CONVOLUTION KERNELS OF \((n + 1)\)-FOLD MARCINKIEWICZ MULTIPLIERS ON THE HEISENBERG GROUP

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We prove a characterisation, in terms of regularity and cancellation conditions, of the convolution kernels of Marcinkiewicz multiplier operators \(m(\mathcal{L}_1, \ldots, \mathcal{L}_n, iT)\) on the Heisenberg group \(H_n\), where \(\mathcal{L}_1, \ldots, \mathcal{L}_n\) are the \(n\) partial sub-Laplacians. The necessity of these regularity and cancellation conditions was established by Fraser (2001); here, we prove their sufficiency.

1. INTRODUCTION

This paper deals with a class of convolution kernels on the Heisenberg group \(\mathbb{H}_n = C^n \times \mathbb{R}\) which in a sense correspond to the product-type Calderon-Zygmund kernels on \(\mathbb{R}^n\), and involve an underlying multi-parameter structure. These kernels were defined in [3], where they arose as the convolution kernels of spectral multiplier operators on the Heisenberg group analogous to the \(n\)-fold Marcinkiewicz multipliers on \(\mathbb{R}^n\). We show here that they always arise as such.

The situation thus corresponds to that of Marcinkiewicz multipliers in \(\mathbb{R}^n\), (see [1, 5, 7]) whose convolution kernels can be seen to be precisely the class of product-type Calderon-Zygmund kernels. Just as multipliers on \(\mathbb{R}^n\) can be viewed as functions of \(i\frac{\partial}{\partial x_1}, \ldots, i\frac{\partial}{\partial x_n}\), we take for our operators on the Heisenberg group functions \(m(\mathcal{L}_1, \ldots, \mathcal{L}_n, iT)\) of \(iT\) and the partial sub-Laplacians \(\mathcal{L}_1, \ldots, \mathcal{L}_n\). The \((n + 1)\)-fold Marcinkiewicz-type condition we require is that

\[
|(\xi_1 \partial_{\xi_1})^i \cdots (\xi_n \partial_{\xi_n})^i (\eta \partial_\eta)^j m(\xi, \eta)| \leq C_{ij}.
\]

Our work uses the methods of Müller, Ricci and Stein who in [6] studied the case of functions \(m(\mathcal{L}, iT)\) of \(iT\) and the sub-Laplacian \(\mathcal{L}\), where \(m\) satisfies a two-fold Marcinkiewicz-type condition,

\[
|(\xi \partial_\xi)^i (\eta \partial_\eta)^j m(\xi, \eta)| \leq C_{ij}.
\]

In [3], by lifting to a larger product group, where multi-parameter methods can be used, and then transferring the results obtained back down to \(\mathbb{H}_n\), we established the
boundedness on $L^p$, for $1 < p < \infty$, of these Marcinkiewicz multipliers and showed that their convolution kernels are polyradial distributions on $\mathbb{H}_n$, which are smooth away from the $z_i = 0$ planes and satisfy the regularity and cancellation conditions (4)–(7) given in Section 3 below. We now show that these conditions in fact characterise the convolution kernels of Marcinkiewicz multipliers $m(\mathcal{L}_1, \ldots, \mathcal{L}_n, iT)$.

The proof is analogous to the proof in $\mathbb{R}^n$ of conditions on the multipliers of Calderón–Zygmund operators, starting from estimates on their convolutions kernels. We do not have to work with the full group Fourier transform on the Heisenberg group, because our kernels are polyradial, and so their Fourier transforms are diagonalisable operators on the representation space. We are thus able to deal simply with their eigenvalues; that is, with the Gelfand transform of the kernels. Thus, making use of Geller's explicit expression ([4]) for the Gelfand transform of a polyradial function on $\mathbb{H}_n$, in terms of Laguerre functions (this and other facts about the Fourier transform on $\mathbb{H}_n$ are outlined below in Section 2), we begin in Section 4 by considering the case of a smooth, compactly supported kernel satisfying (4)–(7), and show that its Gelfand transform is bounded.

In Section 5, we show that a discrete Marcinkiewicz-type condition on the Gelfand transform of $C_c^\infty$ kernels follows from the boundedness proved in Section 4, by the simple observation that certain differential and difference operators, applied to the Gelfand transforms of our kernels, still yield Gelfand transforms of the same kind of kernels. This corresponds to the fact in $\mathbb{R}^n$ that

$$\xi_j \partial_{\xi_j} \hat{f} = (-f - x_j \partial_{x_j} f)^\sim,$$

and $-f - x_j \partial_{x_j} f$ is still a Calderón–Zygmund kernel if $f$ is. The relations on $\mathbb{H}_n$, analogous to (2) are proved in Lemma 5.1. This discrete Marcinkiewicz-type condition on the Gelfand transform is then shown in Proposition 5.3 to extend to give the required Marcinkiewicz condition on the multiplier $m$. This is really a result about interpolation between integers to go from difference conditions on a function on $\mathbb{Z}^n$ to differential conditions on an extended smooth function on $\mathbb{R}^n$. (The argument is an adaptation to $n$-variables of that in [6]).

Thus, in Sections 4 and 5, the main result is established for smooth kernels of compact support. The general case is then proved in Section 6 by an approximation argument.

2. Preliminaries

We denote by $\mathbb{H}_n$ the $2n + 1$-dimensional Heisenberg group, $\mathbb{H}_n = \mathbb{C}^n \times \mathbb{R}$, with multiplication $(z, t) \cdot (w, s) = (z + w, t + s + 2 \text{Im} z \cdot \overline{w})$. The identity under this multiplication is $(0, 0)$, and the inverse $(z, t)^{-1}$ of $(z, t)$ is $(-z, -t)$. The Heisenberg group is a connected, simply connected nilpotent Lie group. We define one-parameter dilations
on $\mathbb{H}_n$, for $r > 0$, by $r(z,t) = (rz, r^2 t)$. These dilations are group automorphisms. A homogeneous norm on $\mathbb{H}_n$ is given by

$$|h| = |(z,t)| = (|z|^2 + |t|)^{1/2}. $$

Using coordinates $h = (z,t) = (x + iy, t)$, for points in $\mathbb{H}_n$, the left-invariant vector fields $X_j, Y_j$ and $T$ on $\mathbb{H}_n$ equal to $\frac{\partial}{\partial x_j}, \frac{\partial}{\partial y_j}$ and $\frac{\partial}{\partial t}$ at the origin are given by

$$X_j = \frac{\partial}{\partial x_j} + 2y_j \frac{\partial}{\partial t}, \quad Y_j = \frac{\partial}{\partial y_j} - 2x_j \frac{\partial}{\partial t} \quad \text{and} \quad T = \frac{\partial}{\partial t}$$

respectively. These $2n + 1$ vector fields form a basis for the Lie algebra $\mathfrak{h}_n$ of $\mathbb{H}_n$ with commutation relations

$$[Y_j, X_j] = 4T,$$

for $j = 1, \ldots, n$, and all other commutators equal to 0.

A differential operator $D$ on $\mathbb{H}_n$ is called homogeneous of degree $d$ if

$$D(f(r \cdot \cdot)) = r^d (Df)(r \cdot \cdot) .$$

Thus, $X_1, \ldots, X_n, Y_1, \ldots, Y_n$ are homogeneous of degree one, and $T$ is homogeneous of degree two. The homogenous dimension of $\mathbb{H}_n$ is $2n + 2$, the sum of the degrees of the homogeneous basis elements $X_1, \ldots, X_n, Y_1, \ldots, Y_n,$ and $T$.

The partial sub-Laplacians $L_1, \ldots, L_n$ on $\mathbb{H}_n$ are given by

$$L_j = -\frac{1}{4}(X_j^2 + Y_j^2),$$

for $j = 1, \ldots, n$. The operators $L_1, \ldots, L_n,$ and $iT$ form a family of commuting self-adjoint operators with commuting spectral measures, and so, by the spectral theorem, for $m \in L^\infty((\mathbb{R}^+)^n \times \mathbb{R})$, we can define the joint spectral multiplier operator $m(L_1, \ldots, L_n, iT)$ which is then a bounded operator on $L^2(\mathbb{H}_n)$. Since $L_1, \ldots, L_n,$ and $iT$ are left-invariant, $m(L_1, \ldots, L_n, iT)$ commutes with left-translations and is therefore given by convolution with a distribution $K \in S'(\mathbb{H}_n)$: $m(L_1, \ldots, L_n, iT) \varphi = \varphi * K$, for all $\varphi \in S(\mathbb{H}_n)$.

Given any $\tau \in \mathbb{T}^n$, the $n$-torus, define the operator $\rho_\tau$ on functions $f$ on $\mathbb{H}_n$ by

$$\rho_\tau f(z,t) = f(\tau \cdot z,t).$$

A function $f$ on $\mathbb{H}_n$ will be called polyradial if $f = \rho_\tau f$ for all $\tau \in \mathbb{T}^n$. A distribution $K \in S'(\mathbb{H}_n)$ is said to be polyradial if $K(\varphi) = K(\rho_\tau \varphi)$ for all $\tau \in \mathbb{T}^n$ and all $\varphi \in S(\mathbb{H}_n)$. Since $L_1, \ldots, L_n,$ and $iT$ commute with all $\rho_\tau$, so does $m(L_1, \ldots, L_n, iT)$. Therefore the convolution kernel $K$ of $m(L_1, \ldots, L_n, iT)$ is polyradial.

We now set down some facts about the Fourier transform for polyradial functions on the Heisenberg group. These are covered in detail in [4].
The Fock–Bargmann realisation of the (infinite-dimensional) irreducible unitary representations of $\mathbb{H}_n$ is as follows. For $\lambda > 0$, the Fock space $\mathcal{H}_\lambda$ is the set of all holomorphic functions $F : \mathbb{C}^n \to \mathbb{C}$ such that

$$\|F\|^2 = \left(\frac{2\lambda}{\pi}\right)^n \int_{\mathbb{C}^n} |F(w)|^2 e^{-2\lambda|w|^2} dw < \infty.$$ 

$\mathcal{H}_\lambda$ is a Hilbert space, with orthonormal basis

$$E^\lambda_k(w) = \left(\frac{\sqrt{2\lambda}}{\sqrt{k!}}\right)^{|k|} w^k$$

for $k = (k_1, \ldots, k_n) \in \mathbb{N}^n$ (where $|k| = k_1 + \cdots + k_n$, $k! = k_1! \cdots k_n!$, and $w^k = w_1^{k_1} \cdots w_n^{k_n}$ for $w \in \mathbb{C}^n$). For $\lambda < 0$, $\mathcal{H}_\lambda = \mathcal{H}_{-\lambda}$, $E^\lambda_k = E^{-\lambda}_k$. The irreducible unitary representations $\pi_\lambda$ of $\mathbb{H}_n$ on $\mathcal{H}_\lambda$ are given by, for $\lambda > 0$,

$$\pi_\lambda(z, t)F(w) = e^{-i\lambda t}e^{-\lambda(|z|^2 + 2z \cdot w)}F(w + \bar{z})$$

for $F \in \mathcal{H}_\lambda$, and for $\lambda < 0$,

$$\pi_\lambda(z, t)F(w) = e^{-i\lambda t}e^{\lambda(|z|^2 - 2z \cdot w)}F(w - z)$$

for $F \in \mathcal{H}_\lambda$. Then $d\pi_\lambda(iT) = \lambda I$, and for $j = 1, \ldots, n$, $d\pi_\lambda(E_j)$ is diagonal on the basis $\{E^\lambda_k\}$, with eigenvalues $(2k_j + 1)|\lambda|$ corresponding to $E^\lambda_k$.

A function $f \in L^1(\mathbb{H}_n)$ is polyradial if and only if $\pi_\lambda(f)$ is diagonal on the basis $\{E^\lambda_k\}$, for all $\lambda \neq 0$. In this case, let

$$\pi_\lambda(f)E^\lambda_k = \tilde{f}(k, \lambda)E^\lambda_k,$$

for $k \in \mathbb{N}^n$, $\lambda \neq 0$, then $\tilde{f}$ is the Gelfand transform, also given by

$$\tilde{f}(k, \lambda) = \int_{\mathbb{H}_n} e^{-i\lambda t} \ell_k(2|\lambda| |z|^2) f(z, t) \, dz \, dt$$

where $|z|^2 = (|z_1|^2, \ldots, |z_n|^2)$ for $z \in \mathbb{C}^n$, and $\ell_k = \ell^0_k$ is a Laguerre function, defined as follows. For $x > 0$, and $k \in \mathbb{N}$, $m \in \mathbb{N} \cup \{0\}$, the Laguerre polynomials are

$$L^m_k(x) = \sum_{i=0}^k \binom{k+m}{k-i} \frac{(-x)^i}{i!}.$$ 

The Laguerre function $\ell^m_k$ on $\mathbb{R}^+$ is then

$$\ell^m_k(x) = \left(\frac{k!}{(k+m)!}\right)^{1/2} x^{m/2} L^m_k(x) e^{-x/2}.$$ 

And finally, for $k \in \mathbb{N}^n$, $m \in (\mathbb{N} \cup \{0\})^n$, $x \in (\mathbb{R}^+)^n$, the Laguerre function $\ell^m_k$ is given by

$$\ell^m_k(x) = \ell^m_{k_1}(x_1) \cdots \ell^m_{k_n}(x_n).$$
The inversion formula for polyradial $f$ is
\[ f(x,t) = c_n \sum_{k \in \mathbb{N}^n} \mathcal{F}(k,\lambda) e^{i \lambda \cdot t} \ell_k(2|\lambda| |x|^2)(2|\lambda|)^n d\lambda. \]

Thus, the convolution kernel $K$ of $m(\mathcal{L}_1, \ldots, \mathcal{L}_n, it)$ for bounded $m$ on $(\mathbb{R}^+)^n \times \mathbb{R}$ is given by
\[ K(z,t) = c_n \sum_{k \in \mathbb{N}^n} m((2k_1 + 1)|\lambda|, \ldots, (2k_n + 1)|\lambda|) e^{i \lambda \cdot t} \ell_k(2|\lambda| |z|^2)(2|\lambda|)^n d\lambda. \]

The joint spectrum of $\mathcal{L}_1, \ldots, \mathcal{L}_n, it$ is the closure in $(\mathbb{R}^+)^n \times \mathbb{R}$ of
\[ \left\{ ((2k_1 + 1)|\lambda|, \ldots, (2k_n + 1)|\lambda|, \lambda) : k \in \mathbb{N}^n, \lambda \neq 0 \right\}. \]

The multiplier $m(\xi, \eta)$ on this joint spectrum is then related to the Gelfand transform $\mu = \widetilde{K}$ by
\[ \mu(k,\lambda) = m((2k_1 + 1)|\lambda|, \ldots, (2k_n + 1)|\lambda|, \lambda) \]
for $k = (k_1, \ldots, k_n) \in \mathbb{N}^n$, $\lambda \in \mathbb{R} \setminus \{0\}$.

The following properties of Laguerre functions $\ell_k = \ell_k^0$ on $\mathbb{R}^+$ (see for example [2]) will be used throughout this paper:

(i) $x \ell_k = k(\ell_k - \ell_{k-1}) - \frac{1}{2}x \ell_k$
(ii) $x \ell''_k + \ell'_k - \frac{x}{4} \ell_k = -(k + \frac{1}{2}) \ell_k$
(iii) $|\ell_k| \leq 1$
(iv) $|\ell'_k| \leq C(k + 1)$

3. KERNELS OF MARCINKIEWICZ MULTIPLIER OPERATORS ON THE HEISENBERG GROUP

Before stating our main result, we first define what will be referred to as a product-type kernel on a product group $G = G_1 \times \cdots \times G_N$. This definition applies in the generality of homogeneous groups; however, for our purposes in this paper we shall only consider products of the homogeneous groups $\mathbb{C}$ (where $Q = 2$), and $\mathbb{R}$ ($Q = 1$).

For $j \in \{1, \ldots, N\}$, we let $G_j$ be a homogeneous group of homogeneous dimension $Q_j$ Then $G_j$ is equipped with an automorphic one-parameter dilation (which, for $r_j > 0$, we denote simply by $x_j \mapsto r_j x_j$, for $x_j \in G_j$) and a homogeneous norm $| \cdot |$. Given a basis $\{X_{j,1}, \ldots, X_{j,n_j}\}$ of left-invariant vector-fields, for $I \in (\mathbb{Z}^+)^{n_j}$ (where $\mathbb{Z}^+$ denotes the set $\{0,1,2,\ldots\}$), the degree of the left-invariant differential operator $X_I^j = X_{j,1}^{i_1} \cdots X_{j,n_j}^{i_{n_j}}$ on $G_j$ will be denoted by $d_I(I)$.

NOTATION: Throughout this paper, we shall adopt the following notational conventions for product groups. For $x$ in a product group $G = G_1 \times \cdots \times G_N$, we let
\[ |x| = (|x_1|, \ldots, |x_N|), \]

\begin{align*}
\text{(i)} & \quad x \ell_k = k(\ell_k - \ell_{k-1}) - \frac{1}{2}x \ell_k \\
\text{(ii)} & \quad x \ell''_k + \ell'_k - \frac{x}{4} \ell_k = -(k + \frac{1}{2}) \ell_k \\
\text{(iii)} & \quad |\ell_k| \leq 1 \\
\text{(iv)} & \quad |\ell'_k| \leq C(k + 1) \\
\end{align*}
so that, given \( J \in (\mathbb{Z}^+)^N \),
\[
|x|^{J'} = |x_1|^{j_1} \cdots |x_N|^{j_N}.
\]

For \( j \in \mathbb{Z}^+ \), we denote \( j \) the multi-index \((j, \ldots, j) \in (\mathbb{Z}^+)^m \) for a dimension \( m \) which will always be clear from the context. We set \( Q = (Q_1, \ldots, Q_N) \), and for a multi-index \( I = (I_1, \ldots, I_N) \), with \( I_j \in (\mathbb{Z}^+)^{n_j}, j = 1, \ldots, N \), we also set
\[
d(I) = (d_1(I_1), \ldots, d_N(I_N)).
\]

The differential operator \( X_I = X_{I_1}^1 \cdots X_{I_N}^N \) on \( G \), with \( X_j^I \) on \( G_j \) defined as above, then has degree \( |d(I)| \).

We denote multi-parameter dilation, given \( r = (r_1, \ldots, r_N) \in (\mathbb{R}^+)^N \), by
\[
d_r(x) = (r_1x_1, \ldots, r_Nx_N)
\]
for \( x \in G \).

Frequently it will be necessary to split a variable \( x \) in a product group \( G \) into two component variables. In such cases we shall write \( x = (x_{\xi}, x_\zeta) \), where
\[
x_{\xi} = (x_1, \ldots, x_\ell), \quad x_\zeta = (x_{\ell+1}, \ldots, x_N)
\]
for \( 1 \leq \ell \leq n \), and
\[
X_J^\xi = X_1^{j_1} \cdots X_\ell^{j_\ell}, \quad X_J^K = X_{\ell+1}^{K_{\ell+1}} \cdots X_N^{K_N}
\]
for \( J = (J_1, \ldots, J_\ell) \in (\mathbb{Z}^+)^{n_1} \times \cdots \times (\mathbb{Z}^+)^{n_\ell} \) and \( K = (K_{\ell+1}, \ldots, K_N) \in (\mathbb{Z}^+)^{n_{\ell+1}} \times \cdots \times (\mathbb{Z}^+)^{n_N} \). We also set
\[
G_\xi = G_1 \times \cdots \times G_\ell, \quad G_\zeta = G_{\ell+1} \times \cdots \times G_N,
\]
with corresponding definitions of \( Q_\xi, Q_\zeta, d_\xi, \) and \( d_\zeta \).

We note that for \( z \in \mathbb{C}^N \), \( |z|^2 = |z_1|^2 \cdots |z_N|^2 \), while \( |z|^2 \) is the square of the usual norm, \( |z_1|^2 + \cdots + |z_N|^2 \).

We shall define the product-type kernel conditions in terms of normalised bump functions. A \( C_\infty^\infty \) function \( \varphi \) is called a normalised bump function if \( \varphi \) is supported in the unit ball, and \( \varphi \) and all first order partial derivatives of \( \varphi \) are bounded by a fixed, pre-determined constant.

A function \( K \) on \( G_1 \times \cdots \times G_N \) is said to be a kernel of product-type (or to satisfy product-type kernel conditions) if it satisfies the following conditions:

(a) the regularity condition:
\[
|X_I^J K(x)| \leq C_I |x|^{-Q-d(I)}
\]
for all \( I = (I_1, \ldots, I_N), \quad I_j \in (\mathbb{Z}^+)^{n_j}, \quad j = 1, \ldots, N; \)
(b) for each $\ell = 1, \ldots, N$, the cancellation condition in $x_\ell$:

$$\left| \int_{G_\ell} X_\ell^I K(x)(\delta_\ell(x_\ell)) \, dx_\ell \right| \leq C_I \left| x_\ell \right|^{-Q_\ell - d_\ell(I)}$$

for all $I = (I_{\ell+1}, \ldots, I_N)$, $I_j \in (\mathbb{Z}^+)^{n_j}$, $j = \ell + 1, \ldots, N$, all normalised bump functions $\varphi$ on $G_\ell$, and all $r \in (\mathbb{R}^+)^t$.

In addition, for each permutation $\sigma \in S_N$ $K$ must satisfy the cancellation condition in $x_{\sigma(\ell)}$ obtained from (b) by permuting the indices $1, \ldots, N$ by $\sigma$.

In the case where $K$ is a tempered distribution, we assume that $K$ is smooth away from the “planes” $\{ x \in G : x_j = 0 \}$, $j = 1, \ldots, N$, and the cancellation conditions are to be understood as follows. Given $\varphi$ in the Schwartz space $\mathcal{S}(G_\ell)$, we define the distribution $K_\varphi$ by $K_\varphi(\psi) = K(\varphi \otimes \psi)$ for all $\psi \in \mathcal{S}(G_\ell)$, where $\varphi \otimes \psi(x_1, \ldots, x_N) = \varphi(x_\ell) \psi(x_\ell)$.

The cancellation condition then states that for all normalised bump functions $\varphi$ on $G_\ell$, and for all $r \in (\mathbb{R}^+)^t$, the distributions $K_{\varphi \otimes \psi} \in \mathcal{S}'(G_\ell)$, are smooth away from the planes $\{ x_\ell \in G_\ell : x_j = 0 \}$, $j = \ell + 1, \ldots, N$, and uniformly satisfy

$$\left| X_\ell^I K_{\varphi \otimes \psi}(x_\ell) \right| \leq C_I \left| x_\ell \right|^{-Q_\ell - d_\ell(I)}$$

for all $I = (I_{\ell+1}, \ldots, I_N) \in (\mathbb{Z}^+)^{n_{\ell+1}} \times \cdots \times (\mathbb{Z}^+)^{n_N}$.

We now state the main result of this paper, the characterisation of kernels of Marcinkiewicz multipliers $m(L_1, \ldots, L_n, iT)$, and observe that these kernels form a subset of the product-type kernels on $\mathbb{C}^n \times \mathbb{R}$. The cancellation conditions (5)-(7) below are to be interpreted for distribution kernels in the same manner as indicated above in the case of the product-type cancellation conditions.

**Theorem 3.1.** Let $m$ be a function on $(\mathbb{R}^+)^n \times \mathbb{R}$ satisfying the Marcinkiewicz-type condition (1) for all $i_1, \ldots, i_n, j \in \mathbb{Z}^+$. Then, the convolution kernel $K$ on $\mathbb{H}^n$ corresponding to the operator $m(L_1, \ldots, L_n, iT)$ is polyradial, smooth away from the $z_i = 0$ planes, and satisfies the size condition

$$|\partial_\ell^I \partial_k^k K(z, t)| \leq C_{I, k} \left| z \right|^{-2-l} (|z|^2 + |t|)^{-1-k}$$

for all $I \in (\mathbb{Z}^+)^n$, $k \in \mathbb{Z}^+$; as well as the following cancellation conditions: for all $\ell$, $1 \leq \ell \leq n$,

$$\left| \int_{C_{\ell}} \partial_\ell^I \partial_k^k K(z, t) \varphi(\delta_\ell(z_\ell)) \, dz_\ell \right| \leq C_{I, k} \left| z_\ell \right|^{-2-l} (|z|^2 + |t|)^{-1-k}$$

for all normalised bump functions $\varphi$ on $C^t$, $r \in (\mathbb{R}^+)^t$, $I \in (\mathbb{Z}^+)^{n-t}$, and $k \in \mathbb{Z}^+$;

$$\left| \int_{C_{\ell}} \int_{\mathbb{R}} \partial_\ell^I \partial_k^k K(z, t) \varphi(\delta_\ell(z_\ell, t)) \, dz_\ell \, dt \right| \leq C_I \left| z_\ell \right|^{-2-l}$$
for all normalised bump functions \( \varphi \) on \( \mathbb{C} \), \( r \in (\mathbb{R}^+)^{t+1} \), and \( I \in (\mathbb{Z}^+)^{n-t} \); all conditions obtained from (5) and (6) by permuting the indices \( 1, \ldots, n \); and

(7) \[
\left| \int_{\mathbb{R}} \partial_z^I K(z,t) \varphi(rt) \, dt \right| \leq C_I |z|^{-2-I}
\]

for all normalised \( \varphi \) on \( \mathbb{R} \), \( r > 0 \), and \( I \in (\mathbb{Z}^+)^n \). The converse also holds: every polyradial distribution \( K \) on \( \mathbb{H}_n \) which is smooth away from the \( z_i = 0 \) planes and satisfies (4)–(7) is the convolution kernel of a multiplier operator \( m(C_1, \ldots, C_n, iT) \) for some \( m \) on \((\mathbb{R}^+)^n \times \mathbb{R} \) satisfying the Marcinkiewicz-type condition (1) for all \( i_1, \ldots, i_n, j \in \mathbb{Z}^+ \).

The forward direction was established in [3]. We prove the converse here.

4. THE RESULT FOR SMOOTH COMPACTLY SUPPORTED KERNELS

**Proposition 4.1.** If \( K \in C^\infty_c(\mathbb{H}_n) \) is polyradial and satisfies (4)–(7), then \( \mu(k, \lambda) = \tilde{K}(k, \lambda) \) (as given in (3)) is bounded by a constant that depends only on the constants in the conditions (4)–(7).

**Proof:** By homogeneity, we may assume \( \lambda = 1 \). We have

\[
\mu(k, 1) = \int_{\mathbb{H}_n} e^{-it \ell_k(2|z|^2)} K(z,t) \, dz \, dt.
\]

We break up the integral according to the size of the \( |z_i| \)'s by introducing a normalised bump function \( \varphi \) on \( \mathbb{R} \), such that \( \varphi \equiv 1 \) on \( [-1/2, 1/2] \). Observe that \( \psi(w) = \varphi(|w|^2) \) is then a normalised bump function on \( \mathbb{C} \). Thus,

\[
\mu(k, 1) = \sum_{\ell=0}^n \sum_{1 \leq i_1 < \cdots < i_\ell \leq n} I_{t(i_1, \ldots, i_\ell)}
\]

with

\[
I_{t(i_1, \ldots, i_\ell)} = \int_{\mathbb{H}_n} \left( 1 - \varphi(2k_{i_1}|z_{i_1}|^2) \right) \cdots \left( 1 - \varphi(2k_{i_\ell}|z_{i_\ell}|^2) \right) \cdot \varphi(2k_{j_1}|z_{j_1}|^2) \cdots \varphi(2k_{j_{n-\ell}}|z_{j_{n-\ell}}|^2) e^{-it \ell_k(2|z|^2)} K(z,t) \, dz \, dt
\]

where \( j_1 < \cdots < j_{n-\ell} \) are such that \( \{i_1, \ldots, i_\ell, j_1, \ldots, j_{n-\ell} \} = \{1, \ldots, n\} \).

Now, given \( \ell, 0 \leq \ell \leq n \), by relabelling of variables, it suffices to consider only the term \( I_{t(i_1, \ldots, i_\ell)} \).

Since \( K \) is polyradial, we may define

\[
f(2|z_1|^2, \ldots, 2|z_\ell|^2, z_{\ell+1}, \ldots, z_n, t) = K(z,t).
\]
Then, changing variables \( r_i = 2|z_i|^2 \) for \( i = 1, \ldots, \ell \), we have,

\[
I_{\ell(1, \ldots, \ell)} = \left( \frac{\pi}{2} \right)^\ell \int_{\mathbb{R}^{n-\ell} \times \mathbb{R}^\ell} (1 - \varphi(k_1 r_1)) \cdots (1 - \varphi(k_\ell r_\ell))
\]

\[
\cdot \varphi(2k_1 |z_1|^2) \cdots \varphi(2k_n |z_n|^2) \ell_{k_1}(2|z_1|^2) \cdots \ell_{k_n}(2|z_n|^2)
\]

\[
= \left( \frac{\pi}{2} \right)^\ell \int_{\mathbb{R}^\ell \times \mathbb{C}^{n-\ell} \times \mathbb{R}} \Phi_1(r) \ell(r)e^{-itf(r_1, \ldots, r_\ell, z_1, \ldots, z_n, t)} \, dr \, dz' \, dt
\]

where \( r = (r_1, \ldots, r_\ell) \), \( z' = (z_{\ell+1}, \ldots, z_n) \),

\[
\Phi_1(r) = (1 - \varphi(k_1 r_1)) \cdots (1 - \varphi(k_\ell r_\ell)), \quad \ell(r) = \ell_{k_1}(r_1) \cdots \ell_{k_\ell}(r_\ell),
\]

and by properties (iii) and (iv) for Laguerre functions,

\[
u(z') = \varphi(2k_1 |z_1|^2) \cdots \varphi(2k_n |z_n|^2) \ell_{k_1}(2|z_1|^2) \cdots \ell_{k_n}(2|z_n|^2)
\]

is a dilate of a normalised bump function on \( \mathbb{C}^{n-\ell} \).

We remark that using (5) directly at this stage to estimate

\[
\int_{\mathbb{C}^{n-\ell}} f(r, z', t)u(z') \, dz'
\]

would yield \( r_1^{-1} \cdots r_\ell^{-1}(r_1 + \cdots + r_\ell + |t|)^{-1} \), which is too large. There are two tricks at our disposal to improve matters. Writing \( e^{-it} \) as its own derivative and integrating by parts in \( t \), we can move as many derivatives \( \partial_t^N \) as we wish onto \( f \), so that (5) then gives the estimate \( r_1^{-1} \cdots r_\ell^{-1}(r_1 + \cdots + r_\ell + |t|)^{-N-1} \), which is integrable (for example, for \( N = \ell \)) on the support of \( (1 - \varphi(k_1 r_1)) \cdots (1 - \varphi(k_\ell r_\ell)) \), but at a cost dependent on \( k \):

\[
|I_{\ell(1, \ldots, \ell)}| \leq C \int_{r_i \geq 1/(2k_i), i=1, \ldots, \ell} r_1^{-1} \cdots r_\ell^{-1} \left( r_1 + \cdots + r_\ell + |t| \right)^{-\ell-1} \, dt \, dr
\]

\[
\leq C k_1 \cdots k_\ell.
\]

The second trick is the corresponding one in the \( r_1, \ldots, r_\ell \) variables, and we begin with this. By property (ii), we can write the Laguerre functions in terms of their derivatives:

\[
\ell_{k_i}(r_i) = \frac{-1}{k_i + 1/2} \left[ r_i \left( \frac{d}{dr_i} \right)^2 + \frac{d}{dr_i} - \frac{1}{4} r_i \right] \ell_{k_i}(r_i).
\]

Setting \( L_i = L_i^2 + L_i^1 + L_i^0 \), with \( L_i^2 = r_i \left( \frac{d}{dr_i} \right)^2, L_i^1 = \frac{d}{dr_i}, \) and \( L_i^0 = -r_i/4 \), then,

\[
\ell = \ell_{k_1} \cdots \ell_{k_\ell} = \frac{(-1)^\ell}{(k_1 + 1/2) \cdots (k_\ell + 1/2)} L_1 \ell_{k_1} \cdots L_\ell \ell_{k_\ell}.
\]
with $L_1 \ell_{k_1} \cdots L_\ell \ell_{k_\ell}$ expanding to

$$(L_1^2 \ell_{k_1} + L_1^1 \ell_{k_1} + L_1^0 \ell_{k_1}) \cdots (L_\ell^2 \ell_{k_\ell} + L_\ell^1 \ell_{k_\ell} + L_\ell^0 \ell_{k_\ell})$$

$$= \sum \ell_{a_{t_1} + 1} \ell_{a_{t_2} + 1} \cdots \ell_{a_{t_\ell} + 1} \ell_{a_{t_2}} \ell_{a_{t_1}} \cdots \ell_{a_{t_\ell}} \ell_{a_{t_2 + 1}} \ell_{a_{t_1 + 1}} \cdots \ell_{a_{t_\ell}} \ell_{a_{t_2 + 1}} \ell_{a_{t_1 + 1}} \cdots \ell_{a_{t_\ell}} \ell_{a_{t_2 + 1}}$$

where the summation is over all $\ell_1, \ell_2, 0 \leq \ell_1 \leq \ell_2 \leq \ell$, and all partitions $a$ of $\{1, \ldots, \ell\}$ into three sets, $\{a_{t_1} + 1 < \cdots < a_{t_2}\}$, $\{a_1 < \cdots < a_{t_1}\}$, and $\{a_{t_2 + 1} < \cdots < a_{t_\ell}\}$ of cardinalities $\ell_2 - \ell_1$, $\ell_1$, and $\ell - \ell_2$. Thus, correspondingly,

$$I_{t_1, \ldots, t_\ell} = \sum_{t_1 = 0}^{t} \sum_{t_2 = t_1}^{t} \sum_{a} A_a^{t_1, t_2}.$$ 

Given $0 \leq \ell_1 \leq \ell_2 \leq \ell$, without loss of generality, we consider only the term $A_{t_1, t_2} = A_{a_1, a_2}^{t_1, t_2}$, with the partition $a$ given by $a_i = i$, for $i = 1, \ldots, \ell$.

The proposition will be proved once we have shown that $A_{t_1, t_2}$ is bounded. Letting

$$A_{t_1, t_2} = \left(\frac{\pi}{2}\right)^t \frac{(-1)^t}{(k_1 + 1/2) \cdots (k_\ell + 1/2)} B_{t_1, t_2},$$

then $|A_{t_1, t_2}| \leq C k_1^{-1} \cdots k_\ell^{-1} |B_{t_1, t_2}|$, and so it suffices to show $|B_{t_1, t_2}| \leq C k_1 \cdots k_\ell$ (thus giving us lee-way to use the integration by parts in $t$ trick if needed).

We thus consider

$$B_{t_1, t_2} = c \int_{(\mathbb{R}^+)^t \times \mathbb{R}^{\ell - t} \times \mathbb{R}} \Phi_1(r) f(r, z', t) e^{-it} u(z')$$

$$\cdot r_{t_1 + 1} \cdots r_{t_\ell} \ell_{k_{t_1 + 1}}(r_{t_1 + 1}) \cdots \ell_{k_{t_\ell}}(r_{t_\ell})$$

$$\cdot \left(\frac{d}{dr_{t_1}}\right) \ell_{k_1}(r_1) \cdots \left(\frac{d}{dr_{t_1}}\right) \ell_{k_\ell}(r_{t_1})$$

$$\cdot \ell_{t_2 + 1} \cdots \ell_{t_{t_\ell + 1}}(r_{t_2 + 1}) \ell_{t_2 + 1} \cdots \ell_{t_{t_\ell + 1}}(r_{t_2 + 1})$$

$$\cdot r_{t_\ell} \ell_{k_{t_\ell}}(r_{t_\ell}) dr dz' dt.$$ 

Since $f$ has compact support, and because of the presence of the factor $\Phi_1(r) = (1 - \varphi(k_i r_i)) \cdots (1 - \varphi(k_{t_\ell} r_{t_\ell}))$, we may integrate by parts in the $r$-variables. Thus, $B_{t_1, t_2}$ becomes

$$c \int_{(\mathbb{R}^+)^t \times \mathbb{R}^{\ell - t} \times \mathbb{R}} \partial_{r_{t_1}} \cdots \partial_{r_{t_\ell}} \partial_{r_{t_1 + 1}}^2 \cdots \partial_{r_{t_\ell}}^2 \left[r_{t_1 + 1} \cdots r_{t_\ell} \Phi_1(r) f(r, z', t)\right]$$

$$\cdot \ell(r) e^{-it} u(z') r_{t_1 + 1} \cdots r_{t_\ell} dr dz' dt.$$ 

In order to expand the derivative expression in the first line of the integrand, we observe that each second order derivative produces five kinds of terms:

$$\partial_{r_i}^2 \left\{r_i (1 - \varphi(k_i r_i)) g\right\} = 2(1 - \varphi(k_i r_i)) \partial_{r_i} g - 2r_i k_i \varphi'(k_i r_i) \partial_{r_i} g$$

$$- 2k_i \varphi'(k_i r_i) g - r_i k_i^2 \varphi''(k_i r_i) g + r_i (1 - \varphi(k_i r_i)) \partial_{r_i}^2 g.$$
Thus, $\partial_{t_{1}+\cdots+t_{5}}^{2} \partial_{r_{1}+\cdots+r_{6}} \Phi_{1} f$ expands into a sum of $5^{6}-4$ terms given by, for $t_{1} = j_{1} \leq j_{2} \leq j_{3} \leq j_{4} \leq j_{5} \leq j_{6} = t_{2}$, all terms obtained by permuting the variables $r_{t_{1}+1}, \ldots, r_{t_{2}}$ in the term

$$
\psi_{1} \psi_{2} \psi_{3} \psi_{4} \psi_{5} \psi_{6} \phi_{2} \partial_{\alpha} f
$$

where $\phi_{2}(r)$ is just $\phi_{1}(r)$ without the factors $(1 - \varphi(k_{t_{1}+1} r_{t_{1}+1})) \cdots (1 - \varphi(k_{t_{2}} r_{t_{2}}))$, and

$$
\psi_{1} = \prod_{i \in \mathcal{J}_{1}} (1 - \varphi(k_{i} r_{i})), \quad \psi_{2} = \prod_{i \in \mathcal{J}_{2}} r_{i} k_{i} \varphi'(k_{i} r_{i}),
$$

$$
\psi_{3} = \prod_{i \in \mathcal{J}_{3}} k_{i} \varphi'(k_{i} r_{i}), \quad \psi_{4} = \prod_{i \in \mathcal{J}_{4}} r_{i} k_{i}^{2} \varphi''(k_{i} r_{i}),
$$

$$
\psi_{5} = \prod_{i \in \mathcal{J}_{5}} r_{i} (1 - \varphi(k_{i} r_{i})),
$$

where for $i = 1, \ldots, 5$, $\mathcal{J}_{i} = \{ j_{i}+1, \ldots, j_{i+1} \}$, $\mathcal{J}_{6} = \emptyset$ if $j_{i} = j_{i+1}$, and $\beta = (b_{1}, \ldots, b_{t}) \in (\mathbb{Z}^{+})^{t}$ is the multi-index with all entries 0, except $b_{i} = 1$, for $i \in \mathcal{J}_{1} \cup \mathcal{J}_{2}$, $b_{i} = 2$, for $i \in \mathcal{J}_{5}$, so that

$$
\partial_{\alpha} = \partial_{r_{t_{1}+1}} \cdots \partial_{r_{3}} \partial_{r_{5}+1}^{2} \cdots \partial_{r_{t_{2}}}^{2}.
$$

Next, each first order derivative produces two kinds of terms:

$$
\partial_{r_{i}} \left[ (1 - \varphi(k_{i} r_{i})) g \right] = -k_{i} \varphi'(k_{i} r_{i}) g + (1 - \varphi(k_{i} r_{i})) \partial_{r_{i}} g
$$

Thus, the full expansion of

$$
\partial_{r_{1}} \cdots \partial_{r_{t_{1}}} \partial_{r_{t_{2}+1}} \cdots \partial_{r_{t_{3}}} \partial_{r_{t_{5}+1}}^{2} \cdots \partial_{r_{t_{2}}}^{2} \left[ r_{t_{1}+1} \cdots r_{t_{2}} \phi(r, z', t) \right]
$$

has $2^{t_{1}} 5^{t_{2} - t_{1}}$ terms given by, for each $0 \leq j_{0} \leq j_{1} \leq j_{2} \leq j_{3} \leq j_{4} \leq j_{5} \leq j_{6}$, the $\binom{t_{1}}{j_{0}} \binom{t_{2}}{j_{2} j_{3} j_{4} j_{5}}$ different terms obtained by permuting the variables $r_{1}, \ldots, r_{t}$ and $r_{t_{1}+1}, \ldots, r_{t_{2}}$ in the term

$$
\psi_{-1} \psi_{0} \psi_{1} \psi_{2} \psi_{3} \psi_{4} \psi_{5} \phi \partial_{\alpha} f,
$$

with $\phi(r) = (1 - \varphi(k_{t_{2}+1} r_{t_{2}+1})) \cdots (1 - \varphi(k_{t} r_{t}))$, and

$$
\psi_{-1} = \prod_{i \in \mathcal{J}_{-1}} k_{i} \varphi'(k_{i} r_{i}), \quad \psi_{0} = \prod_{i \in \mathcal{J}_{0}} (1 - \varphi(k_{i} r_{i})),
$$

where for $i = -1, 0, (j_{-1} = 0)$, $\mathcal{J}_{i}$ is defined as above, and $\alpha = \beta + \gamma$, with $\gamma = (c_{1}, \ldots, c_{t})$, $c_{i} = 1$ for $i \in \mathcal{J}_{0}$, and 0 otherwise, so that

$$
\partial_{\alpha} = \partial_{r_{j_{0}+1}} \cdots \partial_{r_{j_{3}} \partial_{j_{5}+1}}^{2} \cdots \partial_{r_{t_{2}}}^{2}.
$$

We also set $\psi_{6} = r_{t_{2}+1} \cdots r_{t} \phi$, and $\mathcal{J}_{6} = \{ t_{2}+1, \ldots, t \}$. 
The integral $B_j^{1,1}$ therefore splits up accordingly into $2^{\ell_1 5_{\ell_2 - \ell}}$ terms. Given $j_0, j_2, j_3, j_4, j_5$ as above, it suffices to consider the term

$$B_j^{1,1} = \int_{(R^+)^{x} \times C_0^{-\epsilon} \times R} \partial_{r} \left(f(r, z', t) \psi(r) e^{-it} u(z') dr \, dz' \, dtight)$$

with $\psi(r) = \psi_1 \psi_2 \psi_3 \psi_4 \psi_5 \psi_6$. The Proposition will be proved if we show that $|B_j^{1,1}| \leq C k_1 \cdots k_2$. We shall establish this estimate below, in Case 1 (if $j_5 < \ell_2$ or $\ell_2 < \ell$) and Case 2 (otherwise).

**Case 1.** $J_5 \cup J_6 \neq \emptyset$. In this case, we use the integration by parts in $t$ trick to improve the estimate (5) by an extra $r^{-1}_i$ for each $i \in J_5$ and by $r^{-2}_i$ for each $i \in J_6$. That is, we let $N = |J_5| + 2|J_6| = \ell_2 - j_5 + 2(\ell - \ell_2)$. Then, using

$$e^{-it} = (i)^N \left(\frac{d}{dt}\right)^N e^{-it},$$

and integrating by parts $N$ times in $t$, we obtain,

$$|B_j^{1,1}| = C \left| \int_{(R^+)^{x} \times C_0^{-\epsilon} \times R} \partial_{r}^N \partial_{z}^N f(r, z', t) \psi(r) e^{-it} u(z') dr \, dz' \, dt \right| \leq C \int_{\Omega} \int_{R} \left| \int_{C_0^{-\epsilon}} \partial_{r}^N \partial_{z}^N f(r, z', t) u(z') dz' \right| \psi(r) \, dt \, dr,$$

where $\Omega$ is the region

$$\Omega = \left\{ r \in R^\ell : \frac{1}{2k_i} \leq r_i \leq \frac{1}{k_i}, \quad i \in \bigcup_{i \in J_1 \cup J_2 \cup J_3 \cup J_4}; \right. \\
\left. \frac{1}{2k_i} \leq r_i, \quad i \in J_5 \cup J_6 \cup J_7 \cup J_8 \right\}$$

containing the support of $\psi$. Estimating the inner integral by the cancellation condition (5) in $z'$, then

$$|B_j^{1,1}| \leq C \int_{\Omega \times R} \prod_{i \in J_1 \cup J_2 \cup J_3 \cup J_4} r_i^{-1} \prod_{i \in J_5 \cup J_6} r_i^{-3/2} \prod_{i \in J_5} r_i^{-2} \\
\left( r_1 + \cdots + r_\ell + |t| \right)^{-1-N} \left| \psi(r) \right| \, dt \, dr \\
\leq C \int_{\Omega \times R} \prod_{i \in J_1 \cup J_2 \cup J_3} k_i r_i^{-1} \prod_{i \in J_2} k_i r_i^{-1/2} \prod_{i \in J_4} k_i^{2} \prod_{i \in J_5 \cup J_6} r_i^{-3/2} \prod_{i \in J_5} r_i^{-1} \left( r_1 + \cdots + r_\ell + |t| \right)^{-1-N} \, dt \, dr$$

But

$$\int_{R} \left( r_1 + \cdots + r_\ell + |t| \right)^{-1-N} \, dt \leq C (r_1 + \cdots + r_\ell)^{-N} \leq C \prod_{i \in J_5} r_i^{-1} \prod_{i \in J_6} r_i^{-2}$$
since $N = |\mathcal{J}_5| + 2|\mathcal{J}_6|$, whence $|B_j^{t_1,t_2}| \leq Ck_1 \cdots k_\ell$, as required.

**CASE 2.** $\mathcal{J}_5 \cup \mathcal{J}_6 = \emptyset$.

In this case, the cancellation condition (5) in $z'$ gives too large an estimate, which cannot be improved by integrating by parts in $t$. We must therefore take into account the cancellation in $t$. Recall that $\varphi$ is a normalised bump function on $\mathbb{R}$, with $\varphi \equiv 1$ on $[-(1/2), (1/2)]$. Including the factor 1,

$$1 = \varphi(t) + (1 - \varphi(t))$$

in the integrand, we then break $B_j^{t_1,t_2}$ accordingly into two integrals:

$$B_j^{t_1,t_2} = B_j^{t_1,t_2} + B_j^{t_1,t_2}.$$

We first observe that $|B_j^{t_1,t_2}|$ is bounded by

$$\int_\Omega \left| \int_{\mathbb{C}^{n-\ell} \times \mathbb{R}} \partial_\varphi^\alpha f(r,z',t)e^{-it}u(z')\varphi(t) \, dz' \, dt \right| \Psi(r) \, dr,$$

with $\Omega$ as above. By the cancellation condition (6) in $z'$ and $t$, since $e^{-it}u(z')\varphi(t)$ is a dilate of a normalised bump function in $z'$ and $t$,

$$|B_j^{t_1,t_2}| \leq C \int_\Omega \prod_{i \in \mathcal{J}_1 \cup \mathcal{J}_2} r_i^{-1} \prod_{i \in \mathcal{J}_3} r_i^{-3/2} \left| \Psi(r) \right| \, dr$$

$$\leq C \int_\Omega \prod_{i \in \mathcal{J}_1 \cup \mathcal{J}_2} k_i r_i^{-1} \prod_{i \in \mathcal{J}_3} k_i r_i^{-1/2} \prod_{i \in \mathcal{J}_4} k_i^2 \prod_{i \in \mathcal{J}_5} r_i^{-3/2} \, dr$$

$$\leq C k_1 \cdots k_\ell.$$

Next, for $B_j^{t_1,t_2}$, writing $e^{-it} = e^{-it} e^{-it}$, and integrating by parts in $t$, we obtain,

$$|B_j^{t_1,t_2}| = C \int_{(\mathbb{R}^+) \times \mathbb{C}^{n-\ell} \times \mathbb{R}} \left| \Psi(r) e^{-it}u(z') (1 - \varphi(t)) \partial_\varphi^\alpha f(r,z',t) \, dr \, dz' \, dt \right|$$

$$= C \int_{(\mathbb{R}^+) \times \mathbb{C}^{n-\ell} \times \mathbb{R}} \left| \left[ \partial_\varphi^\alpha f(r,z',t) + (1 - \varphi(t)) \partial_t \partial_\varphi^\alpha f(r,z',t) \right] \Psi(r) e^{-it}u(z') \, dt \, dz' \, dr \right|$$

$$\leq C \int_\Omega \left[ \int_{|(1/2) - x| \leq 1} \left| \int_{\mathbb{C}^{n-\ell}} \partial_\varphi^\alpha f(r,z',t) u(z') \, dz' \right| \, dt \right] \Psi(r) \, dr$$

$$+ \int_{|x| \geq (1/2)} \left| \int_{\mathbb{C}^{n-\ell}} \partial_t \partial_\varphi^\alpha f(r,z',t) u(z') \, dz' \right| \, dt \] \Psi(r) \, dr.$$

Then, by the cancellation condition (5) in $z'$, $|B_j^{t_1,t_2}|$ is bounded by

$$C \int_\Omega \prod_{i \in \mathcal{J}_1 \cup \mathcal{J}_2} r_i^{-1} \prod_{i \in \mathcal{J}_3} r_i^{-3/2} \left[ \int_{|(1/2) - x| \leq 1} (r_1 + \cdots + r_\ell + |t|)^{-1} \, dt \right] \Psi(r) \, dr$$

$$+ \int_{|x| \geq (1/2)} (r_1 + \cdots + r_\ell + |t|)^{-2} \, dt \] \Psi(r) \, dr.$$
Since the integrals over $t$ are bounded, we are left with the same integral obtained for $B^i_{j_1, j_2}$, which concludes the proof of Proposition 4.1.

\[ \square \]

REMARK. We notice from the proof that in Proposition 4.1, conditions (4)-(7) were in fact not required for all derivatives on the kernel $K$, but only as follows: (4) for all $\partial^I_x \partial^K_t$ where $I \in (\mathbb{Z}^+)^n$, $k \in \mathbb{Z}^+$ such that $i_j \leq 2$, $j = 1, \ldots, n$ and $k \leq \max\left(1, \sum_{j=0}^{n} (\delta_{i_j,2} + 2\delta_{i_j,0})\right)$; (5) for all $\partial^I_x \partial^K_t$ for $I$, $k$ as above, with $n - \ell$ replacing $n$; (6) for all $\partial^I_x \partial^K_t$ with $I$ as for (5); and (7) for all $\partial^I_x$ with $I$ as for (4).

5. CONDITIONS ON THE GELFAND TRANSFORM

Define the difference operators $\Delta_i$, $i = 1, \ldots, n$ by

$$\Delta_i \mu(k, \lambda) = \mu(k, \lambda) - \mu(k - 1, \lambda),$$

where $1_i = (j_1, \ldots, j_n) \in \mathbb{N}^n$ with $j_\ell = \delta_{i_\ell}.$

**Lemma 5.1.** If $f \in C_c^\infty(\mathbb{H}_n)$ is polyradial, with $\mu(k, \lambda) = \tilde{f}(k, \lambda)$ as in (3), then

\begin{align*}
\lambda \frac{\partial}{\partial \lambda} \mu(k, \lambda) &= - \left( (n+1)f + t \frac{\partial}{\partial t} f + \sum_{j=1}^{n} z_j \frac{\partial}{\partial z_j} f \right) \tilde{f}(k, \lambda), \\
k_j \Delta_j \mu(k, \lambda) &= - \left[ f + z_j \frac{\partial}{\partial z_j} f + i(\text{sgn } \lambda)|z_j|^2 \frac{\partial}{\partial t} f \right] \tilde{f}(k, \lambda).
\end{align*}

**Proof:** Since $f$ is polyradial, we may set $f_0(2|z_1|^2, \ldots, 2|z_n|^2, t) = f(z, t)$. Then by the change of variables $r_i = 2|z_i|^2$, $i = 1, \ldots, n$, we have

$$\mu(k, \lambda) = \int_{\mathbb{H}_n} e^{-i\lambda t} \ell_k(2\lambda|z|^2)f(z, t) \, dz \, dt = \left( \frac{n}{2} \right)^{\frac{n}{2}} \int_{(\mathbb{R}^+)^n \times \mathbb{R}} e^{-i\lambda t} \ell_k(2\lambda|z|^2)f_0(r_1, \ldots, r_n, t) \, dr \, dt.$$

We first prove (8). We notice that $\lambda \partial/\partial \lambda \left[ e^{-i\lambda t} \ell_k(|\lambda|r) \right]$ equals

$$\lambda \frac{\partial}{\partial \lambda} e^{-i\lambda t} \ell_k(|\lambda|r) + \sum_{j=1}^{n} e^{-i\lambda t} \ell_{k_1}(|\lambda|r_1) \cdots \tilde{\ell}_{k_n}(|\lambda|r_n) \lambda \frac{\partial}{\partial \lambda} \ell_{k_j}(|\lambda|r_j)$$

$$= t \frac{\partial}{\partial t} \ell_k(|\lambda|r) + \sum_{j=1}^{n} e^{-i\lambda t} \ell_{k_1}(|\lambda|r_1) \cdots \tilde{\ell}_{k_n}(|\lambda|r_n) r_j \frac{\partial}{\partial r_j} \ell_{k_j}(|\lambda|r_j),$$

where the shorthand $a_1 \cdots \tilde{j} \cdots a_n$ will be used to denote the product $a_1 \cdots a_{j-1} a_{j+1} \cdots a_n$, (that is, without the $j$th factor $a_j$). Observing the presence of the
factor $r_j$, and the fact that $f_0$ has compact support, we integrate by parts in $t$ and the $r_j$ variables, to obtain

$$
\lambda \frac{\partial}{\partial \lambda} \mu(k, \lambda) = -\left(\frac{\pi}{2}\right)^n \int_{(\mathbb{R}^+)^n \times \mathbb{R}} e^{-i\lambda t} \ell_k(|\lambda|r_j) \frac{\partial}{\partial t} (r_j f_0(r, t)) \, dr \, dt
$$

$$
+ \sum_{j=1}^{n} \int_{(\mathbb{R}^+)^n \times \mathbb{R}} e^{-i\lambda t} \ell_k(|\lambda|r) \frac{\partial}{\partial r_j} (r_j f_0(r, t)) \, dr \, dt
$$

$$
= -\left(\frac{\pi}{2}\right)^n \int_{(\mathbb{R}^+)^n \times \mathbb{R}} e^{-i\lambda t} \ell_k(|\lambda|r)
$$

$$
\left[ (n + 1) f_0(r, t) + t \frac{\partial}{\partial t} f_0(r, t) + \sum_{j=1}^{n} r_j \frac{\partial}{\partial r_j} f_0(r, t) \right] \, dr \, dt,
$$

which proves (8), since $r_j \frac{\partial}{\partial r_j} f_0(r, t) = z_j \frac{\partial}{\partial z_j} f(z, t)$.

Now, for (9), using property (i) for the Laguerre function $\ell_k$,

$$
k_j \Delta_j \ell_{k_j}(|\lambda|r_j) = |\lambda|r_j \ell'_{k_j}(|\lambda|r_j) + \frac{1}{2} |\lambda|r_j \ell_{k_j}(|\lambda|r_j)
$$

and thus $k_j \Delta_j \mu(k, \lambda)$ equals

$$
\left(\frac{\pi}{2}\right)^n \int_{(\mathbb{R}^+)^n \times \mathbb{R}} e^{-i\lambda t} \ell_{k_1}(|\lambda|r_1) \cdots \ell_{k_n}(|\lambda|r_n) r_j \frac{\partial}{\partial r_j} \ell_{k_j}(|\lambda|r_j) f_0(r, t) \, dr \, dt
$$

$$
+ \left(\frac{\pi}{2}\right)^n \int_{(\mathbb{R}^+)^n \times \mathbb{R}} \ell_k(|\lambda|r) \frac{1}{2} |\lambda|r_j e^{-i\lambda t} f_0(r, t) \, dr \, dt.
$$

Integrating by parts in $r_j$ for the first term, and in $t$ for the second, where we write $|\lambda| e^{-i\lambda t} = i (\text{sgn} \lambda) \frac{\partial}{\partial t} e^{-i\lambda t}$, then the above is equal to

$$
- \left(\frac{\pi}{2}\right)^n \int_{(\mathbb{R}^+)^n \times \mathbb{R}} e^{-i\lambda t} \ell_k(|\lambda|r_j) \frac{\partial}{\partial r_j} (r_j f_0(r, t)) \, dr \, dt
$$

$$
- \left(\frac{\pi}{2}\right)^n \int_{(\mathbb{R}^+)^n \times \mathbb{R}} e^{-i\lambda t} \ell_k(|\lambda|r) (i \text{sgn} \lambda) \frac{1}{2} r_j \frac{\partial}{\partial t} f_0(r, t) \, dr \, dt
$$

$$
= - \int_{(\mathbb{R}^+)^n \times \mathbb{R}} e^{-i\lambda t} \ell_k(|\lambda|r) \left[ f_0(r, t) + r_j \frac{\partial}{\partial r_j} f_0(r, t) + i \text{sgn} \lambda r_j \frac{\partial}{\partial t} f_0(r, t) \right] \, dr \, dt
$$

which proves (9).

With this lemma and Proposition 4.1 in hand, the Gelfand transform of our kernel is easily seen to satisfy the discrete Marcinkiewicz condition

$$
|\langle k_\alpha \Delta_1 \rangle^{a_1} \cdots \langle k_n \Delta_n \rangle^{a_n} (\lambda \partial_\lambda)^{\beta} \mu(k, \lambda) | \leq C_{ob}
$$

for all $\alpha_i, b \in \mathbb{N} \cup \{0\}$.
COROLLARY 5.2. If $K \in C_c^\infty(\mathbb{H})$ is polyradial, and satisfies (4)--(7), then 
$\mu(k, \lambda) = K(k, \lambda)$ satisfies (10), with constants $C_{a,b}$ that depend only on the constants 
in (4)--(7). 

PROOF: Since $t^{-\lambda} \frac{\partial}{\partial t} K$, $z_j \frac{\partial}{\partial z_j} K$ and $|z_j|^\lambda \frac{\partial}{\partial t} K$ also satisfy (4)--(7) if $K$ does, then by 
Lemma 5.1, applying $\lambda \frac{\partial}{\partial \lambda}$ or $k_j \Delta_j$ to the $\sim$-transform of a $C_c^\infty$ function satisfying (4)--(7) 
still yields a $\sim$-transform of a $C_c^\infty$ function satisfying (4)--(7). Thus the result follows 
from Proposition 4.1. \[ \square \]

REMARK. Further to the remark at the end of Section 4, we observe that it is of course 
here in Corollary 5.2 that conditions (4)--(7) are required for all derivatives on the kernel 
$K$.

The Marcinkiewicz condition (1) on the multiplier $m$ then follows by the following 
argument of interpolation between integers.

PROPOSITION 5.3. If $\mu(k, \lambda) = m((2k_1 + 1)|\lambda|, \ldots, (2k_n + 1)|\lambda|, \lambda)$ satisfies 
(10) then $m$ satisfies (1) with constants $C_{a,b}$ depending only on the constants $C_{a,b}$ in the 
condition (10) on $\mu$.

PROOF: Let $\varphi \in C_c^\infty(\mathbb{R})$ be supported in $[-(3\pi/2), (3\pi/2)]$, with $\varphi \equiv 1$ on 
$[-(\pi/2), (\pi/2)]$, and 
$$
\sum_{k \in \mathbb{Z}} \varphi(x + 2k\pi) = 1
$$
for all $x \in \mathbb{R}$. Then, setting $\Phi(x) = \varphi(x_1) \cdots \varphi(x_n)$ for $x \in \mathbb{R}^n$, we have 
$$
\hat{\Phi}(k) = \delta_{k0} \quad \text{for } k \in \mathbb{Z}^n.
$$
For all $\lambda \neq 0$, we set $a_k(\lambda) = \mu(k, \lambda)$ for $k \in \mathbb{N}^n$, and $a_k(\lambda) = -a_{k'}(\lambda)$ for $k, k' \in \mathbb{Z}^n$, if 
k and $k'$ differ only in one component, say the $j$th, where $k_j = -k'_j$. For $\nu \in \mathbb{R}^n$, $\lambda \neq 0$, 
define 
$$
\hat{\mu}(\nu, \lambda) = \sum_{k \in \mathbb{Z}^n} a_k(\lambda) \hat{\Phi}(\nu - k).
$$
Then $\partial_k^j \hat{\mu}(k, \lambda) = \partial_k^j \mu(k, \lambda)$ for $k \in \mathbb{N}^n$, for all $b \in \mathbb{N} \cup \{0\}$, so that $\hat{\omega}$ is a smooth extension 
of $\mu$.

For fixed $\lambda \neq 0$, and $b \in \mathbb{N} \cup \{0\}$, let $\tilde{a}_k = (\lambda \partial_k)^b a_k(\lambda)$. Then by definition, the $\tilde{a}_k$ 
are odd in each $k_j$-component. By (10),
$$
|\Delta_1^{\alpha_1} \cdots \Delta_n^{\alpha_n} a_{k}| \leq C_{a,b}
$$
for all $\alpha \in (\mathbb{N} \cup \{0\})^n$. And consequently 
$$
|\Delta_1^{\alpha_1} \cdots \Delta_n^{\alpha_n} \tilde{a}_{\alpha_k}| \leq C_{a,b}
$$
holds for all \( \alpha \in (\mathbb{N} \cup \{0\})^n \). Since in particular, \( |\tilde{a}_k| \leq C_b \), then the series

\[
\sum_{k \in \mathbb{Z}^n} \tilde{a}_k e^{ikx}
\]

converges in the sense of distributions to a distribution \( H \) on the \( n \)-torus \( \mathbb{T}^n = (-\pi, \pi)^n \).

We observe that, viewing \( H \) as a periodic distribution on \( \mathbb{R}^n \), then the compactly supported distribution \( \Phi H \) has Fourier transform

\[
(\Phi H)^\wedge(\nu) = \sum_{k \in \mathbb{Z}^n} \tilde{a}_k \tilde{\Phi}(\nu - k) = (\lambda \partial_\lambda)^b \tilde{\mu}(\nu, \lambda).
\]

We define

\[
h_N(x) = \sum_{k_1=-N}^N \ldots \sum_{k_n=-N}^N \tilde{a}_k e^{ikx}.
\]

Given \( \alpha \in (\mathbb{N} \cup \{0\})^n \), if there are only finitely many non-zero \( \tilde{a}_k \)'s, then fixing \( x \), and for \( i = 1, \ldots, n \), splitting the summation in \( k_i \) according to the size of \( |k_i| \) with respect to \( |x_i|^{-1} \), it follows from summation by parts and (11) that

\[
\left| \sum_{\text{finite}} (\Delta_1^{\alpha_1} \ldots \Delta_n^{\alpha_n} k_1^{\alpha_1} \ldots k_n^{\alpha_n} \tilde{a}_k) e^{ikx} \right| \leq C_{\alpha,\beta} |x_1|^{-1} \ldots |x_n|^{-1}
\]

independently of the number of nonzero \( \tilde{a}_k \)'s. But the sum in absolute value on the left-hand side above equals

\[
(1 - e^{ix_1})^{\alpha_1} \ldots (1 - e^{ix_n})^{\alpha_n} (-i)^{\lfloor \alpha_1 \rfloor \partial_{x_1}^{\alpha_1} \ldots \partial_{x_n}^{\alpha_n} \sum_{\text{finite}} \tilde{a}_k e^{ikx}
\]

and therefore

\[
(12) \quad \left| \partial_{x_1}^{\alpha_1} \ldots \partial_{x_n}^{\alpha_n} h_N(x) \right| \leq C_{\alpha,\beta} |x_1|^{-1-\alpha_1} \ldots |x_n|^{-1-\alpha_n}
\]

holds uniformly in \( N \), for all \( \alpha \in (\mathbb{N} \cup \{0\})^n \).

Viewing \( h_N \) as periodic functions on \( \mathbb{R}^n \), then for \( \beta, \gamma \in (\mathbb{N} \cup \{0\})^n \), the functions

\[
g_N = x^{\gamma} (\partial_{x_1} x_1)^{\beta_1} \ldots (\partial_{x_n} x_n)^{\beta_n} (\Phi h_N)
\]

also satisfy the same condition (12) uniformly in \( N \) for all \( \alpha \in (\mathbb{N} \cup \{0\})^n \).

Since the \( \tilde{a}_k \) are odd in each \( k_i \), then the periodic functions \( h_N \) are odd in each \( x_i \). Therefore, given \( 1 \leq i \leq n \), if \( \gamma_i \in \mathbb{N} \cup \{0\} \) is even, then for fixed \( x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n \), the functions

\[
(\Phi h_N)(x_1, \ldots, x_{i-1}, x_i+1, \ldots, x_n)
\]

are odd on \( \left[ -(\pi/2), (\pi/2) \right] \), where \( \varphi \equiv 1 \). Consequently, by (12), if \( \beta, \gamma \in (\mathbb{N} \cup \{0\})^n \) with \( \gamma_i \) even for all \( i = 1, \ldots, n \), then the functions \( g_N \) satisfy the product-type cancellation conditions on \( \mathbb{R}^n \) (see Section 3), uniformly in \( N \).
Since $g_N$ converge in the sense of distributions to the compactly supported distribution $x^\gamma (\partial_x, x_1)^{\beta_1} \ldots (\partial_x, x_n)^{\beta_n} (\Phi H)$, then by Lemma 5.5 below,

$$\left| \partial_{\nu_1} \nu_1 \partial_{\nu_2} \nu_2 \ldots \partial_{\nu_n} \nu_n \Phi H(\nu) \right| \leq C_{\beta, \gamma, \delta}$$

holds for all $\gamma, \beta \in \{N \cup \{0\}\}^n$ with each $\gamma_i$ even ($i = 1, \ldots, n$), and consequently by Lemma 5.4, for all $\gamma, \beta \in \{N \cup \{0\}\}^n$. Therefore

$$\left| \left( \nu_1 + \frac{1}{2} \right) \partial_{\nu_1} \right|^\alpha_1 \ldots \left( \nu_n + \frac{1}{2} \right) \partial_{\nu_n} \right|^\alpha_n \Phi H(\nu) \right| \leq C_{\alpha, \beta}$$

for all $\alpha \in \{N \cup \{0\}\}^n$, and since $(\Phi H)(\nu) = (\lambda \partial_\lambda)^b \mu(\nu, \lambda)$, then

$$\left| \left( \nu_1 + \frac{1}{2} \right) \partial_{\nu_1} \right|^\alpha_1 \ldots \left( \nu_n + \frac{1}{2} \right) \partial_{\nu_n} \right|^\alpha_n (\lambda \partial_\lambda)^b \mu(\nu, \lambda) \right| \leq C_{\alpha, \beta}$$

for all $\alpha \in \{N \cup \{0\}\}^n, b \in N \cup \{0\}$. Hence,

$$m(\xi, \eta) = \tilde{\mu} \left( \frac{1}{2} \left( \frac{\xi_1}{|\eta|} - 1 \right), \ldots, \frac{1}{2} \left( \frac{\xi_n}{|\eta|} - 1 \right) \right)$$

satisfies (1), concluding the proof of Proposition 5.3.

**Lemma 5.4.** Let $f$ be a smooth function on $\mathbb{R}$ such that

$$|(d/dt)^i (td/dt)^j f(t)| \leq C_{i,j}$$

for all $j \in \mathbb{Z}^+$ and all $i \in 2\mathbb{Z}^+$. Then (13) holds for all $i, j \in \mathbb{Z}^+$.

**Proof:** Given $i, j \in \mathbb{Z}^+$ with $i$ even, we let $g(t) = (td/dt)^j f(t)$, and show that

$$|(d/dt)^{i+1} g(t)| \leq C_{i,j}.$$  

From (13), we have

$$|(d/dt)^i (td/dt) g(t)| \leq C_{i,j}, \quad \text{and} \quad |(d/dt)^i g(t)| \leq C_{i,j}. \quad (14)$$

Using

$$(d/dt)^i (td/dt) = (td/dt)(d/dt)^i \pm i \frac{d}{dt}^i,$$  

then

$$|(d/dt)^{i+1} g(t)| \leq C_{i,j} |t|^{-1}; \text{ proving (14) for } |t| \geq 1. \text{ Now, by (13), } |(d/dt)^{i+1} g(t)| \leq C_{i,j}, \text{ whence }$$

$$|(d/dt)^{i+1} g(t) - (d/dt)^{i+1} g(1)| \leq C_{i,j} |t - 1| .$$

Thus for $|t| < 1$

$$|(d/dt)^{i+1} g(t)| \leq 2C_{i,j} + |(d/dt)^{i+1} g(1)|,$$

establishing (14) for all $t \in \mathbb{R}$.
**Lemma 5.5.** Let $K$ be a distribution on $\mathbb{R}^n$ satisfying product-type kernel conditions for all multi-indices $I$ with $i_j \leq 1$ for all $j$. Then $\hat{K}$ is bounded (by a constant depending only on those in the regularity and cancellation conditions $K$ satisfies).

The proof in the case of $K$ a smooth function of compact support (which is in fact all that we require in Proposition 5.3) uses a similar (but much simpler) argument to that in Proposition 4.1 in Section 4. The result for general $K$ can then be obtained by approximation, as in Theorem 3.1 in Section 6.

6. APPROXIMATION OF A DISTRIBUTION KERNEL

We now conclude the proof of Theorem 3.1. We reduce to the case of $K$ smooth, with compact support as follows. Let $\varphi$ be a polyanalogue normalised bump function on $\mathbb{H}_n$ of the product form given in Lemma 6.1 below, with $\int \varphi = 1$, and $\varphi(0) = 1$. Denote by $\varphi_{\varepsilon_1}$, the normalised dilated function

$$\varphi_{\varepsilon_1}(z, t) = \varepsilon_1^{-2n-2} \varphi(z/\varepsilon_1, t/\varepsilon_1^2)$$

for $\varepsilon_1 > 0$, and by $\varphi(\varepsilon_2 \cdot z, \varepsilon_2 t)$, the dilated function

$$\varphi(\varepsilon_2 \cdot (z, t)) = \varphi(\varepsilon_2 z, \varepsilon_2^2 t)$$

for $\varepsilon_2 > 0$. Then the functions $K_{\varepsilon_1, \varepsilon_2} = \varphi(\varepsilon_2 \cdot ) (K * \varphi_{\varepsilon_1})$ are smooth, compactly supported, polyanalogue, converge to $K$ in the sense of distributions, as $\varepsilon_1, \varepsilon_2 \to 0$, and by Lemmas 6.1 and 6.2 below, satisfy (4)-(7) uniformly in $\varepsilon_1$ and $\varepsilon_2$.

By Corollary 5.2 and Proposition 5.3, the multipliers $m_{\varepsilon_1, \varepsilon_2}$, such that

$$m_{\varepsilon_1, \varepsilon_2}(\mathcal{L}_1, \ldots, \mathcal{L}_n, i T) f = f * K_{\varepsilon_1, \varepsilon_2}$$

for $f \in S(\mathbb{H}_n)$, satisfy the Marcinkiewicz condition (1) for all $i_1, \ldots, i_n, j \in \mathbb{Z}^+$, uniformly in $\varepsilon_1, \varepsilon_2 > 0$.

It follows by Ascoli's Theorem that a subsequence $m_{\varepsilon_{1j}, \varepsilon_{2j}}$ with $\varepsilon_{1j} \to 0$, $\varepsilon_{2j} \to 0$ as $j \to \infty$ is uniformly bounded, and converges pointwise to a function $m$ on $(\mathbb{R}^+)^n \times \mathbb{R}$ satisfying (1). But since $K_{\varepsilon_1, \varepsilon_2}$ converge to $K$ in $S'(\mathbb{H}_n)$, then $K$ is the convolution kernel of $m(\mathcal{L}_1, \ldots, \mathcal{L}_n, i T)$.

**Lemma 6.1.** Let $\psi$ be a radial normalised bump function on $\mathbb{C}$, $\psi_0$ a normalised bump function on $\mathbb{R}$, and define $\varphi$ on $\mathbb{H}_n$ by

$$\varphi(z, t) = \psi(z_1) \cdots \psi(z_n) \psi_0(t).$$

If $\varphi_{\varepsilon}$ denotes the normalised dilate,

$$\varphi_{\varepsilon}(z, t) = \varepsilon^{-2n-2} \varphi(\varepsilon^{-1} z, \varepsilon^{-2} t)$$

of $\varphi$, for $\varepsilon > 0$, then the smooth functions $K *_{\mathbb{R}^{2n+1}} \varphi_{\varepsilon}$ obtained by convolving $K$ as a distribution on $\mathbb{R}^{2n+1}$ with $\varphi_{\varepsilon}$, are polyanalogue, and satisfy (4)-(7) uniformly in $\varepsilon > 0$. 

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PROOF: All convolutions in the proof of this lemma are Euclidean, rather than Heisenberg, and thus henceforth we shall drop the subscripts to $\ast$.

Let $\epsilon > 0$ be given. The function $K \ast \varphi_\epsilon$ is easily seen to be polyradial. For $f$ on $\mathbb{R}^N$, and $u \in \mathbb{R}^N$, we let $\tilde{f}(x) = f(-x)$ and denote by $f^{[u]}$ the Euclidean translate by $u$ of $f$. That is, $f^{[u]}(v) = f(u - v_1, \ldots, u_N - v_N)$. Thus, $K \ast \varphi_\epsilon(z, t) = K(\varphi_\epsilon^{[z, t]})$.

Given $\tau \in \mathbb{T}^n$, since by construction $\varphi$ is polyradial,

$$
\varphi_\epsilon^{[\tau, t]}(\zeta, s) = \varphi_\epsilon^{[z, t]}(\tau^{-1}\zeta, s) = \rho_{\tau^{-1}}\varphi_\epsilon^{[z, t]}(\zeta, s).
$$

But as $K$ is a polyradial distribution, $K(\rho_{\tau^{-1}}\varphi_\epsilon^{[z, t]}) = K(\varphi_\epsilon^{[z, t]})$, from which it follows that $K \ast \varphi_\epsilon$ is polyradial.

We now show that $K \ast \varphi_\epsilon$ satisfies the regularity condition (4). Let $(z, t) \in \mathbb{H}_n$, with $z_i \neq 0$ for $i = 1, \ldots, n$. For any $I \in (\mathbb{Z}^+)^n$, and $k \in \mathbb{Z}^+$,

$$
\partial_z^I \partial_t^k(K \ast \varphi_\epsilon)(z, t) = K(\partial_z^I \partial_t^k \varphi_\epsilon)(z, t) = K((\partial_z^I \partial_t^k \varphi_\epsilon)^{[z, t]})
$$

with

$$
(\partial_z^I \partial_t^k \varphi_\epsilon)(z, t) = (\partial_z^I \partial_t^k \varphi_\epsilon)(z - \zeta, t - s)
$$

$$
= (\partial_z^I \psi(\zeta_1) \cdots (\partial_z^{n} \psi \varphi_\epsilon)(\zeta_n)(\partial_t^k \psi_0)(t)](s)
$$

where $\psi$ and $(\psi_0)_{[t]}$ are the usual (Euclidean) normalised dilates of $\psi$ on $\mathbb{C}^n$ and of $\psi_0$ on $\mathbb{R}$.

We shall consider separately the cases corresponding to the sizes of the $|z_j|$, $j = 1, \ldots, n$, with respect to $\epsilon$. When all $|z_j|$ are large with respect to $\epsilon$, then (15) is just the convolution of a smooth function, $\partial_z^I \partial_t^k K$, with $\varphi_\epsilon$, and we shall use the regularity condition (4) on $K$. (Case 1 below.) When some of the $|z_j|$ are small compared to $\epsilon$, then the corresponding $(\partial_z^I \partial_t^k \varphi_\epsilon)^{[z, t]}$ are essentially normalised bump functions, and so we shall use the relevant cancellation condition (5) or (6) on $K$. (Cases 2 and 3 below.)

CASE 1. If $|z_j| > 2\epsilon$ for all $j = 1, \ldots, n$, then since $\psi_\epsilon$ is supported in the $\epsilon$-disc, $(w \in \mathbb{C} : |w| \leq \epsilon)$ in $\mathbb{C}$, $(\partial_z^I \partial_t^k \varphi_\epsilon)^{[z, t]}$ is supported in the region

$$
\{ (\zeta, s) \in \mathbb{H}_n : |\zeta_i| > \epsilon, \ i = 1, \ldots, n \},
$$

in which $K$ is a smooth function. Therefore

$$
|\partial_z^I \partial_t^k(K \ast \varphi_\epsilon)(z, t)| \leq C_{I, k} \int_{\mathbb{R}^{2n+1}} |\partial_z^I \partial_t^k \varphi_\epsilon(\zeta, s)| |\varphi_\epsilon(z - \zeta, t - s)| d\zeta ds
$$

by (4). But for $(\zeta, s)$ in the support of $\varphi_\epsilon(z - \zeta, t - s)$, we have $|\zeta_i| > |z_i|/2$ for $i = 1, \ldots, n$. If $|t| > 2\epsilon^2$, then for $(\zeta, s)$ in the support of $\varphi_\epsilon(z - \zeta, t - s)$ we also have $|s| > |t|/2$. Thus

$$
|\partial_z^I \partial_t^k(K \ast \varphi_\epsilon)(z, t)| \leq C_{I, k} |z|^{-2-l}(|z|^2 + |t|)^{-1-k} \int |\varphi_\epsilon^{[z, t]}(\zeta, s)| d\zeta ds
$$

$$
\leq C_{I, k} |z|^{-2-l}(|z|^2 + |t|)^{-1-k}.
$$
If \(|t| \leq 2\varepsilon^2\), then \(|z|^2 \geq (|z|^2 + |t|)/2\), and therefore
\[
|\partial_z^l \partial_t^k (K \ast \varphi_{\varepsilon})(z, t)| \leq C_{l,k} |z|^{-2-l} |z|^{-2-k} \int |\varphi_{\varepsilon}^{[l,k]}(\zeta, s)| d\zeta ds
\]
\[
\leq C_{l,k} |z|^{-2-l} (|z|^2 + |t|)^{-1-k}
\]

**CASE 2.** If only \(\ell = 0\) of the \(|z_i| \leq 2\varepsilon\), \(0 < \ell < n\), we may assume, by relabelling variables if necessary, that \(|z_1|, \ldots, |z_{\ell+1}| \leq 2\varepsilon\) and \(|z_{\ell+1}|, \ldots, |z_n| > 2\varepsilon\). We split \(\varphi(\zeta, s) = \varphi_1(\zeta, s) \varphi_2(\zeta, s)\). Now,
\[
\partial_w^l \psi_{\varepsilon} = \varepsilon^{-2-l}(\partial_w^l \psi_{\varepsilon}) \circ \varepsilon^{-1}
\]
for \(i \in \mathbb{Z}^+\). For \(j = 1, \ldots, \ell\), since \((\partial_w^j \psi_{\varepsilon}) \circ \varepsilon^{-1}\) is supported in the \(\varepsilon\)-disc in \(\mathbb{C}\), and \(|z_j| \leq 2\varepsilon\), then \((\partial_w^j \psi_{\varepsilon}) \circ \varepsilon^{-1})^{(z_j)}\) is supported in the \(3\varepsilon\)-disc in \(\mathbb{C}\), and is thus a dilate by \((3\varepsilon)^{-1}\) of the normalised bump function \(h_j(w) = (\partial_w^j \psi_{\varepsilon})(\varepsilon^{-1} z_j - 3w)\) on \(\mathbb{C}\). Therefore
\[
\Phi(\zeta, s) = ((\partial_w^j \varphi_1) \circ \varepsilon_{\varepsilon})(z, t) = h_1((3\varepsilon)^{-1} \zeta_1) \cdots h_\ell((3\varepsilon)^{-1} \zeta_\ell)
\]
is a dilate of a normalised bump function on \(\mathbb{C}^\ell\). Here, \(e = (\varepsilon^{-1}, \ldots, \varepsilon^{-1}) \in (\mathbb{R}^+)^\ell\).

Consequently, by the cancellation condition (5), the distribution \(K_*\) on \(\mathbb{R}^{n-\ell}\), defined as in Section 4, is smooth away from the \(\zeta_{\ell+1} = 0, \ldots, \zeta_n = 0\) planes and satisfies
\[
|\partial_L^j \partial_S^k K_*(\zeta, s)| \leq C_{l,j} |\zeta|^{-2-j} (|\zeta|^2 + |s|)^{-1-j}
\]
for all \(J \in (\mathbb{Z}^+)^{n-\ell}, j \in \mathbb{Z}^+\), where the constant does not depend on \(\varepsilon\). But
\[
\partial_L^j \partial_S^k (K \ast \varphi_{\varepsilon})(z, t) = \varepsilon^{-2l-j} L \partial_S^k K_{\varepsilon}((\partial_L^j \partial_S^k (\varphi_{\varepsilon}))^{(z, t)}),
\]
and since \(|z_{\ell+1}|, \ldots, |z_n| > 2\varepsilon\), we are therefore reduced to the situation of Case 1 on \(\mathbb{R}^{n-\ell}\). Thus,
\[
|\partial_L^j \partial_S^k (K \ast \varphi_{\varepsilon})(z, t)| = \varepsilon^{-2l-j} L \partial_S^k (K \ast (\varphi_{\varepsilon})_{\varepsilon})(z, t)
\]
\[
\leq C_{l,k} \varepsilon^{-2l-j} L |z|^{-2l-j} (|z|^2 + |t|)^{-1-k}
\]
\[
\leq C_{l,k} |z|^{-2l-j} (|z|^2 + |t|)^{-1-k},
\]
using the fact that \(|z_1|, \ldots, |z_{\ell}| < 2\varepsilon < |z_{\ell+1}|, \ldots, |z_n|\).

**CASE 3(a).** If \(|z_i| \leq 2\varepsilon\) for all \(i = 1, \ldots, n\), and \(|t| > 2\varepsilon^2\), we define \(\Phi\) on \(\mathbb{C}^n\) as in Case 2, with \(\ell = n\).

Then \(\Phi\) is a dilate of a normalised bump function on \(\mathbb{C}^n\), and so by the cancellation condition (5) on \(K\), the distribution \(K_*\) on \(\mathbb{R}\) is smooth away from 0, and satisfies
\[
|\partial_S^j K_{\varepsilon}(s)| \leq C_{l,j} |s|^{-1-j}
\]
for all $j \in \mathbb{Z}^+$, where the constant does not depend on $\varepsilon$.

Since $|t| > 2\varepsilon^2$, and $\partial^k_\varepsilon(\psi_0)_\varepsilon$ is supported in $[-\varepsilon^2, \varepsilon^2]$, then $(\partial^k_\varepsilon(\psi_0)_\varepsilon)^{|t|}$ is supported outside $[-\varepsilon^2, \varepsilon^2]$. Therefore

$$
|\partial^l_\varepsilon \partial^k_\varepsilon(K * \varphi_\varepsilon)(z, t)| = \varepsilon^{-2n-|l|} |K_\Phi((\partial^k_\varepsilon(\psi_0)_\varepsilon)^{|t|})|
$$

$$
= \varepsilon^{-2n-|l|} \left| \int_\mathbb{R} \partial^k_\varepsilon K_\Phi(s)(\psi_0)_{\varepsilon^2}(t - s) \, ds \right|
$$

$$
\leq C_{l, k} \varepsilon^{-2n-|l|} \int_\mathbb{R} |s|^{-1-k} |(\psi_0)_{\varepsilon^2}(t - s)| \, ds .
$$

But for $s$ in the support of $(\psi_0)_{\varepsilon^2}(t - s)$, $|s| \geq |t|/2$, and since $|t| > 2\varepsilon^2$, and $|z_i| \leq 2\varepsilon$ for $i = 1, \ldots, n$, then $|t| \geq c(|z|^2 + |t|)$. Thus, the result follows.

**CASE 3(b).** If all $|z_j| \leq 2\varepsilon$, and $|t| \leq 2\varepsilon^2$, then $(\partial_l^d \partial_k^n \varphi_\varepsilon)^{|z|, |t|} = \varepsilon^{-2n-|l|} e^{-2k} \Phi$ where

$$
\Phi = (\partial^l_\varepsilon \partial^k_\varepsilon)^z_\varepsilon \psi \varepsilon_\varepsilon^{-1} \psi \varepsilon_\varepsilon^{-1} \psi \varepsilon_\varepsilon^{-1}
$$

is a dilate of a normalised bump function on $\mathbb{H}_n$, and so the cancellation condition (6) on $K$ yields

$$
|\partial^l_\varepsilon \partial^k_\varepsilon(K * \varphi_\varepsilon)(z, t)| = \varepsilon^{-2n-|l|} e^{-2k} |K(\Phi)| \leq C e^{-2n-|l|} e^{-2k}
$$

$$
\leq C |z|^{-2-|l|} (|z|^2 + |t|)^{-1-k} ,
$$

concluding the proof of (4).

Before proceeding with the proof of the cancellation conditions (5)–(7), we recall that for a distribution $K$ on $\mathbb{R}^{N_1+N_2}$ and $f_1 \in S(\mathbb{R}^{N_1})$, the distribution $K_{f_1}$ on $\mathbb{R}^{N_2}$ is defined by

(16) \hspace{1cm} K_{f_1}(f_2) = K(f_1 \otimes f_2) \text{ for all } f_2 \in S(\mathbb{R}^{N_2}).

We observe that if $g_1, f_1 \in S(\mathbb{R}^{N_1}), g_2 \in S(\mathbb{R}^{N_2})$, then letting $g = g_1 \otimes g_2$,

(17) \hspace{1cm} (K * g)_{f_1} = K_{f_1 \ast g_1} * g_2 .

The cancellation conditions (5)–(7) are defined using (16). To consider these cancellation conditions, we write $\mathbb{H}_n = H_0 \times \cdots \times H_n$, where exactly one $H_i = \mathbb{R}$ and all others are $\mathbb{C}$. Abusing terminology somewhat, we shall refer to a kernel on $\mathbb{H}_n$ satisfying (4)–(7) as a *product-type* kernel on $H_0 \times \cdots \times H_n$. A product-type kernel on $\mathbb{C}^n$ or on $\mathbb{R}$, however, will mean the usual product-type kernel on $\mathbb{C} \times \cdots \times \mathbb{C}$ or respectively on $\mathbb{R}$ (one-fold product) defined in Section 3.

We let $G_1 = H_1 \times \mathbb{R}$, $G_2 = H_2 \times \mathbb{R}$, and split $\varphi$ into $\varphi_1 \otimes \varphi_2$, with $\varphi_i$ on $G_i$, $i = 1, 2$. For the cancellation in the $G_1$-variables on $K * \varphi_\varepsilon$, given a normalised bump function $\eta$ on $G_1$, and $r \in (\mathbb{R}^+)^{t+1}$, we must show that $(K * \varphi_\varepsilon)_{t0\delta}$ satisfies the product-type regularity condition on $G_2$ independently of $\eta$ or $r$. 

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But by (17),
\[ (K \ast \varphi_\varepsilon)_{\eta_0 \delta} = K_{(\eta_0 \delta)} \ast \varphi_{1 \varepsilon}, \]
and so if we show that \( K_{(\eta_0 \delta)} \ast \varphi_{1 \varepsilon} \) is a product-type kernel on \( G_2 \), then it will suffice to prove that the regularity condition on \( G_2 \) is satisfied by \( K \ast \varphi_{1 \varepsilon} \), whenever \( K \) is a product-type kernel on \( G_2 \). But we proved this above for the case where \( G_2 = \mathbb{H}_n \). For \( G_2 = \mathbb{C}^n - \ell \), the result follows as in Cases 1 and 2 above; for \( G_2 = \mathbb{R} \), as in Cases 1 and 3(b).

To see that \( K_{(\eta_0 \delta)} \ast \varphi_{1 \varepsilon} \) is a product-type kernel on \( G_2 \), we let \( \Phi = (\eta_0 \delta) \ast \varphi_{1 \varepsilon} \). Then
\[ \Phi(x) = \int_{G_1} \eta(\delta_r(x - \varepsilon y)) \varphi_{1 \varepsilon}(y) dy = \int_{G_1} \eta(\delta_r(\varepsilon y)) \varphi_{1 \varepsilon} \left( \frac{x}{\varepsilon} - y \right) dy, \]
and we see that \( |\partial_z \Phi(x)\| \leq C s_i \), where \( s_i = \min(r_i, \varepsilon^{-1}) \), while \( \Phi \) is supported in the region where \( |x_i| \leq \varepsilon + r_i^{-1} \leq 2s_i^{-1} \). Thus \( \Phi(x) \) is a dilate by \( s = (s_0, \ldots, s_\varepsilon) \) of a normalised bump function on \( G_1 \). It follows easily from the product-type cancellation conditions on \( K \) that \( K_{\Phi} \) is thus of product-type on \( G_2 \), which concludes the proof of Lemma 6.1.

**Lemma 6.2.** Let \( \varphi \) be as in Lemma 6.1, and denote by \( \varphi(\varepsilon \cdot) \) the dilate
\[ \varphi(\varepsilon \cdot (z, t)) = \varphi(\varepsilon z, \varepsilon^2 t) \]
of \( \varphi \), for \( \varepsilon > 0 \). If \( K \) is a polyradial distribution on \( \mathbb{H}_n \) satisfying (4)-(7), then \( K_\varepsilon = \varphi(\varepsilon \cdot) K \) are polyradial, and satisfy (4)-(7) uniformly in \( \varepsilon > 0 \).

**Proof:** We first prove (4). This follows directly from (4) on \( K \). Since \( K \) is smooth away from the \( z_i = 0 \) planes, then so is \( K_\varepsilon \). We set \( \varepsilon = (\varepsilon, \ldots, \varepsilon, \varepsilon^2) \in (\mathbb{R}^+)^{n+1} \), so that \( \varphi(\varepsilon \cdot) = \varphi_\delta \), where \( \delta \) denotes multi-parameter dilation on \( \mathbb{H}_n = \mathbb{C}^n \times \mathbb{R} \), and let \( \Omega \) be the region in \( \mathbb{H}_n \), containing the support of \( \varphi_\delta \), where \( |z_1|, \ldots, |z_n| \leq \varepsilon^{-1} \), and \( |t| \leq \varepsilon^{-2} \). Then given any \( (z, t) \in \mathbb{H}_n \) with \( z_i \neq 0 \) for \( i = 1, \ldots, n \),
\[ |\partial_z^I \partial_t^j K_\varepsilon(z, t)| \leq \sum_{U \leq I} \sum_{v=0}^j c_{U, v} |\partial_z^U \partial_t^v (\varphi_\delta)(z, t)| \partial_z^I - U \partial_t^I - v K(z, t)| \]
\[ \leq C_{1, j} \sum_{U \leq I} \sum_{v=0}^j \varepsilon^{|U|+|2v|} \chi_\Omega(z, t) \partial_z^I - U \partial_t^I - v K(z, t)| \]
\[ \leq C_{1, j} \sum_{U \leq I} \sum_{v=0}^j |z|^{|U|} (|z|^2 + |t|)^{-v} \partial_z^I - U \partial_t^I - v K(z, t)| \]
\[ \leq C_{1, j} |z|^{-2 - j} (|z|^2 + |t|)^{-1 - j} \]
by (4) on \( K \). The summations in \( U \) above are over multi-indices \( U \in (\mathbb{Z}^+)^n \) such that \( u_j \leq i_j \) for \( j = 1, \ldots, n \).
For the cancellation conditions (5)-(7) on $K_\varepsilon$, we write $H = H_0 \times \cdots \times H_n$, as in the proof of the previous lemma. Given $\ell$, $0 \leq \ell \leq n$, we let $G_1 = H_\ell$, $G_2 = H_{\ell+1}$, and split $\varphi$ and $\epsilon$ accordingly into $\varphi_1 \otimes \varphi_2$ and $\epsilon_1 \otimes \epsilon_2$ respectively. Given a normalised bump function $\eta$ on $G_1$, and $r \in (\mathbb{R}^+)^{\ell+1}$, we observe that

$$(K_\varepsilon)_{\eta \delta_r} = (\varphi_2 \delta_{\varepsilon_2})(\eta \delta_r)$$

as distributions on $G_2$. But $(\varphi_1 \delta_{\varepsilon_1})(\eta \delta_r)$ is a dilate of a normalised bump function on $G_1$. Consequently, by the cancellation condition in the $G_1$-variables on $K$, $K_{(\varphi_1 \delta_{\varepsilon_1})(\eta \delta_r)}$ satisfies the regularity condition on $G_2$. We are therefore reduced to the situation of estimate (4) proved above for the case of $G_2 = \mathbb{H}^n$. The cases $G_2 = \mathbb{C}^{n-\ell}$ and $G_2 = \mathbb{R}$ follow in the same manner. This concludes the proof of Lemma 6.2.

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