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AN ANALYTICAL SOLUTION FOR PARISIAN UP-AND-IN CALLS

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Abstract

We derive an analytical solution for the value of Parisian up-and-in calls by using the "moving window" technique for pricing European-style Parisian up-and-out calls. Our pricing formula can be applied to both European-style and American-style Parisian up-and-in calls, due to the fact that with an "in" barrier, the option holder cannot do or decide on anything before the option is activated, and once the option is activated it is just a plain vanilla call, which could be of American style or European style.

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1. Introduction

Barrier options are cheaper alternatives to vanilla options for hedging and speculating, but the "one-touch" knock-in or knock-out feature is prone to market manipulations. To eliminate these manipulations, Parisian options are introduced, while the underlying asset price has to continually stay above or below the asset barrier for a prescribed amount of time before the knock-out or knock-in feature is activated. However, the introduction of the "time barrier" turns the option valuation into a three-dimensional problem, which is more complicated to solve. This is especially true in the case of American-style Parisian knock-out options, since the corresponding optimal exercise boundary is a three-dimensional surface.

Fortunately, this difficulty disappears in the valuation of American-style Parisian knock-in options. In fact, by definition, before the knock-in feature is activated, the option holder cannot do anything regardless of the exercise style of the option and, once the "knock-in" feature is activated, the value of the Parisian option takes on the value of the embedded American-style vanilla option. Therefore, the solution procedure for the valuation of an American-style Parisian knock-in option and that of

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its European-style counterpart should be very similar. The only difference is that upon activation the former becomes an American-style vanilla option, and the latter becomes a European-style vanilla option. Thus, the technique proposed by Zhu and Chen [13] for their solution of European-style Parisian up-and-out calls could be applied to find analytical solutions for both American-style and European-style Parisian knock-in options. Recently, this technique was used to find a simple analytical solution for Parisian down-and-in calls [15]. The current paper aims to apply the same technique again for the derivation of an analytical solution for Parisian up-and-in calls.

The paper is organized as follows. In Section 2, we introduce the partial differential equation (PDE) systems governing the price of a Parisian up-and-in call. The solution procedure is presented in Section 3, while Section 4 provides a numerical example to illustrate the implementation of our formulas. The paper ends with some concluding remarks in Section 5.

2. Formulation

By definition, a Parisian up-and-in call is knocked in and becomes the embedded vanilla call, which could be of American or European style, if the underlying asset price continually stays above the barrier $\bar{S}$ for a prescribed time period $\bar{J}$. Otherwise, the Parisian up-and-in call expires worthless.

For some extreme values of $\bar{S}$ and $\bar{J}$, we observe that a Parisian up-and-in call becomes worthless, or degenerates to either a one-touch barrier option or a vanilla option. For other nondegenerate cases, the price of a Parisian up-and-in call depends on the underlying asset price S, the current time t and the barrier time J, in addition to other parameters such as the volatility rate σ, the risk-free interest rate r and the expiry time T.

We now assume that the underlying asset price S with a continuous dividend yield D follows a lognormal Brownian motion governed by

$$dS = (r - D)S\,dt + \sigma S\,dZ,$$

where Z is a standard Brownian motion.

Based on financial arguments similar to those of Zhu and Chen [13], the pricing domains of those nondegenerate cases can be elegantly reduced to the regions

$$\begin{aligned} &\text{I: } \{0 \le S \le \bar{S},\ 0 \le t \le T - \bar{J},\ J = 0\}, \\ &\text{II: } \{\bar{S} \le S < \infty,\ J \le t \le J + T - \bar{J},\ 0 \le J \le \bar{J}\}. \end{aligned}$$

Let $V_1(S, t)$ and $V_2(S, t, J)$ denote the option prices in the regions I and II, respectively. Following the arguments of Haber et al. [5] and Zhu and Chen [13], we show that V_1 and V_2 should satisfy the following two PDE systems $\mathcal{A}_1$ and $\mathcal{A}_2$ defined in region I

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and region II, respectively:

$$\mathcal{A}_1 = \begin{cases} \dfrac{\partial V_1}{\partial t} + \mathbb{L}V_1 = 0, \\ V_1(S, T - \bar{J}) = 0, \\ V_1(0, t) = 0, \\ V_1(\bar{S}, t) = V_2(\bar{S}, t, 0), \end{cases} \qquad \mathcal{A}_2 = \begin{cases} \dfrac{\partial V_2}{\partial t} + \dfrac{\partial V_2}{\partial J} + \mathbb{L}V_2 = 0, \\ V_2(S, t, \bar{J}) = C(S, t), \\ V_2(S, t, J) \backsim S \quad \text{as } S \to +\infty, \\ V_2(\bar{S}, t, J) = V_2(\bar{S}, t, 0), \end{cases} \tag{2.1}$$

with the connectivity condition

$$\frac{\partial V_1}{\partial S}(\bar{S}, t) = \frac{\partial V_2}{\partial S}(\bar{S}, t, 0). \tag{2.2}$$

Here, $C = C_A$ (the embedded American-style vanilla option price) if the Parisian option is of American style, or $C = C_E$ (the embedded European-style vanilla option price) if the Parisian option is of European style, and the operator $\mathbb{L}$ is defined as

$$\mathbb{L} = \frac{\sigma^2 S^2}{2} \frac{\partial^2}{\partial S^2} + (r - D)S \frac{\partial}{\partial S} - rI, \tag{2.3}$$

with I being the identity operator.

First, we point out that the option will expire worthless if the asset price still remains below or at the asset barrier when t reaches $T - \bar{J}$, because there is not enough time left for J to reach $\bar{J}$. Therefore, $V_1(S, t) = 0$ for all $t \geq T - \bar{J}$ and $S \leq \bar{S}$. This fact explains the "terminal condition" in $\mathcal{A}_1$ at $t = T - \bar{J}$. Secondly, the terminal condition, with respect to J, in $\mathcal{A}_2$ corresponds to the "knock-in" feature that the option price is equal to that of the embedded call, denoted by $C_A(S, t)$ or $C_E(S, t)$, at the time t the option is activated. Thirdly, we have the inhomogeneous boundary condition in $\mathcal{A}_2$ when S approaches infinity, because, in this case, the knock-in feature will be surely triggered and thereby the knock-in option price would be the same as its embedded option price, which is equivalent to the asset price S. Finally, the last equation in $\mathcal{A}_2$ holds only for $0 \leq J < \bar{J}$, that is, before the knock-in feature is triggered.

The above two PDE systems (2.1) resemble those of Zhu and Chen [13], so their "moving window" technique can be adopted to obtain the solution for our problem. In the next section, we shall discuss the solution procedure.

3. Solution procedure

Following the method of Zhu and Chen [13], the three-dimensional system in (2.1) and (2.2) can be reduced to a two-dimensional system by replacing the sum of the partial derivatives of V_2, that is, $\partial V_2/\partial t + \partial V_2/\partial J$, with its directional derivative $\sqrt{2}(\partial V_2/\partial l)$ in the direction of $(\sqrt{2}, \sqrt{2})$. After a further change of variable by $l = \sqrt{2}l'$, $\mathcal{A}_1$ and $\mathcal{A}_2$ in (2.1) and (2.2) are transformed to $\mathcal{A}_3$ and $\mathcal{A}_4$, respectively, as

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follows:

$$\mathcal{A}_3 = \begin{cases} \dfrac{\partial V_1}{\partial t} + \mathbb{L}V_1 = 0, \\ V_1(S, T - \bar{J}) = 0, \\ V_1(0, t) = 0, \\ V_1(\bar{S}, t) = W(t), \end{cases} \qquad \mathcal{A}_4 = \begin{cases} \dfrac{\partial V_2}{\partial l'} + \mathbb{L}V_2 = 0, \\ V_2(S, \bar{J}; t) = C(S, t + \bar{J}), \\ V_2(S, l'; t) \backsim S \quad \text{as } S \to +\infty, \\ V_2(\bar{S}, l'; t) = W(t + l'), \end{cases} \tag{3.1}$$

with the connectivity condition

$$\frac{\partial V_1}{\partial S}(\bar{S}, t) = \frac{\partial V_2}{\partial S}(\bar{S}, 0; t). \tag{3.2}$$

Here, $\mathcal{A}_3$ is defined on $t \in [0, T - \bar{J}]$, $S \in [0, \bar{S}]$; $\mathcal{A}_4$ is defined on $l' \in [0, \bar{J}]$, $S \in [\bar{S}, \infty)$, with the parameter $t \in [0, T - \bar{J}]$. The unknown function $W(t) = V_2(\bar{S}, 0; t)$, which provides the coupling between $\mathcal{A}_3$ and $\mathcal{A}_4$, needs to be solved as part of the solution.

To solve the newly established pricing systems (3.1) and (3.2) effectively, we shall first nondimensionalize all variables by introducing the following variables:

$$\begin{gathered} x = \ln \frac{S}{\bar{S}}, \quad \tau = (T - \bar{J} - t)\frac{\sigma^2}{2}, \quad \tilde{l} = \frac{\sigma^2}{2}(\bar{J} - l'), \quad \bar{J}' = \frac{\sigma^2 \bar{J}}{2}, \\ T' = \frac{\sigma^2 T}{2}, \quad W'(\tau) = \frac{W(t)}{\bar{S}}, \quad V_1'(x, \tau) = \frac{V_1(S, t)}{\bar{S}}, \\ V_2'(x, \tilde{l}; \tau) = \frac{V_2(S, l'; t)}{\bar{S}}, \quad C'(x, \tau) = \frac{C(S, t + \bar{J})}{\bar{S}}. \end{gathered} \tag{3.3}$$

With all primes and tildes dropped from now on, $\mathcal{A}_3$ and $\mathcal{A}_4$ in (3.1) and (3.2) are transformed to $\mathcal{B}_1$ and $\mathcal{B}_2$, respectively,

$$\mathcal{B}_1 = \begin{cases} \dfrac{\partial V_1}{\partial \tau} = \mathcal{K}V_1, \\ V_1(x, 0) = 0, \\ \lim\limits_{x \to -\infty} V_1(x, \tau) = 0, \\ V_1(0, \tau) = W(\tau), \end{cases} \qquad \mathcal{B}_2 = \begin{cases} \dfrac{\partial V_2}{\partial l} = \mathcal{K}V_2, \\ V_2(x, 0; \tau) = C(x, \tau), \\ V_2(x, l; \tau) \backsim e^x \quad \text{as } x \to +\infty, \\ V_2(0, l; \tau) = W(\tau - \bar{J} + l), \end{cases} \tag{3.4}$$

with the connectivity condition

$$\frac{\partial V_1}{\partial x}(0, \tau) = \frac{\partial V_2}{\partial x}(0, \bar{J}; \tau). \tag{3.5}$$

Here, $\mathcal{B}_1$ is defined on $\tau \in [0, T - \bar{J}]$, $x \in (-\infty, 0]$; $\mathcal{B}_2$ is defined on $l \in [0, \bar{J}]$, $x \in [0, \infty)$, with the parameter $\tau \in [0, T - \bar{J}]$. The operator $\mathcal{K}$ is defined as

$$\mathcal{K} = \frac{\partial^2}{\partial x^2} + k\frac{\partial}{\partial x} - \gamma I, \tag{3.6}$$

with $\gamma = 2r/\sigma^2$, $q = 2D/\sigma^2$ and $k = \gamma - q - 1$. Note that S, C, W, V_1 and V_2 are nondimensionalized by $\bar{S}$ here. As a result, the x-domains in $\mathcal{B}_1$ and $\mathcal{B}_2$ are semi-infinite.

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By applying the Laplace transform technique, the solution of $\mathcal{B}_1$ is obtained as

$$V_1(x,\tau) = \int_0^{\tau} W(s) g_1(x,\tau - s)\,ds \quad \text{for all } x \le 0, \tag{3.7}$$

where

$$g_1(x,\tau) = -\frac{x}{2\sqrt{\pi}\tau^{3/2}} e^{\alpha x + \beta\tau - (x^2/4\tau)}, \quad \alpha = -\frac{k}{2}, \quad \beta = -\frac{k^2}{4} - \gamma. \tag{3.8}$$

Since the PDE in $\mathcal{B}_2$ is linear, its solution can be found by superposition of the solutions of the following two systems:

$$\mathcal{B}_3 = \begin{cases} \dfrac{\partial V_2}{\partial l} = \mathcal{K} V_2, \\ V_2(x,0;\tau) = 0, \\ \lim\limits_{x\to+\infty} V_2(x,l;\tau) = 0, \\ V_2(0,l;\tau) = W(\tau - \bar{J} + l), \end{cases} \qquad \mathcal{B}_4 = \begin{cases} \dfrac{\partial V_2}{\partial l} = \mathcal{K} V_2, \\ V_2(x,0;\tau) = C(x,\tau), \\ V_2(x,l;\tau) \backsim e^x \quad \text{as } x \to +\infty, \\ V_2(0,l;\tau) = 0. \end{cases}$$

Since system $\mathcal{B}_3$ is very similar to $\mathcal{B}_1$, its solution is

$$V_2^{(1)}(x,l;\tau) = \int_0^{l} W(\tau - \bar{J} + s) g_2(x, l - s)\,ds \quad \text{for all } x \le 0,$$

where $g_2(x,l) = -g_1(x,l)$, with g_1 being defined in (3.8).

By using the variable transform $V_2(x,l;\tau) = e^{\alpha x + \beta\tau} u(x,l;\tau)$ (with α, β as in (3.8)), $\mathcal{B}_4$ can be transformed to a standard heat problem on a semi-infinite domain, whose solution can be found in [6]. As a result, the solution for $\mathcal{B}_4$ can be obtained as

$$V_2^{(2)}(x,l;\tau) = \int_0^{+\infty} \frac{1}{2\sqrt{\pi l}} e^{\alpha(x-z)+\beta l}[e^{-(x-z)^2/4l} - e^{-(x+z)^2/4l}] C(z,\tau)\,dz$$

and, for $\mathcal{B}_2$ in (3.4), as

$$V_2(x,l;\tau) = V_2^{(1)}(x,l;\tau) + V_2^{(2)}(x,l;\tau). \tag{3.9}$$

Applying the connectivity condition (3.5) to (3.7) and (3.9), we obtain an integral equation governing $W(\tau)$:

$$\begin{aligned} &\int_0^{\tau} W(s) \frac{\partial g_1}{\partial x}(x,\tau - s)\,ds \bigg|_{x=0} \\ &\quad = \frac{\partial V_2^{(2)}}{\partial x}(x,\bar{J};\tau)\bigg|_{x=0} + \int_0^{\bar{J}} W(\tau - \bar{J} + s) \frac{\partial g_2}{\partial x}(x,\bar{J} - s)\,ds \bigg|_{x=0}, \end{aligned} \tag{3.10}$$

where

$$\frac{\partial V_2^{(2)}}{\partial x}(x,\bar{J};\tau)\bigg|_{x=0} = \int_0^{+\infty} \frac{z C(z,\tau)}{2\sqrt{\pi \bar{J}^3}} e^{-\alpha z + \beta\bar{J} - z^2/4\bar{J}}\,dz.$$

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Now, a simple coordinate transform, $\xi = \tau - \bar{J} + s$, in the last integral on the right-hand side of equation (3.10) yields

$$\int_0^\tau W(s)\frac{\partial g_1}{\partial x}(x,\tau - s)\,ds\bigg|_{x=0} = \frac{\partial V_2^{(2)}}{\partial x}(x,\bar{J};\tau)\bigg|_{x=0} + \int_{\tau-\bar{J}}^\tau W(\xi)\frac{\partial g_2}{\partial x}(x,\tau-\xi)\,d\xi\bigg|_{x=0}. \tag{3.11}$$

We observe that the left-hand side of (3.11) contains $W(s)$ for $s \in [0,\tau]$, while its right-hand-side integral involves $W(\xi)$ for $\xi \in [\tau - \bar{J}, \tau]$, which coincides with the projection of the "slide" (a plane is of 45° angle to both of the plane $t = 0$ and $J = 0$) passing through $(\bar{S}, \tau, 0)$ on the plane $J = 0$. As in [13], we also name such a projection a "window".

We now solve the integral equation (3.11) for $\tau \in [0, \bar{J}]$ to obtain $W_1(\tau)$, which is the value of W in the first window. Note that $W(\xi) = 0$ for all $\xi \in [-\bar{J}, 0]$, because $V_1(S,t) = 0$ for all $t \geq T - \bar{J}, S \leq \bar{S}$ (as already explained in Section 2). Therefore, for $\tau \in [0, \bar{J}]$, equation (3.11) reduces to

$$\int_0^\tau W_1(s)\Big(\frac{\partial g_1}{\partial x} - \frac{\partial g_2}{\partial x}\Big)(x,\tau-s)\,ds\bigg|_{x=0} = \frac{\partial V_2^{(2)}}{\partial x}(x,\bar{J};\tau)\bigg|_{x=0}. \tag{3.12}$$

Clearly, the left-hand side of the last equation is a convolution integral involving the unknown function W_1. Taking the Laplace transform of equation (3.12) with respect to τ yields

$$\mathcal{L}[W_1(\tau)]\mathcal{L}\Big[\Big(\frac{\partial g_1}{\partial x} - \frac{\partial g_2}{\partial x}\Big)(x,\tau)\Big]\bigg|_{x=0} = \mathcal{L}\Big[\frac{\partial V_2^{(2)}}{\partial x}(x,\bar{J};\tau)\Big]\bigg|_{x=0},$$
$$\text{where}\quad \mathcal{L}\Big[\Big(\frac{\partial g_1}{\partial x} - \frac{\partial g_2}{\partial x}\Big)(x,\tau)\Big]\bigg|_{x=0} = 2\sqrt{p-\beta},$$

with p being the Laplace parameter [13]. Thus,

$$\mathcal{L}[W_1(\tau)] = \frac{1}{2\sqrt{p-\beta}}\,\mathcal{L}\Big[\frac{\partial V_2^{(2)}}{\partial x}(x,\bar{J};\tau)\Big]\bigg|_{x=0}. \tag{3.13}$$

Taking the inverse Laplace transform on both sides of (3.13) yields

$$\begin{aligned} W_1(\tau) &= \int_0^\tau \frac{\partial V_2^{(2)}}{\partial x}(x,\bar{J};s)\bigg|_{x=0} \frac{e^{\beta(\tau-s)}}{2\sqrt{\pi(\tau-s)}}\,ds \\ &= \int_0^{+\infty} \frac{z e^{-\alpha z + \beta\bar{J} - z^2/4\bar{J}}}{4\pi \bar{J}^{3/2}} \int_0^\tau \frac{C(z,s)e^{\beta(\tau-s)}}{\sqrt{\tau-s}}\,ds\,dz. \end{aligned}$$

Similar to the case of Zhu and Chen [13], for a state point (S,τ,J), one can evaluate W forwards, window by window, until the value at the required time τ is found. In fact, assuming that W_n is known for $n \geq 1$, we can then calculate the option price V_1 or V_2 in the nth window from the formula (3.7) or (3.9), respectively. However, the determination of W_{n+1}, assuming that W_n is known for $n \geq 1$, is slightly different from

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that of W_1. The two-dimensional coupled PDE systems governing the option price in the $(n+1)$th window are expressed as

$$C_1 = \begin{cases} \dfrac{\partial V_1}{\partial t} + \mathbb{L} V_1 = 0, \\ V_1(S, T-(n+1)\bar{J}) = \bar{S} f_n\Big(\ln \dfrac{S}{\bar{S}}\Big), \\ V_1(0,t) = 0, \\ V_1(\bar{S}, t) = W(t), \end{cases} \qquad C_2 = \begin{cases} \dfrac{\partial V_2}{\partial l'} + \mathbb{L} V_2 = 0, \\ V_2(S, \bar{J}; t) = C(S, t+\bar{J}), \\ V_2(S, l'; t) \backsim S \quad \text{as } S \to +\infty, \\ V_2(\bar{S}, l'; t) = W(t+l'), \end{cases} \tag{3.14}$$

with the connectivity condition

$$\frac{\partial V_1}{\partial S}(\bar{S}, t) = \frac{\partial V_2}{\partial S}(\bar{S}, 0; t), \tag{3.15}$$

where

$$f_n(x) = \sum_{i=1}^{n} \int_{(i-1)\bar{J}}^{i\bar{J}} W_i(s) g_1(x, n\bar{J} - s)\, ds. \tag{3.16}$$

Here, C_1 is defined on $t \in [T-(n+2)\bar{J}, T-(n+1)\bar{J}]$, $S \in [0, \bar{S}]$; C_2 is defined on $l' \in [0, \bar{J}]$, $S \in [\bar{S}, \infty)$, the parameter $t \in [T-(n+2)\bar{J}, T-(n+1)\bar{J}]$; the operator $\mathbb{L}$ is defined in (2.3).

Note that the system (3.14) and (3.15) are very similar to (3.1) and (3.2), respectively, except for the inhomogeneous initial condition in C_1; $\bar{S} f_n(\ln(S/\bar{S})) > 0$ for all $S < \bar{S}$. To nondimensionalize the systems (3.14) and (3.15), we use the same variables introduced in (3.3), except that τ and $W'(\tau)$ are replaced by $\tilde{\tau} = (T-(n+1)\bar{J}-t)\sigma^2/2 = \tau - n\bar{J}'$ and $U(\tilde{\tau})$, respectively. Dropping all primes from now on, C_1 and C_2 in (3.14) and (3.15) are transformed to C_3 and C_4, respectively, as follows:

$$C_3 = \begin{cases} \dfrac{\partial V_1}{\partial \tilde{\tau}} = \mathcal{K} V_1, \\ V_1(x, 0) = f_n(x), \\ \lim\limits_{x \to -\infty} V_1(x, \tilde{\tau}) = 0, \\ V_1(0, \tilde{\tau}) = U(\tilde{\tau}), \end{cases} \qquad C_4 = \begin{cases} \dfrac{\partial V_2}{\partial l} = \mathcal{K} V_2, \\ V_2(x, 0; \tilde{\tau}) = C(x, \tilde{\tau}), \\ V_2(x, l; \tilde{\tau}) \backsim e^x \quad \text{as } x \to +\infty, \\ V_2(0, l; \tilde{\tau}) = U(\tilde{\tau} - \bar{J} + l), \end{cases}$$

with the connectivity condition

$$\frac{\partial V_1}{\partial x}(0, \tilde{\tau}) = \frac{\partial V_2}{\partial x}(0, \bar{J}; \tilde{\tau}),$$

where $f_n(x)$ is defined in (3.16); C_3 is defined on $\tilde{\tau} \in [0, \bar{J}]$, $x \in (-\infty, 0]$; C_4 is defined on $l \in [0, \bar{J}]$, $x \in [0, \infty)$ with parameter $\tilde{\tau} \in [0, \bar{J}]$; the operator $\mathcal{K}$ is defined in (3.6).

The inhomogeneous initial condition of C_3 makes its solution procedure more complicated than that of $\mathcal{B}_1$. The solution for C_3 can be found by splitting the linear problem into two sub-problems: one with homogeneous boundary conditions

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but a nonzero initial condition and the other with a zero initial condition but an inhomogeneous boundary condition at $x = 0$. The first can be transformed to a standard heat problem on a semi-infinite domain, which has a standard solution [6], while the solution of the second problem can be obtained by applying the Laplace transform technique, as we did to solve $\mathcal{B}_1$. Without going through the lengthy derivation process, the solution of C_3 is

$$V_1(x, \tilde{\tau}) = G(x, \tilde{\tau}) + \int_0^{\tilde{\tau}} U(s) g_1(x, \tilde{\tau} - s)\, ds,$$

where

$$G(x, \tilde{\tau}) = \int_{-\infty}^{0} \frac{1}{2\sqrt{\pi \tilde{\tau}}} e^{\alpha(x-z)+\beta\tilde{\tau}} [e^{-(x-z)^2/4\tilde{\tau}} - e^{-(x+z)^2/4\tilde{\tau}}] f_n(z)\, dz.$$

Consequently, the corresponding integral equation governing $U(\tilde{\tau})$ is

$$\begin{aligned} &\frac{\partial G}{\partial x}(x, \tilde{\tau})\Big|_{x=0} + \int_0^{\tilde{\tau}} U(s) \frac{\partial g_1}{\partial x}(x, \tilde{\tau} - s)\, ds\Big|_{x=0} \\ &\quad = \frac{\partial V_2^{(2)}}{\partial x}(x, \bar{J}; \tilde{\tau})\Big|_{x=0} + \int_0^{\bar{J}} U(\tilde{\tau} - \bar{J} + s) \frac{\partial g_2}{\partial x}(x, \bar{J} - s)\, ds\Big|_{x=0}. \end{aligned} \tag{3.17}$$

Now, taking a simple coordinate transform, $\xi = \tilde{\tau} - \bar{J} + s$, in the integral on the right-hand side of the above equation yields

$$\begin{aligned} &\frac{\partial G}{\partial x}(x, \tilde{\tau})\Big|_{x=0} + \int_0^{\tilde{\tau}} U(s) \frac{\partial g_1}{\partial x}(x, \tilde{\tau} - s)\, ds\Big|_{x=0} \\ &\quad = \frac{\partial V_2^{(2)}}{\partial x}(x, \bar{J}; \tilde{\tau})\Big|_{x=0} + \int_{\tilde{\tau}-\bar{J}}^{\tilde{\tau}} U(\xi) \frac{\partial g_2}{\partial x}(x, \tilde{\tau} - \xi)\, d\xi\Big|_{x=0}. \end{aligned} \tag{3.18}$$

Let $U_0(\xi) = W_n(\xi + n\bar{J})$ for all $\xi \in [-\bar{J}, 0]$; then (3.18) reduces to

$$\begin{aligned} &\int_0^{\tilde{\tau}} U(s) \Big(\frac{\partial g_1}{\partial x} - \frac{\partial g_2}{\partial x}\Big)(x, \tilde{\tau} - s)\, ds\Big|_{x=0} \\ &\quad = \frac{\partial V_2^{(2)}}{\partial x}(x, \bar{J}; \tilde{\tau})\Big|_{x=0} + \int_{\tilde{\tau}-\bar{J}}^{0} U_0(\xi) \frac{\partial g_2}{\partial x}(x, \tilde{\tau} - \xi)\, d\xi\Big|_{x=0} - \frac{\partial G}{\partial x}(x, \tilde{\tau})\Big|_{x=0} \end{aligned} \tag{3.19}$$

and, taking the Laplace transform on both sides of (3.19) with respect to $\tilde{\tau}$,

$$\begin{aligned} \mathcal{L}[U(\tilde{\tau})] \mathcal{L}\Big[\Big(\frac{\partial g_1}{\partial x} - \frac{\partial g_2}{\partial x}\Big)(x, \tilde{\tau})\Big]\Big|_{x=0} &= \mathcal{L}\Big[\frac{\partial V_2^{(2)}}{\partial x}(x, \bar{J}; \tilde{\tau})\Big]\Big|_{x=0} - \mathcal{L}\Big[\frac{\partial G}{\partial x}(x, \tilde{\tau})\Big]\Big|_{x=0} \\ &\quad + \mathcal{L}\Big[\int_{\tilde{\tau}-\bar{J}}^{0} U_0(\xi) \frac{\partial g_2}{\partial x}(x, \tilde{\tau} - \xi)\, d\xi\Big]\Big|_{x=0}. \end{aligned}$$

Therefore,

$$\begin{aligned} \mathcal{L}[U(\tilde{\tau})] = \frac{1}{2\sqrt{p-\beta}} \Big(&\mathcal{L}\Big[\frac{\partial V_2^{(2)}}{\partial x}(x, \bar{J}; \tilde{\tau})\Big] \\ &+ \mathcal{L}\Big[\int_{\tilde{\tau}-\bar{J}}^{0} U_0(\xi) \frac{\partial g_2}{\partial x}(x, \tilde{\tau} - \xi)\, d\xi\Big] - \mathcal{L}\Big[\frac{\partial G}{\partial x}(x, \tilde{\tau})\Big]\Big)\Big|_{x=0}. \end{aligned} \tag{3.20}$$

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By taking the inverse Laplace transform on both sides of (3.20), we can obtain the solution of equation (3.17) as follows:

$$
\begin{aligned}
U(\tilde{\tau}) = &\int_{-\infty}^{0} \frac{e^{-\alpha z+\beta\tilde{\tau}}}{2\sqrt{\pi\tilde{\tau}}} e^{-z^2/4\tilde{\tau}} f_n(z)\,dz - \frac{e^{\beta\bar{J}}}{2\pi\sqrt{\bar{J}}}\int_0^{\tilde{\tau}} \frac{e^{\beta(\tilde{\tau}-s)}}{\sqrt{\tilde{\tau}-s}} U_0(s-\bar{J})\,ds \\
&+ \frac{U_0(0)}{2}e^{\beta\tilde{\tau}} + \int_0^{+\infty} \frac{z}{4\pi\bar{J}^{3/2}} e^{-(z^2/4\bar{J})+\beta\bar{J}-\alpha z} \int_0^{\tilde{\tau}} \frac{C(z,s)}{\sqrt{\tilde{\tau}-s}} e^{\beta(\tilde{\tau}-s)}\,ds\,dz \\
&- \frac{1}{\pi}\int_0^{\tilde{\tau}} \frac{e^{\beta(\tilde{\tau}-s)}}{\sqrt{\tilde{\tau}-s}} \int_{\sqrt{s}}^{\sqrt{\bar{J}}} e^{\beta t^2}[(-\beta)U_0(s-t^2) + U_0'(s-t^2)]\,dt\,ds,
\end{aligned}
$$

where $U_0(\tilde{\tau}) = W_n(\tilde{\tau} + n\bar{J})$ for all $\tilde{\tau} \in [-\bar{J}, 0]$.

Note that the inverse Laplace transform of the first term on the right-hand side of (3.20) is the same as that in the calculation of W_1, while the inverse Laplace transforms of the last two terms on the right-hand side of (3.20) are also carried out analytically; the detailed calculation can be found in Appendices A and B in [13].

Consequently, for $\tau \in [n\bar{J}, (n+1)\bar{J}]$, $n \geq 1$,

$$
\begin{aligned}
W_{n+1}(\tau) = &\int_{-\infty}^{0} \frac{e^{-\alpha z+\beta(\tau-n\bar{J})}}{2\sqrt{\pi(\tau-n\bar{J})}} e^{-z^2/4(\tau-n\bar{J})} f_n(z)\,dz - \frac{e^{\beta\bar{J}}}{2\pi\sqrt{\bar{J}}}\int_{n\bar{J}}^{\tau} \frac{e^{\beta(\tau-s)}}{\sqrt{\tau-s}} W_n(s-\bar{J})\,ds \\
&+ \frac{W_n(n\bar{J})}{2}e^{\beta(\tau-n\bar{J})} + \int_0^{+\infty} \frac{z}{4\pi\bar{J}^{3/2}} e^{(-z^2/4\bar{J})+\beta\bar{J}-\alpha z} \int_{n\bar{J}}^{\tau} \frac{C(z,s)}{\sqrt{\tau-s}} e^{\beta(\tau-s)}\,ds\,dz \\
&- \frac{1}{\pi}\int_{n\bar{J}}^{\tau} \frac{e^{\beta(\tau-s)}}{\sqrt{\tau-s}} \int_{\sqrt{s-n\bar{J}}}^{\sqrt{\bar{J}}} e^{\beta t^2}[(-\beta)W_n(s-t^2) + W_n'(s-t^2)]\,dt\,ds.
\end{aligned}
$$

Thus, we have obtained an analytical formula for Parisian up-and-in calls. This formula can be used for the valuation of American-style and European-style Parisian up-and-in calls, once C is substituted by C_A and C_E in the above formulas of V_1, V_2 and W, respectively. It should not be too difficult to calculate C_A or C_E, because the valuations of European-style vanilla options and American-style vanilla options have been thoroughly studied in the literature [1–4, 7–12, 14].

4. Numerical example and discussion

In this section, we provide an example of pricing an American-style Parisian up-and-in call. This example illustrates the implementation of our analytical solution as well as reveals some interesting features of a Parisian up-and-in call.

Note that the calculation procedure for an American-style Parisian up-and-in call option is similar to that for a European-style Parisian up-and-out call as presented by Zhu and Chen [13], except that we have replaced the values of the vanilla European option by the numerical values of its American counterpart, which are obtained by using the highly efficient integral equation method [8]. Once the value of the embedded vanilla American option is determined, the integrals in our analytical formula are

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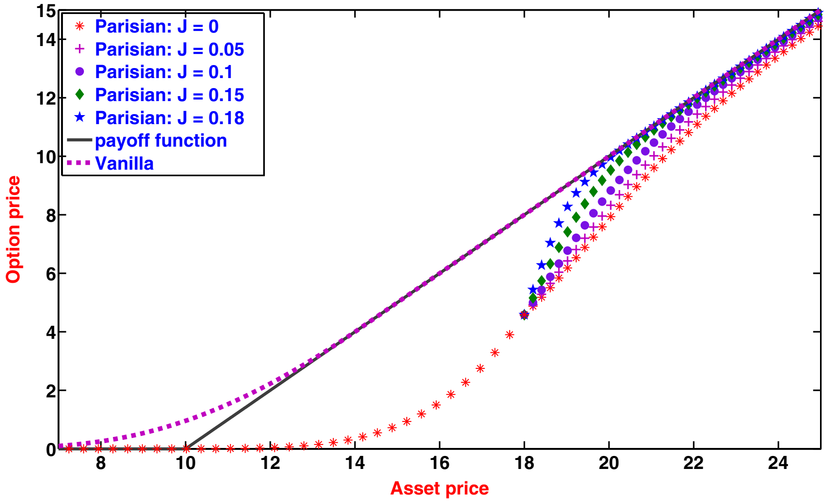

FIGURE 1. Price of an American-style up-and-in call with parameters $E = 10$, $T - t = 0.8$, $\bar{S} = 18$, $\bar{J} = 0.2$, $\sigma = 0.3$, $r = 0.05$, $D = 0.1$ (colour available online).

computed by using quadrature rules, such as Gauss–Laguerre, Gauss–Legendre or Gauss–Jacobi rules, in a very similar way as that done by Zhu and Chen [13].

Figure 1 compares the values of an American-style Parisian up-and-in call for various J values with the value of its embedded vanilla American call. The parameters used in our calculations are $E = 10$, $T - t = 0.8$, $\bar{S} = 18$, $\bar{J} = 0.2$, $\sigma = 0.3$, $r = 0.05$, $D = 0.1$. As can be seen clearly from the figure, the value of the Parisian option is always less than that of its embedded vanilla option. This makes sense financially as a holder of the Parisian up-and-in call has to wait until the knock-in feature is activated to obtain the same exercise right as the holder of the embedded vanilla option. This waiting period, with the risk that the “knock in” may never occur, would definitely devalue the Parisian up-and-in call in comparison with its embedded vanilla counterpart.

Figure 1 also reveals some interesting properties of a Parisian up-and-in call with respect to changes in S and J. Observe that when J is fixed, the Parisian call price is an increasing function of asset price. In fact, when the asset price increases, the knock-in feature is more likely to be activated and thus the value of the Parisian call increases and finally approaches the value of its embedded vanilla option. Similarly, with a fixed value of S, the knock-in feature is more likely to be activated when J gets closer to $\bar{J}$. As a result, the Parisian option price increases when J increases.

5. Conclusion

In this paper, we have derived a simple analytical formula for Parisian up-and-in calls by using the technique proposed by Zhu and Chen [13]. Unlike “knock-out” cases, the evaluation of American-style Parisian up-and-in calls is very similar to that

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of its European counterpart, and both can be handled with the same solution procedure. As a result, we have obtained a pricing formula that can be used to evaluate both American-style and European-style Parisian up-and-in calls. We have also provided an example to illustrate the implementation of our analytical solution as well as to reveal some interesting features of a Parisian up-and-in call.

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