general additive mappings. Our first result is

**Theorem 1.** Let $R$ be a prime ring of characteristic not 2. If an additive mapping $f: R \to R$ is skew-commuting on some ideal $I$ of $R$, then $f(x) = 0$ for all $x \in I$.

Clearly, the requirement that the characteristic of $R$ is not 2 cannot be removed (consider, for instance, the identity on $R$). In fact, if the characteristic of a ring $R$ is 2, then the notion of skew-commuting mappings coincides with the notion of commuting mappings, that is, the mappings $f$ satisfying $f(x)x = xf(x)$. In [1] we showed that every additive commuting mapping of a prime ring $R$ (of arbitrary characteristic) is of the form $x \to \lambda x + \zeta(x)$ where $\lambda$ is an element in $C$, the extended centroid of $R$, and $\zeta$ is an additive mapping of $R$ into $C$ (see also [2, 3] for similar results). The fact that the structure of commuting mappings can be described has been one of the main motivations for this research.

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Suppose a ring $R$ contains nonzero ideals $I$ and $J$ such that $IJ = 0 =JI$ (thus $R$ is not prime). Any mapping $f$ of $R$ with range contained in $J$ is certainly skew-commuting on $I$; however, it does not necessarily vanish on $I$. Thus Theorem 1 does not hold for semiprime rings in general. Nevertheless, the following is true:

**Theorem 2.** Let $R$ be a 2-torsion free semiprime ring. If an additive mapping $f: R \to R$ is skew-commuting on $R$, then $f = 0$.

Theorem 2 will follow easily from Theorem 1. In order to prove Theorem 1 we define $I_n = \{z^n \mid z \in I\}$ ($n$ is a positive integer), and let us prove

**Lemma 1.** Let $R$ be a prime ring, $I$ be a nonzero ideal of $R$, and $a \in R$. If there exists a positive integer $n$ such that $I_n a = 0$ (or $a I_n = 0$), then $a = 0$.

**Proof:** Suppose $a \neq 0$. Since $R$ is prime there exists $w \in I$ such that $aw \neq 0$. For any $z \in R$, the element $awz$ lies in $I$, hence $(awz)^n a = 0$ for all $z \in R$. But then $(awz)^{n+1} = 0$, $z \in R$, and so $[5$, Lemma 1.1$]$ yields $aw = 0$, contrary to the assumption. Similarly one discusses the case when $a I_n = 0$. \(\square\)

**Proof of Theorem 1:** For the proof we need several steps. We begin with

**Lemma A.** For $x, y \in I$,

1. $f(x)y + yf(x) + f(y)x + xf(y) = 0$ for all $x, y \in I$.
2. $x^4 f(x) = 0 = f(z) z^4$.

**Proof:** Linearising $f(x)x + xf(x) = 0$ we obtain (1). Let us prove (2). From the initial hypothesis we see that for any $z \in I$, $f(z)$ commutes with $x^2$. Therefore, replacing $y$ by $x^2$ in (1) we obtain

$$2x^2 f(x) + f(x^2) x + xf(x^2) = 0 \text{ for all } x \in I.$$  

Multiply (3) from the right by $x^2$; since $f(x)x^2 = x^2 f(x)$ and since, by the initial hypothesis, $f(x^2) x^2 + x^2 f(x^2) = 0$, it follows that

$$2x^4 f(x) = x^2 f(x^2) x + x^3 f(x^2).$$

On the other hand, by (3) we see that

$$2x^4 f(x) = x^2 (2x^2 f(x)) = -x^2 f(x^2) x - x^3 f(x^2).$$

Comparing the last two relations we arrive at $4x^4 f(x) = 0$. We have assumed that the characteristic of $R$ is not 2, and so $x^4 f(z) = 0$. Since $f(z)z = -zf(z)$, we also have $f(z)z^4 = 0$. \(\square\)
**Lemma B.** For $u \in I_{10}$, $y \in I$, $uf(y)u = 0$.

**Proof:** Multiply (1) from the left and from the right by $x^4$. According to (2) we obtain

\[(4) \quad x^4f(y)x^5 + x^5f(y)x^4 = 0 \quad \text{for all} \quad x, y \in I.\]

Taking $x^2$ for $x$ in (4) we get

\[x^8f(y)x^{10} + x^{10}f(y)x^8 = 0.\]

But from (4) if follows that

\[x^8f(y)x^{10} = x^4(x^4f(y)x^5)x^5\]
\[= -x^4(x^5f(y)x^4)x^5\]
\[= -x^5(x^5f(y)x^4)x^4\]
\[= x^5(x^5f(y)x^4)x^4\]
\[= x^{10}f(y)x^8.\]

Comparing the last two identities one concludes that $x^8f(y)x^{10} = 0$ for all $x, y \in I$. But then also $x^{10}f(y)x^{10} = 0$, which is the assertion of the lemma. $\square$

There is nothing to prove if $I = 0$. Therefore, we assume henceforth that $I \neq 0$.

**Lemma C.** There exists a nonzero left ideal $L$ of $R$, contained in $I$, such that $f(L) = 0$.

**Proof:** As a special case of (1) we have

\[(5) \quad f(x)u + uf(x) + f(u)x + xf(u) = 0 \quad \text{for all} \quad x \in I, \ u \in I_{10}.\]

Multiplying (5) from the right by $u$, and then using Lemma B, we arrive at

\[(6) \quad f(x)u^2 + f(u)xu + xf(u)u = 0 \quad \text{for all} \quad x \in I, \ u \in I_{10}.\]

Suppose $x \in I_{10}$. By Lemma B we then see that $zf(u)x = 0$, and also $x^2f(x) = -zf(x)x = 0$. Therefore it follows from (6) that $x^3f(u)u = 0$. That is, $uf(u)u = 0$ for all $u \in I_{10}, \ u \in I_{10}$. By Lemma 1 we then have $f(u)u = 0$. Thus (6) reduces to

\[(7) \quad f(x)u^2 + f(u)xu = 0 \quad \text{for all} \quad x \in I, \ u \in I_{10}.\]

Substituting $zu$ for $z$ in (7) we obtain $f(zu)u^2 + f(u)zu^2 = 0$. On the other hand, $f(u)zu^2 = (f(u)xu)u = -f(x)u^3$. Consequently we have

\[(8) \quad f(zu)u^2 = f(x)u^3 \quad \text{for all} \quad x \in I, \ u \in I_{10}.\]
Now, multiply (5) from the left by \(u\). Since \(uf(x)u = 0\) and \(uf(u) = -f(u)u\), it follows that \(u^2f(x) + uxf(u) = 0\), \(x \in I\), \(u \in I_{10}\). Replacing \(x\) by \(xu\) in this relation, and applying \(uf(u) = 0\), we then get

\[
(9) \quad u^2f(xu) = 0 \text{ for all } x \in I, u \in I_{10}.
\]

As a special case of (1) we have

\[
f(x)yu + yuf(x) + f(yu)x + xf(yu) = 0
\]

for all \(x, y \in I\), \(u \in I_{10}\). Multiply this identity from the left and from the right by \(u^2\). In view of Lemma B, (9) and (8), we then get \(u^2f(x)yv^3 + u^2xf(y)v^3 = 0\). Hence

\[
vf(x)yv + vxf(y)v = 0
\]

holds for all \(v \in I_{50}\), \(x, y \in I\). Replace in this relation \(y\) by \(yvf(z)\) where \(y, z \in I\), \(v \in I_{50}\). Then the first term is zero by Lemma B, so we have \(vxf(yvf(z))v = 0\). Since \(R\) prime it follows that

\[
(10) \quad f(yvf(z))v = 0 \text{ for all } y, z \in I, v \in I_{50}.
\]

Substituting \(yvf(z)\) for \(y\) in (1) we obtain

\[
f(x)yvf(z) + yvf(z)f(x) + f(yvf(z))x + xf(yvf(z)) = 0.
\]

Multiplying from the right by \(v\), and using Lemma B and (10), we then obtain

\[
(11) \quad yvf(z)f(x)v + f(yvf(z))xv = 0 \text{ for all } x, y, z \in I, v \in I_{50}.
\]

Taking \(ry\) for \(y\), where \(r \in R\) and \(y \in I\), we get

\[
ryvf(z)f(x)v + f(ryvf(z))xv = 0.
\]

On the other hand we see from (11) that

\[
ryvf(z)f(x)v = -rf(yvf(z))xv.
\]

Comparing we obtain

\[
\{f(ryvf(z)) - rf(yvf(z))\}xv = 0
\]

for all \(r \in R\), \(x, y, z \in I\), \(v \in I_{50}\). The primeness of \(R\) yields

\[
(12) \quad f(ryvf(z)) = rf(yvf(z)) \text{ for all } r \in R, y, z \in I, v \in I_{50}.
\]
Multiply (12) from the left and from the right by \( u \in I_0 \). In view of Lemma B it follows that \( urf(yv(z))u = 0 \). Thus \( f(yv(z))u = 0 \), and so, by Lemma 1,

\[
(13) \quad f(yv(z)) = 0 \text{ for all } y, z \in I, \ v \in I_0.
\]

We may assume that \( f(z) \neq 0 \) for some \( z \in I \). By Lemma 1, \( vrf(z) \neq 0 \) for some \( v \in I_0 \). Hence \( a = zrvf(z) \neq 0 \) for some \( x \in I \). Thus \( L = Ra \) is a nonzero left ideal of \( R \), and since \( a \in I, L \) is contained in \( I \). By (13), \( f(L) = 0 \).

**Lemma D.** \( f(I) = 0 \).

**Proof:** From \( f(L) = 0 \) and (1) it follows at once that

\[
(14) \quad f(x)t + tf(x) = 0 \text{ for all } t \in L, x \in I.
\]

Replacing \( t \) by \( rt \), where \( r \in R \) and \( t \in L \), it follows that \( f(x)rt + rtf(x) = 0 \). By (14), the second term is equal to \( -rf(x)t \), therefore \( (f(x)r - rf(x))t = 0 \) for all \( r \in R, x \in I, t \in L \). Since \( R \) is prime we then have \( f(x)r - rf(x) = 0 \) for all \( r \in R, x \in I \). That is, \( f(x) \) lies in the centre of \( R \) for every \( x \) in \( I \). But then (14) implies that \( f(x)L = 0, x \in I \), and therefore \( f(x) = 0 \). With this the theorem is proved.

**Proof of Theorem 2:** Since \( R \) is semiprime, the intersection of all prime ideals in \( R \) is zero.

Now pick a prime ideal \( P \) such that \( R/P \) is of characteristic not 2. We want to show that \( P \) is invariant under \( f \). A linearisation of \( f(x)x + xf(x) = 0 \) gives \( f(x)y + yf(x) + f(y)x + xf(y) = 0, x, y \in R \). Hence we see that

\[
(15) \quad f(p)x + xf(p) \in P \text{ for all } p \in P, x \in R.
\]

In particular, \( f(p)zx + zxf(p) \in P \) for all \( p \in P, x, y \in R \). That is, \( (f(p)x + zxf(p))y + x(yf(p) - f(p)y) \in P \). The first term is contained in \( P \) by (15), hence \( x(yf(p) - f(p)y) \in P, p \in P, x, y \in R \). Since \( P \) is a prime ideal it follows that \( yf(p) - f(p)y \in P \) for all \( p \in P, y \in R \). Combining this statement with (15) we obtain \( 2f(p)x \in P \). Since the characteristic of \( R/P \) is not 2 it follows that \( f(p)x \in P \) for all \( p \in P, x \in R \). The ideal \( P \) is prime, therefore, \( f(p) \in P \) for every \( p \in P \).

Since \( f(P) \subseteq P, f \) induces an additive mapping \( F \) on \( R/P \), defined by \( F(x + P) = f(x) + P \). Of course, \( F \) is skew-commuting. Hence \( F = 0 \) by Theorem 1.

Thus we have proved that the range of \( f \) is contained in any prime ideal \( P \) such that \( R/P \) is of characteristic not 2. The theorem will be proved by showing that the intersection of all such ideals is equal to zero. There exist prime ideals \( \{P_a \mid a \in A\} \) such that \( \bigcap_a P_a = 0 \). Let \( B = \{b \in A \mid \text{the characteristic of } R/P_b \text{ is not 2}\} \) and \( C = \{c \in A \mid \text{the characteristic of } R/P_c \text{ is 2}\} \). Thus \( 2x \in \bigcap_c P_c \) for every \( x \in R \). Therefore, given \( x \in \bigcap_b P_b \), we have \( 2x \in (\bigcap_c P_c) \cap (\bigcap_b P_b) = \bigcap_a P_a = 0 \), and so \( x = 0 \) since \( R \) is 2-torsion free. Thus \( \bigcap_b P_b = 0 \).
REMARK. A mapping $f$ of a ring $R$ is called semi-commuting on a subset $S$ of $R$ if for any $x \in S$, either $f(x)x + xf(x) = 0$ or $f(x)x - xf(x) = 0$. Suppose that $R$ is $2$-torsion free and $3$-torsion free, and suppose that $f$ is an additive mapping of $R$ which is semi-commuting on some additive subgroup $S$ of $R$. We claim that in this case $f$ is either commuting on $S$ or skew-commuting on $S$. Indeed, introducing biadditive mappings $A: S \times S \to R$ and $B: S \times S \to R$ by $A(x, y) = f(x)y + xf(y)$ and $B(x, y) = f(x)y - xf(y)$, we have $S = P \cup Q$ where $P = \{x \in S \mid A(x, x) = 0\}$, $Q = \{x \in S \mid B(x, x) = 0\}$. Suppose our assertion is not true, thus $P \neq S$ and $Q \neq S$. This means that $A(x, x) \neq 0$ and $B(y, y) \neq 0$ for some $x, y \in S$. Then, of course, $A(y, y) = 0$ and $B(x, x) = 0$. Now, consider the element $x + y$. If $x + y \in P$ then we have $A(x, x) + A(x, y) + A(y, x) = 0$, and if $x + y \in Q$ then $B(x, x) + B(y, x) + B(y, y) = 0$. Similarly we consider the elements $x - y$ and $x + 2y$. But then one can easily see that (since $R$ is $2$-torsion free and $3$-torsion free) either $A(x, x) = 0$ or $B(y, y) = 0$, contrary to the assumption. This proves our assertion. According to Theorem 1 we then obtain the following result: Let $f$ be an additive mapping of a prime ring of characteristic not $3$. If $f$ is semi-commuting on some ideal $I$ of $R$, then $f$ is commuting on $I$. Note that this result fairly generalises a theorem in [4].

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