PASCAL'S THEOREM IN n-SPACE *

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Abstract

An analogue in a solid of the well known Pascal's theorem (Baker, [1], p. 219) for a conic is established by Baker ([2], pp. 53–54, Ex. 15) after Chasles [6] and by Salmon ([2], p. 142). The same is discussed in detail by Court [8]. The purpose of this paper is to extend it to a projective space of n dimensions or briefly to an n-space S_n. To prove it, we introduce here once again the idea of a set of $n+1$ associated lines in S_n as indicated in an earlier work (Mandan, [12]) in analogy with a set of 5 associated lines in S_4 (Baker, [4], p. 122), and make use of the method of induction.

Associated spaces

1. DEFINITIONS. A *set of* $n+1$ *lines* x_i in S_n $(n > 3)$ are said to be *associated*, if through every point of every x_i there pass ∞^{n-3} $(n-2)$-spaces meeting all of them such that every $(n-2)$-space meeting n of them meets the $(n+1)$th too (cf. Coxeter & Todd, [10]).

Dually, a *set of* $n+1$ $(n-2)$*-spaces* y_i in S_n are said to be *associated*, if in every prime or S_{n-1} through every y_i there lie ∞^{n-3} lines meeting all of them such that every line meeting n of them meets the $(n+1)$th too.

Thus there are ∞^{n-2} $(n-2)$-spaces meeting all x_i, one $(n-2)$-space in each prime through each x_i, and ∞^{n-2} lines meeting all y_i, one line through each point of each y_i (cf. Baker, [4]; Mandan, [12]).

For $n = 3$, we take $\infty^0 = 1$ such that *any four generators of one system of a quadric in a solid form an associated set.*

For $n = 2$, we take *any triad of concurrent lines in a plane and any triad of collinear points to form an associated set.*

2. Now we establish by induction the following

LEMMA 1. *If through the* $n+1$ *vertices* A_i $(i = 0, \cdots, n)$ *of a simplex* S *in* S_n $(2 < n)$, $n+1$ *lines* x_i *be drawn such that* ∞^{n-3} $(n-2)$*-spaces pass through every* A_i *meeting them, then* x_i *form an associated set.*

PROOF. If P be an arbitrary point on the line x_0, then through every join PA_r $(r = 1, \cdots, n)$ there pass ∞^{n-4} $(n-2)$-spaces meeting all the

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$n+1$ lines x_i. If A'_r be the projection of A_r from P in an S_{n-1} and x'_r of x_r, the n lines x'_r pass through the n points A'_r forming a simplex in the S_{n-1} such that ∞^{n-4} $(n-3)$-spaces pass through every A'_r meeting them. If we assume the lemma to be true in the S_{n-1}, x'_r form an associated set therein such that they are met by ∞^{n-3} $(n-3)$-spaces which joined to P give us ∞^{n-3} $(n-2)$-spaces through P meeting all the lines x_i. The same is similarly true for every point of every x_i taken in place of P and hence the lemma follows if our assumption be true. But it is true for $n = 3$ by definition (§ 1) as a property of four generators of one system of a quadric in a solid. Therefore the lemma follows by induction.

3. Dually we have the following

LEMMA 2. *If in the $n+1$ primes a_i of a simplex S in S_n $(2 < n)$ $n+1$ $(n-2)$-spaces y_i be taken such that ∞^{n-3} lines lie in every a_i meeting them, y_i form an associated set.*

4. As an immediate consequence of definitions (§ 1), we have

LEMMA 3. *A prime through one of $n+1$ associated $(n-2)$-spaces in S_n meets the rest of them in n associated $(n-3)$-spaces therein.*

LEMMA 4. *Any n of $n+1$ associated lines in S_n project in a prime from an arbitrary point on the $(n+1)$th line into n associated lines therein.*

Pascal's theorem

5. The analogue of the Pascal's theorem takes the form of

THEOREM 1. *The $n(n+1)$ points of intersection of a quadric W with the $\binom{n+1}{2}$ edges of a simplex S in S_n lie, in $2^{n(n+1)/2}$ ways, in $n+1$ hyperplanes b_i, each hyperplane determined by n points on the n edges through a vertex A_i of S, which meet the $n+1$ primes a_i of S opposite its corresponding vertices A_i, in general, in $n+1$ associated $(n-2)$-spaces y_i.*

PROOF. Let W meet an edge A_iA_j of S in a pair of points B_{ij}, B_{ji} such that the n points B_{ij} on the n edges through a vertex A_i of S determine a hyperplane b_i meeting a non-corresponding prime a_j of S in an $(n-2)$-space b_{ij} $(\neq b_{ji})$. Let W_i be an $(n-2)$-quadric section of W by a_i, and let a b_{ji} meet the $(n-2)$-space a_{ij} $(= a_{ji})$ of S opposite A_iA_j in the $(n-3)$-space y_{ji} $(\neq y_{ij})$. Then, in the notations of Coxeter (1955), we have

$$\text{(i)} \qquad y_{ji} = a_{ji} \cdot b_{ji} = (a_i \cdot a_j) \cdot (a_i \cdot b_j) = a_i \cdot (a_j \cdot b_j) = a_i \cdot y_j \cdots.$$

If we assume the theorem to be true in S_{n-1}, say for W_i and the $(n-1)$-dimensional simplex of S in a_i, the n $(n-3)$-spaces y_{ji} therein are associated

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such that they and therefore, by virtue of the relation (i), the n $(n-2)$-spaces y_j are met by ∞^{n-3} lines, in a_i (§ 1), which obviously meet y_i too. For y_i lies in a_i. Hence, by the lemma 2, the theorem follows, if our assumption be true. But it is true for $n = 3$ (Baker, [2], pp. 53—54, Ex. 15) and so for all $n \geqq 3$.

Again for the determination of the $n+1$ hyperplanes b_i, either point on every edge of S has two choices independent of one another and hence there are $2^{n(n+1)/2}$ possible ways to determine them.

Thus for $n = 3$, there are 64 ways as stated by Court [8], and *not* 32 as stated by Baker ([2], p. 53), to choose the 4 planes b_i.

6. Conversely we have the following

THEOREM 2. *Any $n+1$ hyperplanes b_i through $n+1$ associated $(n-2)$-spaces y_i in the $n+1$ primes a_i of a simplex S in S_n meet its edges in $n(n+1)$ points, each b_i containing n points on its n edges through its vertex A_i opposite its corresponding prime a_i, which lie on a quadric W.*

PROOF. Let a b_i meet the n edges A_iA_j of S through A_i in the n points B_{ij}, (a_i) be the $(n-1)$-dimensional simplex of S in an a_i, and b_{ij}, b_{ji}, y_{ij}, y_{ji} as in § 5. The n $(n-3)$-spaces y_{ji} in an a_i then form an associated set by lemma 3 and lie in the n $(n-2)$-spaces b_{ji} in a_i.

If we assume the theorem to be true in an S_{n-1}, say in an a_i, the $\binom{n}{2}$ pairs of the points B_{jk}, B_{kj} $(i, j, k = 0, \cdots, n;\ i \neq j \neq k)$ on the $\binom{n}{2}$ edges A_jA_k of (a_i) lie on a quadric W_i therein. Every two such $(n-2)$-quadrics W_i, W_j obviously have a common $(n-3)$-quadric section W_{ij} by the $(n-2)$-space a_{ij} (§ 5) of S. The $n+1$ such $(n-2)$-quadrics W_i are then easily seen to lie on a quadric W in S_n as required, if our assumption be true. But it is true for $n = 3$ (Court, [8]) and so holds for all $n \geqq 3$.

7. The preceding two theorems can now be summed up as the following

THEOREM 3. *The $n(n+1)$ points, two on each edge of a simplex S in S_n, lie on a quadric, if and only if they lie, in $2^{n(n+1)/2}$ ways, in $n+1$ hyperplanes, each hyperplane containing n points on the n edges through a vertex of S, which meet the $n+1$ primes of S opposite its corresponding vertices, in general, in $n+1$ associated $(n-2)$-spaces* (cf. Mandan, [12]).

8. The $n+1$ hyperplanes b_i of theorem 2 may form a simplex, concur or have a line common (cf. Court, [8]). Thus the theorem 2 leads to

THEOREM 4. *If the primes of a simplex in S_n meet those of another in $n+1$ asscoiated $(n-2)$-spaces in a certain one-to-one correspondence, the primes of either simplex cut the non-corresponding primes of the other along $n+1$ $(n-1)$-dimensional simplexes inscribed in the same quadric W. The two such*

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simplexes are polar reciprocals of each other with respect to a quadric Q (Mandan, [12]) *in* $2^{n(n+1)/2}$ *ways.*

THEOREM 5. *The $n+1$ hyperplanes determined by $n+1$ associated $(n-2)$-spaces located in the $n+1$ primes of a simplex S in S_n and any arbitrary point or any line meeting them cut the non-corresponding primes of S along $n+1$ $(n-1)$-dimensional simplexes inscribed in the same quadric.*

Brianchon's theorem

9. Now we are in a position to state the dual of the Pascal's theorems 3—5 as the following

THEOREM 6. *The $n(n+1)$ hyperplanes, two through each $(n-2)$-space of a simplex S in S_n, envelope a quadric, if and only if they pass through, in $2^{n(n+1)/2}$ ways, $n+1$ points, each point determined by the n hyperplanes through the n $(n-2)$-spaces in a prime of S, which join the $n+1$ vertices of S opposite its corresponding primes, in general, into $n+1$ associated lines* (cf. Mandan, [12]).

THEOREM 7. *If the joins of the vertices of a simplex in S_n to those of another in a certain one-to-one correspondence form a set of $n+1$ associated lines, the vertices of either simplex join the $(n-2)$-spaces in the primes of the other opposite its corresponding vertices into $n(n+1)$ hyperplanes tangent to the same quadric W'. The two such simplexes are polar reciprocals of each other with regard to a quadric Q* (Mandan, [12]) *in* $2^{n(n+1)/2}$ *ways.*

THEOREM 8. *The $n+1$ points common to $n+1$ associated lines through the vertices of a simplex S in S_n and any arbitrary hyperplane or any $(n-2)$-space meeting them join the $(n-2)$-spaces in the primes of S opposite its corresponding vertices into $n(n+1)$ hyperplanes tangent to the same quadric.*

Special cases

10. Court [8] proves that the four lines y_i of theorem 1 in a solid (when $n = 3$) are hyperbolic (forming four generators of one system of a quadric) or coplanar. But following his arguments, we can also establish that they may lie two by two in two planes whose common line contains the two points of their intersection which obviously lie on two opposite edges of the tetrahedron S. The four planes b_i in such a case may form a tetrahedron or concur but are never coaxal. Hence corresponding to the theorems 4, 5 and their dual theorems 7, 8, we have

THEOREM 9. *If four lines lying two by two in two planes whose common line contains the two points of their intersection are defined to form a 'semi-*

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associated' set, and the four faces of a tetrahedron meet those of another, lying in the same solid, in four semi-associated lines in a certain one-to-one correspondence, the four faces of either tetrahedron cut the non-corresponding faces of the other along four triangles inscribed in the same quadric W. The two such tetrahedra are polar reciprocals of each other in regard to a quadric Q, in 64 *ways* (cf. Mandan, [12], *S-theorem*), *for which a pair of opposite edges of either tetrahedron are 'conjugate'* (Baker, [2], p. 34; Mandan, [13], [14]). *Hence the four joins of their corresponding vertices also form a set of semi-associated lines such that the vertices of either tetrahedron join the sides in the faces of the other opposite its corresponding vertices into twelve planes tangent to the same quadric W'* (Mandan, [12], *s-theorem*).

THEOREM 10. *The four planes determined by four semi-associated lines located in the four faces of a tetrahedron T and any arbitrary point in its solid cut the non-corresponding faces of T along four triangles inscribed in the same quadric W. Dually the four points common to four semi-associated lines through the vertices of T and any arbitrary plane in its solid join the sides in the faces of T opposite its corresponding vertices into twelve planes tangent to the same quadric W'* (cf. Theorems 5, 8).

11. The truth of lemma 1 is based on the assumption that it is true for $n = 3$. We may observe here that it holds for $n = 4$ even if the four lines x'_r (§ 2) in the S_3 form a semi-associated set. The vertex of projection P then happens to become a *Cremona point of the self-dual Segre's figure* 15_3 of 15 lines and 15 points (Baker, [1], p. 226; [3], pp. 113–14; Mandan, [12]) generated by the five associated lines x_i in the S_4. Thus it holds for higher values of n too.

12. Again when $n = 4$, it may happen that the four lines x'_r (§ 2) in S_3 are neither associated nor semi-associated as considered in the preceding section, but are *concurrent* (Court, [8]). The five lines x_i in the S_4 then have a common transversal t through the vertex of projection P and are no longer associated (Mandan, [12]) as defined by Baker ([3], pp. 113–14). In S_4, ∞^2 planes pass through every line therein and therefore through t too. Thus there pass through every one of the five points P_i of x_i on t, ∞^2 (not ∞^1 as required) planes meeting x_i. If this situation does not arise for any other point on any x_i, the lines x_i satisfy our definition (§ 1) for five associated lines in S_4 except for P_i and therefore may be said to form a *semi-associated set.*

Similarly we may define dually in S_4 a *set offive semi-associated planes* y_i which meet a common plane p in five lines and satisfy our definition for five associated planes therein except for the five solids py_i. Thus: *The theorems* 2, 4, 5, 7, 8 *hold good in S_4 even if the associated lines and planes therein are replaced by semi-associated ones.*

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Again the lemma 1 and its dial lemma 2 too then hold good in S_5 even if the five lines x'_r (§ 2) in S_4 are semi-associated but for one and only one point P on x_0, and therefore the theorems 1—5 also hold in S_5 even if they are true in S_4 with five semi-associated planes y_{ji} (§ 5) but in one and only one of the five primes a_i of the simplex S. Hence they and their duals (Theorems 6—8) are true in S_n for all values of n greater than five.

Remark. This shows the possibility of plane transversals of six associated lines in S_5. Thus it may form a basis for more thorough investigation of the theory of such associated lines in spaces of dimensions higher than four; these have not been much studied before except as an introduction of them by Baker [4], followed by S. Beatty [5], Coxeter and Todd [10] and lastly by Mandan [12], but then only as the joins of the corresponding vertices of a pair of simplexes polar reciprocal of each other in regard to a quadric.

13. It may happen that the situation of the preceding section repeats for a second point P'_i on x_i. The five lines x_i then lie in the solid determined by t (§ 12) and their second transversal t' through P'_i. But then the lines lie in an S_3. Therefore they must concur. Thus five associated lines in S_4 may degenerate into concurrent ones.

Now repeating the argument of the preceding section for $n = 5, 6, \cdots$ successively, we may arrive in S_n at $n+1$ lines x_i which have a common transversal t satisfying our definition (§ 1) for an associated set except for their $n+1$ points on t and therefore may be said to form a *semi-associated set.*

Dually, we may then define in S_n $(3 < n)$ a *set of* $n+1$ *semi-associated* $(n-2)$*-spaces* y_i which meet a common $(n-2)$-space p in $n+1$ $(n-3)$-spaces and satisfy our definition for $n+1$ associated $(n-2)$-spaces except for the $n+1$ primes py_i. Thus: *The theorems* 2, 4, 5, 7, 8 *hold good in* S_n $(2 < n)$ *even if the associated lines and* $(n-2)$*-spaces there are replaced by semi-associated ones* (§§ 10, 12).

14. We have remarked above (§ 5) that the Pascal's theorem 1 is true in S_n when the $n+1$ $(n-2)$-spaces y_i are *in general* associated. We then observe (§§ 10—13) that y_i may degenarate into a semi-associated set or be coprimal according as the n $(n-3)$-spaces y_{ji} (§ 5) lie in an $(n-2)$-space in one prime a_i of the simplex S or in two such primes and therefore in all its primes.

In fact, we have already deduced the Pascal's theorem 4 independently from the *Dandelin's figure in n-space* (Mandan, [18]) when y_i are coprimal in which case the $n+1$ primes b_i (§ 5) form a simplex perspective to S.

A special case arises when the pair of points B_{ij}, B_{ji} (§ 5) on each edge of S in theorem 1 coincide. It has been established in an earlier work

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(Mandan, [11]) that W becomes the *cevian quadric* of S for a point M through which its $\binom{n+1}{2}$ *bicevians* meet its edges in the $\binom{n+1}{2}$ points $B_{ij} = B_{ji}$ where therefore W touches them such that b_i form a simplex S' polar reciprocal of S for W as its *transform in the homology* $(M, m, (1-n)/2)$, m being the *polar hyperplane* of both S and W. Thus S, S' are in perspective from M such that y_i lie in their prime of perspectivity m.

15. As a limiting case of theorem 1, we have the following

THEOREM 11. *The joins of the vertices V_i of a simplex S in S_n to the corresponding ones of the simplex S' formed by the $n+1$ tangent primes at the V_i to a quadric W circumscribed to S form a set of $n+1$ associated lines x_i. Thus S and S' are polar reciprocals of each other for W, and consequently the $n+1$ $(n-2)$-spaces y_i common to their corresponding primes form an associated set* (cf. Baker, [2], p. 53, Ex. 14 for $n = 3$).

16. The lines x_i of the preceding theorem may concur (§ 14) at a point M say. The simplex S' is then perspective to S from M and therefore *anticevian* (Mandan, [11]) of S for M. Consequently the $(n-2)$-spaces y_i lie in the *polar prime* m of M for the simplex S or the quadric W. W is then seen to be the *polar quadric* (Mandan, [11]) of M for S. S becomes *isodynamic* with M, m as its *Lemoine point* and *Lemoine hyperplane*, and S' *isogonic* with M as its *Fermat point* when W is taken to be the circum-hypersphere of S or the inhypersphere of S' (Court, [8]; Mandan, [19]) as established in the author's two papers [21] and '*Polarity for a simplex*' (see *Abstract* in the Proceedings of the 48th Session of the Indian Science Congress Association held at Roorkee in January 1961).

Degenerate cases

17. The cases of degeneration multiply in S_n as n increases. For example, in S_4 the five lines x_i (§ 2), besides degenerating into a semi-associated or a corpimal set (§§ 12, 13), may lie two in a plane and three in a solid such that the three are general or concurrent, or only two of them meet (Mandan, [13], [15], [16]). Similarly we may enumerate several such cases in higher spaces too (Mandan, [17], [20]). The significant degeneration of such associated lines occurs when they have in S_6 and S_7 a plane transversal, in S_8 and S_9 a plane *or* a solid transversal, in S_{10} and S_{11} a plane, a solid or an S_4 transversal, and in S_n $(11 < n)$ a plane, a solid, $\cdots$ or an S_{m-1} transversal for $n = 2m$ and $2m+1$ (Mandan, [12]).

The discussion of the various cases like these forms the subject matter for another paper on '*Associated spaces*' (in progress) as remarked already (§ 12).

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