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A NUMERICAL CRITERION OF QUASI-ABELIAN SURFACES

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§ 1. Statement of the result

At first, we fix the notation. Let $k = \mathbf{C}$ and we shall work in the category of schemes over k. For an algebraic variety V of dimension n, we have the following numerical invariants:

$P_m(V)$ = the m-genus of V,
$q(V)$ = the irregularity of V,
$\kappa(V)$ = the Kodaira dimension of V;
$\bar{P}_m(V)$ = the logarithmic m-genus of V,
$\bar{q}(V)$ = the logarithmic irregularity of V,
$\bar{\kappa}(V)$ = the logarithmic Kodaira dimension of V.

Note that the latter three invariants have been introduced in [1], [2]. About seventy years ago, F. Enriques obtained the following numerical criterion of abelian surfaces: Let V be an algebraic surface (i.e., $n = 2$). Then V is birationally equivalent to an abelian surface if and only if $P_1(V) = P_4(V) = 1$ and $q(V) = 2$.

A slightly weaker version of this criterion is the following: V is birationally equivalent to an abelian surface if and only if $\kappa(V) = 0$, $q(V) = 2$.

Our purpose here is to prove the following numerical criterion of quasi-abelian surfaces, which is a counterpart of the Enriques criterion in proper birational geometry.

THEOREM I. *Let V be a non-singular algebraic surface. The quasi-Albanese map $\alpha_V: V \to \tilde{\mathscr{A}}_V$ is birational and there is an open subset V^0 of V such that $\alpha_V|V^0: V^0 \to \tilde{\mathscr{A}}_V - \{p_1, \cdots, p_r\}$ is proper birational, if and only if $\bar{\kappa}(V) = 0$, $\bar{q}(V) = 2$.*

We have introduced *WWPB*-equivalence in [5]. By definition,

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$\alpha_V : V \to \tilde{\mathscr{A}}_V$ is the $WWPB$-map. Thus, Theorem I is restated as follows:

THEOREM I*. *Let V be an algebraic surface. V is $WWPB$-equivalent to a quasi-abelian surface if and only if $\bar{\kappa}(V) = 0$ and $\bar{q}(V) = 2$.*

$WWPB$-equivalence seems very unnatural. However, a $WWPB$-map φ between affine normal varieties turns out to be an isomorphism. Hence if we restrict ourselves to affine normal surfaces, we obtain the following more natural

THEOREM II. *Let V be an affine normal surface. Then V is isomorphic to G_m^2 if and only if $\bar{\kappa}(V) = 0$ and $\bar{q}(V) = 2$.*

Remark. Recently, K. Ueno [9] has obtained the following numerical criterion of abelian varieties of dimension 3: Let V be an algebraic variety of dimension 3. Then V is birationally equivalent to an abelian variety of dimension 3 if and only if $\kappa(V) = 0$ and $q(V) = 3$.

We make the following

CONJECTURE. Let V be an affine normal algebraic variety of dimension n. Then V is isomorphic to G_m^n if and only if $\bar{\kappa}(V) = 0$ and $\bar{q}(V) = n$.

A partial solution of this conjecture is Theorem 12 [3], by which we prove

THEOREM III. *Let V be an algebraic variety of dimension n with $\bar{\kappa}(V) = 0$. Suppose that there is a dominant strictly rational map of V into G_m^n. Then the quasi-Albanese map $\alpha_V : V \to G_m^n$ is birational. V is $WWPB$-equivalent to G_m^n via α_V. Moreover, if V is affine and normal, α_V is an isomorphism.*

We recall the following genera. $\bar{P}_1(V)$ is called the logarithmic geometric genus and denoted by $\bar{p}_g(V)$. When $\dim V = 1$, $\bar{p}_g(V)$ coincides with $\bar{q}(V)$, which is indicated by $\bar{g}(V)$. $\bar{g}(V)$ is the logarithmic genus of the algebraic curve V. If $V = \boldsymbol{P}^1 - \{a_0, \cdots, a_m\}$, then $\bar{g}(V) = m$.

Let $\bar{V}$ be a complete non-singular algebraic variety and $\bar{D} = \sum D_j$ a reduced divisor on $\bar{V}$. We say that $\bar{D}$ is a divisor of simple normal crossing type if each D_j is non-singular and $\sum D_j$ has only normal crossings. If $\bar{D}$ is a divisor of simple normal crossing type, then we

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say that $\bar{V}$ is a completion of $V = \bar{V} - \bar{D}$ with smooth boundary. Note that $\mathrm{Reg}(\bar{D}) = \bigcup (D_i - \bigcup_{j \neq i} D_j)$, which consists of non-singular points of $\bar{D}$. By definition, letting $K(\bar{V})$ be a canonical divisor on $\bar{V}$, we have

$$\bar{P}_m(V) = \dim H^0(\bar{V}, \mathcal{O}(m(\bar{K} + \bar{D}))) \quad \text{and}$$
$$\bar{\kappa}(V) = \kappa(K(\bar{V}) + \bar{D}, \bar{V}) .$$

The main tools of this paper are the universality of quasi-Albanese map [2] and fundamental theorems on logarithmic Kodaira dimension ([1] and [3]). For instance,

1. Let $f: V_1 \to V_2$ be a dominant morphism with connected fibers. Then $\bar{\kappa}(V_1) \leqq \bar{\kappa}(f^{-1}(v)) + \dim V_2$, v being a general point.

2. Furthermore, when $\dim f^{-1}(v) = 1$, we have

$$\bar{\kappa}(f^{-1}(v)) + \bar{\kappa}(V_2) \leqq \bar{\kappa}(V_1) .$$

This is Kawamata's Theorem [7].

3. Let $f: V \to W$ be a dominant morphism with $\dim V = \dim W$. Then $\bar{\kappa}(V) \geqq \bar{\kappa}(W)$, $\bar{q}(V) \geqq \bar{q}(W)$, and $\bar{P}_m(V) \geqq \bar{P}_m(W)$.

4. Moreover, if f is proper and birational and $\bar{\kappa}(W) \geqq 0$, then for any closed set Δ, we have

$$\bar{\kappa}(V - \Delta) = \bar{\kappa}(W - f(\Delta)) .$$

This follows from Theorem 13 [3].

§ 2. Half-point attachment

Let S be a non-singular algebraic surface. There exists a completion $\bar{S}$ of S with smooth boundary $\bar{D}$. Take a non-singular point p of $\bar{D}$ and perform a monoidal transformation with center p, which we write $\mu: \bar{S}_1 = Q_p(S) \to \bar{S}$. Then $\mu^*(\bar{D}) = \mu^{-1}(\bar{D}) = \bar{D}_1 + E$, where $\bar{D}_1$ is the proper transform of $\bar{D}$ by μ. Write $S_1 = \bar{S}_1 - D_1$, which contains S as an open subset, for $\bar{S}_1 - \bar{D}_1 \supset \bar{S}_1 - \bar{D}_1 - E = \bar{S} - \bar{D} = S$. We say that S_1 is a half-point attachment to S or that S is obtained from S_1 by deleting one half-point. Then

$$K(\bar{S}_1) + \bar{D}_1 = \mu^*(K(S) + D) ,$$

where $K(\bar{S})$ denotes a canonical divisor on $\bar{S}$. Hence $\bar{P}_m(S) = \bar{P}_m(S_1)$ for any $m \geqq 1$ and $\bar{\kappa}(S) = \bar{\kappa}(S_1)$. We have $\bar{q}(S) = \bar{q}(S_1)$ or $\bar{q}(S) = \bar{q}(S_1) + 1$, according to the property of the irreducible component C_1 containing

p. In fact, let $\bar{D} = C_1 + C_2 + \cdots + C_s$ be a sum of prime divisors C_j. Then $D_1 = C_1^* + C_2 + \cdots + C_s$, C_1^* being the proper transform of C_1 by μ. Furthermore, put $S_2 = \bar{S}_1 - C_2 - \cdots - C_s = Q_p(\bar{S} - C_2 - \cdots C_s)$. Then $q(S_2) = q(\bar{S} - C_2 - \cdots - C_s) = \bar{q}(S)$ or $\bar{q}(S) - 1$. Since $S_2 \supset S_1$, if $\bar{q}(S_2) = \bar{q}(S)$, then $\bar{q}(S_1) = \bar{q}(S)$. If $\bar{q}(S_2) = \bar{q}(S) - 1$, then in view of Theorem 1 [2], there are $m_1 \neq 0$, $m_2, \cdots, m_s$ such that

$$m_1C_1 + \cdots + m_sC_s = 0 \qquad \text{in } H^2(\bar{S}, \boldsymbol{Z}) .$$

From this, it follows that

$$m_1(C_1^* + E) + \cdots + m_sC_s = 0 \qquad \text{in } H^2(\bar{S}_1, \boldsymbol{Z}) .$$

By Theorem 1 in [2], we conclude that $\bar{q}(S_1) = \bar{q}(S) - 1$. Thus we obtain

THEOREM 1. *Let S_1 be a half-point attachment to S at $P \in C_1 \subset D$ in which $\bar{D}$ is the smooth boundary of S. Then $\bar{P}_m(S_1) = \bar{P}_m(S)$, for $m = 1, 2, \cdots$. Moreover, if C_1 is cohomologically independent of C_2, $\cdots$, and C_s, then $\bar{q}(S_1) = \bar{q}(S)$. Otherwise, $\bar{q}(S_1) = \bar{q}(S) - 1$.*

Conversely, let E be a closed curve in S. If $E \approx \boldsymbol{P}^1$ and $E^2 = -1$, then E is contracted to a non-singular point. E is called an exceptional curve of the first kind in S. Furthermore, if $\bar{E}$ (the closure of E in $\bar{S}$) is an exceptional curve of the first kind and if $(\bar{E}, \bar{D}) = 1$, then E is called a $\bar{D}$-exceptional curve in S (See Sakai [8]). Contracting the $\bar{E}$ to a non-singular point, we obtain a complete surface $\bar{S}_0$ and a divisor $\bar{D}_0 = C_1' + C_2 + \cdots + C_s$, C_1' being the image of C_1. Putting $S_0 = \bar{S}_0 - \bar{D}_0$, we see that S is a half-point attachment to S_0.

Let $\mathscr{D}_j$ be the connected component of supp$(\bar{D})$ and denote by the same symbol $\mathscr{D}_j$ the reduced divisor whose support is $\mathscr{D}_j$. Then we have

$$D = \mathscr{D}_1 + \cdots + \mathscr{D}_r .$$

We assume that $\kappa(\mathscr{D}_1, \bar{S}) \geqq \cdots \geqq \kappa(\mathscr{D}_r, \bar{S})$. We have three cases.

Case a: $\kappa(\mathscr{D}_1, \bar{S}) = 2$. We use the following

PROPOSITION 1. *Let $\bar{D}$ be a reduced divisor $\sum C_j$ on $\bar{S}$. Then $\kappa(\bar{D}, S) = 2$ if and only if there exists an effective divisor $m_1C_1 + \cdots + m_sC_s$ with positive self-intersection number.*

Proof. The proof of if-part is easy. We assume that $\kappa(\bar{D}, \bar{S}) = 2$.

Then there is $m > 0$ such that $|mD| - |mD|_{\text{fix}}$ is not composite with a pencil. Writing $\mathscr{E}_m = |mD|_{\text{fix}}$ we have $|mD| = |D_m| + \mathscr{E}_m, D_m$ being the general member of $|mD| - \mathscr{E}_m$. Then $D_m^2 > 0$. Hence

$$D_m = \sum a_i C_i \in |mD| - \mathscr{E}_m . \qquad \text{Q.E.D.}$$

PROPOSITION 2. *Notations being as in Proposition* 1, *the intersection matrix* $[(C_i, C_j)]$ *is not negative semi-definite if and only if* $\kappa(\bar{D}, \bar{S}) = 2$. *If* $[(C_i, C_j)]$ *is negative semi-definite, then* $\kappa(\bar{D}, \bar{S}) \leqq 1$. *Conversely, if* $\kappa(\bar{D}, \bar{S}) = 1$, *then* $[(C_i, C_j)]$ *is negative semi-definite that has* 0 *eigen value.*

The proof is easy and omitted.

In the case a, choose $D_1 = a_1C_1 + \cdots + a_sC_s$ whose support $\subset \mathscr{D}_1$ with $a_j > 0$ and $D_1^2 > 0$ by Proposition 1. Then $(D_1, \mathscr{D}_2) = \cdots = (D_1, \mathscr{D}_s) = 0$. By the algebraic index theorem due to Hodge, we see that the intersection matrices of $\mathscr{D}_2, \cdots, \mathscr{D}_s$ are negative definite. Hence any irreducible component E in $\mathscr{D}_2 + \cdots + \mathscr{D}_s$ is cohomologically independent of $\mathscr{D}_1 + \cdots + \mathscr{D}_s - E$. Therefore, by Theorem 1, if a $\bar{D}$-exceptional curve E has a common point with $\mathscr{D}_2$, then $\bar{q}(S) = \bar{q}(S_0)$. Note that $\kappa(\mathscr{D}_2, \bar{S}) = \cdots = \kappa(\mathscr{D}_s, \bar{S}) = 0$.

Case b: $\kappa(\mathscr{D}_1, \bar{S}) = 1$. There is $t > 0$ such that

$$\kappa(\mathscr{D}_1, \bar{S}) = \cdots = \kappa(\mathscr{D}_t, \bar{S}) = 1,\ \kappa(\mathscr{D}_{t+1}, \bar{S}) = \cdots = \kappa(\mathscr{D}_s, \bar{S}) = 0 .$$

Then consider the $\mathscr{D}_1$-canonical fiber space $\psi: \bar{S} \to \Delta$. Since $\mathscr{D}_1$ is connected, $\mathscr{D}_1 = \psi^{-1}(a_1)$ for some a_1. Moreover $(\mathscr{D}_j, \mathscr{D}_1) = (\mathscr{D}_j, \psi^{-1}(u)) = 0$ for a general $u \in \Delta$. Hence $\mathscr{D}_j \leqq \psi^{-1}(a_j)$. If $j \leq t$, then $\psi^{-1}(a_j) = \mathscr{D}_j$. If $t > j$, then $\mathscr{D}_j$ is an incomplete fiber $\subsetneqq \psi^{-1}(a_j)$. In this case $\kappa(\bar{D}, \bar{S}) = 1$.

Case c: $\kappa(\mathscr{D}_1, \bar{S}) = \cdots = \kappa(\mathscr{D}_r, \bar{S}) = 0$. Then $\kappa(\bar{D}, \bar{S}) = 0$.

§ 3. Surfaces with $\bar{\kappa} = 0$ and $\bar{q} = 2$

Let S be a non-singular surface with $\bar{\kappa}(S) = 0$ and $\bar{q}(S) = 2$. Consider the quasi-Albanese map α_S of S. By B we denote the closed image of S in the quasi-Albanese variety $\tilde{\mathscr{A}}_S$ of S. We prove that $B = \tilde{\mathscr{A}}_S$. Actually if $B \neq \tilde{\mathscr{A}}_S$, then $\bar{\kappa}(B) > 0$ by Theorem 4 in [2]. Since $\tilde{\mathscr{A}}_S$ is 2-dimensional by $\bar{q}(S) = 2$, $B \neq \tilde{\mathscr{A}}_S$ implies that B is a non-singular curve by Proposition 5 and Corollary 1 in [2]. In view of Kawamata's theorem [7], we have

$$\bar{\kappa}(\alpha^{-1}(s)) + 1 \geqq \bar{\kappa}(s) = 0 \geqq \bar{\kappa}(\alpha^{-1}(b)) + \bar{\kappa}(B) \qquad \text{for a general } b \in B\,.$$

This implies that $\bar{\kappa}(B) = 0$, a contradiction. Therefore, $B = \tilde{\mathscr{A}}$. In other words, α_S is dominant. Hence $\bar{p}_g(S) = \bar{P}_2(S) = \cdots = 1$.

Case 1: $q(S) = 2$. Then $\tilde{\mathscr{A}}_S$ is an abelian surface. Let $\bar{S}$ be a completion of S with smooth boundary $\bar{D}$. $\alpha = \alpha_S$ defines a rational map $\bar{\alpha}: \bar{S} \to \mathscr{A}_S$, which turns out to be a morphism by the minimality of $\mathscr{A}_S$. Hence $0 \leqq \kappa(\bar{S}) \leqq \bar{\kappa}(S) = 0$ and so $\bar{\alpha}$ is the Albanese map of $\bar{S}$. By the classification theory of algebraic surfaces by Enriques-Kodaira, we see that $\bar{\alpha}$ is birational and hence α_S is birational. By Theorem 5 [3] (§ 1.4), we see that

$$\bar{\kappa}(S) = 0 \qquad \text{if and only if } \bar{\alpha}_*(\bar{D}) = 0\,.$$

Hence $\alpha_S(S)$ is $\mathscr{A}_S$ or a complement of a finite set of points in $\mathscr{A}_S$. Since $\bar{\alpha}(\bar{D})$ is a finite set of points $\{p_1, \cdots, p_s\}$, $\bar{D} \subset \alpha^{-1}\{p_1, \cdots, p_r\}$ and $\bar{S} - \cup\, \bar{\alpha}^{-1}(p_j) \subset S$. We can say that $\alpha = \bar{\alpha}|S: S \to \mathscr{A}$ is a *WWPB*-map (see [5]). Hence S is *WWPB*-equivalent to an abelian surface.

Case 2: $q(S) = 0$. Then $\tilde{\mathscr{A}}_S$ turns out to be an algebraic torus G_m^2. Since $G_m^2 \simeq G_m \times G_m$, we have the projection π of the product $G_m^2 \to G_m$. Then $\varphi = \pi\alpha_S: S \to G_m$ is a dominant morphism. Moreover, for a general $u \in G_m$, $\alpha_S|\pi^{-1}(u): \varphi^{-1}(u) \to G_m = \pi^{-1}(u)$ is dominant and so $\varphi^{-1}(u)$ is not complete. Consider the Stein factorization $\varphi_1: S \to \varDelta$, $\tau: \varDelta \to G_m$ of $\varphi: S \to G_m$. Applying Kawamata's Theorem [7] we obtain

$$0 = \bar{\kappa}(S) \geqq \bar{\kappa}(\varphi_1^{-1}(u)) + \bar{\kappa}(\varDelta)\,.$$

In general, we have

$$0 = \bar{\kappa}(S) \leqq \bar{\kappa}(\varphi_1^{-1}(u)) + \dim \varDelta \quad \text{and} \quad \bar{\kappa}(\varDelta) \geqq \bar{\kappa}(G_m) = 0\,.$$

From these, it follows that $\bar{\kappa}(\varDelta) = 0$ and $\bar{\kappa}(\varphi_1^{-1}(u)) = 0$ and hence $\varDelta = G_m$ and $\varphi_1^{-1}(u) = G_m$. By the universality of quasi-Albanese map, we have a morphism $\varphi_2: G_m^2 \to \varDelta = G_m$ and the commutative diagram Fig. 2. Since $\varphi_1: S \to \varDelta$ has connected fibers, φ_2 has connected fibers, too. Therefore, in view of Theorem 4 [2] and its corollary, we see that $\varphi_2: G_m^2 \to G_m$ is

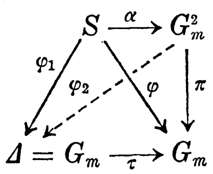

Fig. 1.

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the projection of a decomposition: $G_m^2 \rightrightarrows G_m \times G_m$. Thus we have shown that $\varphi: S \to G_m$ has connected fibers. Let $G_m \times G_m \subset \boldsymbol{P}^1 \times \boldsymbol{P}^1$ be the natural open immersion and let $\bar{\pi}$ denote the natural projection: $\boldsymbol{P}^1 \times \boldsymbol{P}^1 \to \boldsymbol{P}^1$ which is the rational map defined by π. Choosing a suitable completion $\bar{S}$ of S with smooth boundary $\bar{D}$, we have a proper morphism $\bar{\alpha}: \bar{S} \to \boldsymbol{P}^1 \times \boldsymbol{P}^1$ whose restriction to S is α_S.

We assume that α_S is proper and that $\bar{D}$ is connected. Write $\psi = \bar{\pi} \cdot \bar{\alpha}$, which is a completion of φ (Fig. 2). Denote by H the horizontal component of $\bar{D}$ with respect to ψ. Then $(\psi^*(a), H) = 2$ for any $a \in \boldsymbol{P}^1$, because $\psi^{-1}(u) - \bar{D} = \psi^{-1}(u) - H \rightrightarrows G_m$ for a general $u \in \boldsymbol{P}^1$.

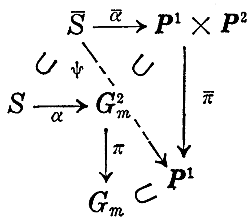

Fig. 2

We shall study singular fibers of φ.

LEMMA 1. *Let $\bar{S}$ be a completion of a non-singular surface S with connected smooth boundary $\bar{D}$. Suppose that there is a surjective morphism $\psi: \bar{S} \to \Delta$ whose general fiber $\psi^{-1}(u)$, u being a general point of Δ, is $\boldsymbol{P}^1$ and $(\bar{D}, \psi^{-1}(u)) = m$. Then any singular fiber $\psi^{-1}(a) \cap S = \sum \Gamma_j$ has the property that $\sum \bar{g}(\Gamma_j) \leqq m - 1$ where the Γ_j are irreducible components.*

Proof. Denote by $\bar{\Gamma}_j$ the closure of Γ_j in $\bar{S}$. Then $\psi^{-1}(a) = \bar{\Gamma}_1 + \cdots + \bar{\Gamma}_s + D_1 + \cdots + D_r$ is a sum of irreducible components in which $D_j \leqq \bar{D}$. Let H be the horizontal component of $\bar{D}$. Then $\mathscr{D} = D_1 + \cdots + D_r + H + \psi^{-1}(u)$ is connected. We indicate by $G(\mathscr{D})$ the (dual) graph of $\mathscr{D}$: Letting α_0 be the number of vertices of $G(\mathscr{D})$ (=the number of irreducible components of $\mathscr{D}$) and α_1 the number of edges and $h(\mathscr{D})$ the cyclotomic number of $G(\mathscr{D})$ (=the number of loops in $G(\mathscr{D})$), we have

$$\alpha_0 - \alpha_1 = 1 - h(\mathscr{D}) .$$

It is clear that $h(\mathscr{D} + \Gamma_1 + \cdots + \Gamma_s) = \bar{p}_g(\bar{S} - H - \psi^{-1}(a) - \psi^{-1}(u)) = m - 1$. Counting α_0 and α_1 of $G(\mathscr{D} + \Gamma_1 + \cdots + \Gamma_s)$, we get

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$$\alpha_0 - \alpha_1 + s - \sum(\mathscr{D}, \bar{\Gamma}_j) = 1 - (m-1) = 2 - m .$$

Moreover, by $-\sum \bar{g}(\Gamma_j) = s - \sum(\mathscr{D}, \bar{\Gamma}_j)$, we obtain

$$\sum \bar{g}(\Gamma_j) \leqq m - 1 . \qquad \text{Q.E.D.}$$

In our case m in Lemma 1 is one. Hence $\bar{g}(\Gamma_j) \leqq 1$ and $\sharp\{j\,;\, g(\Gamma_j) = 1\} \leqq 1$.

Let $a \in G_m = \boldsymbol{P}^1 - \{0, \infty\}$ and use the following notation:

$$\psi^*(a) = m_1 C_1 + \cdots + m_\sigma C_\sigma ,$$
$$\psi^{-1}(a) = C_1 + \cdots + C_\sigma ,$$
$$I = \{i \in [1, \cdots, \sigma]\,;\, C_i \subset \bar{D}\} ,$$
$$I^c = [1, \cdots, \sigma] - I .$$

We assume that $\sigma \geqq 2$. Then there is a component, say C_1, which is an exceptional curve of the first kind.

Case (i): $1 \in I$. Contracting C_1 to a non-singular point p, we have a projective surface $\bar{S}_1$ and a birational morphism $\mu: \bar{S} \to \bar{S}_1$ such that $C_1 = \mu^{-1}(p)$. We claim that

(*) $\bar{\alpha}(C_j)$ is a point, if $j \in I$.

Actually, since α is proper, letting $X = \boldsymbol{P}^1 \times \boldsymbol{P}^1 - G_m^2$, we have $\bar{\alpha}^{-1}(X) = \bar{D}$. Hence $\bar{\alpha}(C_j) \subset X \cap (\boldsymbol{P}^1 \times (a)) =$ a finite set. In particular, $\bar{\alpha}(C_1)$ is a point. Therefore, $\bar{\alpha}_1 = \bar{\alpha} \cdot \mu^{-1}: \bar{S}_1 \to \boldsymbol{P}^1 \times \boldsymbol{P}^1$ is a morphism. It is clear that $\bar{S}_1 - S$ is a divisor of simple normal crossing type. $\bar{\alpha}_1 | S = \alpha$ is proper. Hence we can replace $\bar{S}$ by $\bar{S}_1$. Repeating such contractions, we arrive at the following

Case (ii): $1 \in I^c$. Since $C_1 \not\subset D$, we know $\bar{g}(C_1 - C_1 \cap \bar{D}) \leqq 1$ by Lemma 1. Hence $(C_1, \bar{D}) = 0, 1, 2$.

Case (ii-a): $(C_1, \bar{D}) = 0$. Contracting C_1 to a non-singular point, we obtain a non-singular surface S_1 and a proper birational morphism $\mu: S \to S_1$. Since $\alpha(C_1)$ is complete in $G_m^2, \alpha(C_1)$ is a point and hence $\alpha_1 = \alpha \cdot \mu^{-1}$ is a proper morphism. Replacing S by S_1, we can assume that such C_1 does not exist.

Case (ii-b): $(C_1, \bar{D}) = 1$. Then $\Gamma_1 = C_1 - C_1 \cap \bar{D} \simeq G_a$. Hence $\alpha(\Gamma_1)$ is a point in G_m^2. In fact, if $\alpha(\Gamma_1)$ were a curve, $\bar{\kappa}(\alpha(\Gamma_1)) \leqq \bar{\kappa}(\Gamma_1) = \bar{\kappa}(G_a) = -\infty$. This contradicts the Ueno-type theorem (Theorem 4 [2]) to the effect that $\bar{\kappa}(B) \geqq 0$ if $B \subset G_m^n$. Therefore $\bar{\alpha}(\bar{\Gamma}_1) =$ a point on $X = \boldsymbol{P}^1 \times \boldsymbol{P}^1 - G_m^2$. Hence $\bar{\Gamma}_1 \leqq D = \bar{\alpha}^{-1}(X)$ for α is proper. This contradicts the assumption $1 \in I^c$. Hence the case (ii-b) does not occur.

Case (ii-c): $(\bar{C}_1, \bar{D}) = 2$. We divide the case in the following way:

Subcase I: $(H, C_1) = 2$. Since $2 = (H, \psi^*(a)) = m_1(H, C_1) + m_2(H, C_2) + \cdots$, it follows that $m_1 = 1$, $(H, C_2) = \cdots = 0$. Then, there exists an exceptional curve of the first kind, say C_2. In fact, if $C_j^2 \leqq 0$ for $j = 2, \cdots, \sigma$, then

$$-2 = (K(\bar{S}), \psi^*(a)) = (K(\bar{S}), C_1) + m_2(K(\bar{S}), C_2) + \cdots \geqq -1 .$$

This is a contradiction. By assumption, $2 \in I^c$. Moreover, by Lemma 1 we have $\bar{g}(C_2 - C_2 \cap \bar{D}) = 1$. Hence $(C_2, \bar{D}) = 0$ or 1. Thus we arrive at the case (ii-a) or (ii-b).

Subcase II: $(H, C_1) = 1$. By the same argument as in Subcase II, we have an exceptional curve of the first kind $C_2, 2 \in I^c$. Hence $(C_2, \bar{D}) = 0$ or 1.

Subcase III: $(H, C_1) = 0$. In view of $(C_1, \bar{D}) = 2$, there exist $2, 3 \in I$ satisfying that $(C_1, C_2) = (C_1, C_3) = 1$. By the logarithmic ramification formula for $\alpha : S \to G_m^2$, we obtain

$$K(\bar{S}) + \bar{D} = \bar{R}_\alpha .$$

Write $\Gamma_1 = C_1 - \bar{D} \simeq G_m$ and consider the singular fiber:

$$\varphi^{-1}(a) = \Gamma_1 + \Gamma_2 + \cdots + \Gamma_s .$$

Since $\bar{D}$ is connected, by Lemma 1, we see that

$$\Gamma_j \simeq G_a \quad \text{or} \quad \boldsymbol{P}^1 \quad \text{for } j \geqq 2 .$$

Hence $\alpha(\Gamma_j) =$ a point. This implies that $\bar{\Gamma}_j \leqq \bar{R}_\alpha$ for $j \geqq 2$. Moreover, for any $i \in I$, we infer that $C_i \leqq \bar{R}_\alpha$ from the following

LEMMA 2. *Let $f : V_1 \to V_2$ be a dominant morphism of an n-dimensional non-singular algebraic variety V_1 into another n-dimensional algebraic variety V_2. Let $\bar{V}_i$ be a completion of V_i with smooth boundary $\bar{D}_i$ for each i such that $\bar{f} : \bar{V}_1 \to \bar{V}_2$ defined by f is a morphism. Let $p \in \bar{V}_1$ and $q = \bar{f}(p)$ be closed points and choose systems of regular parameters $(z_1, \cdots, z_n)$ and $(w_1, \cdots, w_n)$ around p and q, respectively as follows: $\bar{D}_1$ is defined by $z_1 \cdots z_r = 0$ locally at p and D_2 is defined by $w_1 \cdots w_s = 0$ locally at q. Let Γ_i be a local divisor defined by $z_i = 0$ and Δ_j a local divisor defined by $w_j = 0$. Denote by W_j a local divisor defined by $w_j = 0$ for $j \geqq s + 1$. We have*

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$$\bar{f}^*(W_j) = \sum n_{ji}\Gamma_i + \textit{some effective divisor}\ .$$

Then

$$\bar{R}_f \geqq \sum_i \Big(\sum_{j=s+1} n_{ji}\Big)\Gamma_i \qquad \textit{locally at } p\ .$$

Proof. By the assumption, for $j \geqq s+1$ we have

$$w_j = \eta_j \cdot \prod z_i^{n_{ji}}\ .$$

Hence

$$\begin{aligned} dw_j &= d\eta_j \prod z_i^{n_{ji}} + \eta_j \prod z_i^{n_{ji}} n_{ji} \frac{dz_i}{z_i} \\ &= \prod z_i^{n_{ji}} \Big\{ d\eta_j + \eta_j \sum n_{ji} \frac{dz_i}{z_i} \Big\}\ . \end{aligned}$$

Therefore, combining this with (dL/L) in § 3 of [1], we obtain

$$\begin{aligned} &\frac{dw_1}{w_1} \wedge \cdots \wedge \frac{dw_s}{w_s} \wedge dw_{s+1} \wedge \cdots \wedge dw_n \\ &\quad = \prod z_i^{\sum n_{ji}} \varphi(z) \frac{dz_1}{z_1} \wedge \cdots \wedge \frac{dz_r}{z_r} \wedge dz_{r+1} \wedge \cdots \wedge dz_n\ , \end{aligned}$$

where $\varphi(z)$ is a regular function at p.

A local equation defining $\bar{R}_f$ at p is $\prod z_i^{n_{ji}} \varphi(z)$. This implies that

$$\bar{R}_f \geqq \sum_i \Big(\sum_{j=s+1} n_{ji}\Big)\Gamma_i \qquad \text{locally at } p\ . \qquad \text{Q.E.D.}$$

We claim that $\bar{R}_\alpha \geqq C_1$. Otherwise,

$$\bar{R}_\alpha = aC_2 + bC_3 + \Theta \qquad (\Theta > 0)$$

induces that

$$(\bar{R}_\alpha, C_1) = a + b + (\Theta, C_1) \geqq 2\ .$$

On the other hand,

$(\bar{R}_\alpha, C_1) = (K(\bar{S}), C_1) + (\bar{D}, C_1) = -1 + 2 = 1$. This is a contradiction. Therefore, $\bar{R}_\alpha \geqq \psi^{-1}(a)$. From this it follows that

$$\kappa(\bar{R}_\alpha, \bar{S}) \geqq \kappa(\psi^{-1}(a), \bar{S}) = \kappa(a, \boldsymbol{P}^1) = 1\ .$$

This is a contradiction. Therefore, the Subcase III does not occur.

Accordingly, after contracting exceptional curves of the first kind in $\psi^{-1}(a)$, we conclude that $\psi^*(a) = \boldsymbol{P}^1$. This implies that $\psi^{-1}(G_m)$ is a $\boldsymbol{P}^1$-bundle over G_m, which turns out to be the product $\boldsymbol{P}^1 \times G_m$. Therefore $S = \varphi^{-1}(G_m) = G_m \times G_m$. Thus we can summarize the above result as follows: If α_S is proper and $\bar{D}$ is connected, then S is obtained from G_m^2 by successive blowing ups.

Consider the general case in which α_S may not be proper. But, assume that $\bar{D}$ is connected. Using the notation at the beginning of Case (2), put $\hat{S} = \bar{\alpha}^{-1}(G_m^2)$ and $\hat{\alpha} = \bar{\alpha} | \hat{S}$. Since $S \subset \hat{S}$, it follows that $\bar{\kappa}(\hat{S}) \leqq \bar{\kappa}(S) = 0$. There is a dominant morphism $\hat{S} \to G_m^2$. Hence $\bar{P}_m(\hat{S}) \geqq 1$ and so $\bar{P}_m(\hat{S}) = 1$ for any $m \geqq 1$. Let $\hat{D} = \bar{S} - \hat{S}$ and $\mathscr{D}_1$ the connected component of $\hat{D}$ containing $H + \psi^{-1}(0) + \psi^{-1}(\infty)$. Then $\kappa(\mathscr{D}_1, \bar{S}) \geqq \kappa(H + \psi^{-1}(0) + \psi^{-1}(\infty), \bar{S}) = 2$. Hence writing $\hat{D}$ as a sum of connected components $\mathscr{D}_1, \mathscr{D}_2, \cdots, \mathscr{D}_r$, we have $\kappa(\mathscr{D}_2, \bar{S}) = \cdots = \kappa(\mathscr{D}_r, \bar{S}) = 0$. Moreover, any E is cohomologically independent of $\mathscr{D}_1, \mathscr{D}_2 - E, \cdots, \mathscr{D}_r$. Hence $\bar{q}(\bar{S} - \mathscr{D}_1) = \bar{q}(\hat{S}) = 2$. Consider the quasi-Albanese maps of the inclusion $\hat{S} \to S' = \bar{S} - \mathscr{D}_1$. First we shall prove that the quasi-Albanese map α_1 of $\hat{S}$ is $\hat{\alpha}$. Denoting by i the inclusion $S \subset \hat{S}$, we have the homomorphism $i_* : G_m^2 \to G_m^2$ such that $i_* \cdot \alpha = \alpha_1 \cdot i$ (Fig. 3).

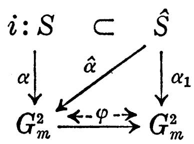

Fig. 3

By the universality of quasi-Albanese map, we have a morphism $\varphi : G_m^2 \to G_m^2$ such that $\varphi \cdot \alpha_1 = \hat{\alpha}$. Then

$$i_* \cdot \varphi \cdot \alpha_1 = i_* \cdot \hat{\alpha} = i_* \cdot \hat{\alpha} = \alpha_1 .$$

Hence $i_* \cdot \varphi = \mathrm{id}$. This implies that φ is injective. Since $\hat{\alpha}$ is dominant, φ is the étale covering. Therefore φ is an isomorphism. Hence $\alpha_1 = \hat{\alpha}$. Then denote by α' the quasi-Albanese map of $S' = \bar{S} - \mathscr{D}_1$. We have the following diagram:

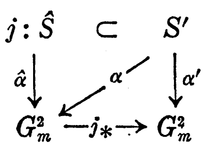

Fig. 4

Since j_* is a homomorphism and G_m^2 is an algebraic torus, j_* turns out to be the étale covering, which is proper. Recalling that $\hat{\alpha}$ is proper, we have a proper morphism $j_* \cdot \hat{\alpha} = \alpha' \cdot j$. Hence $\hat{S} = S'$. Therefore, we can conclude that $\hat{D}$ is connected.

By the previous result, $\hat{\alpha}$ is a proper birational morphism. Moreover, write $F = \hat{\alpha}(\bar{D} \cap \hat{S})$, which is a closed set. Then by Theorem 12 [3], we have

$$\bar{\kappa}(S) = \bar{\kappa}(\hat{S} - \hat{\alpha}^{-1}(F)) = \bar{\kappa}(G_m^2 - F) .$$

Hence $\bar{\kappa}(S) = 0$ implies that F is a finite set of points by Proposition 10 [2]. Then $\bar{D} \subset \bar{\alpha}^{-1}(X) \cup \hat{\alpha}^{-1}(F) = \hat{D} \cup \hat{\alpha}^{-1}(F)$. Since $\bar{D}$ is connected, this means that $F = \phi$ and $\bar{D} = \hat{D}$. Thus we establish the following

THEOREM 2. *Let S be a non-singular surface with connected smooth boundary. Suppose that $\bar{\kappa}(S) = q(S) = 0$ and $\bar{q}(S) = 2$. Then S is obtained from G_m^2 by successive blowing ups.*

We shall study the general case in which $\bar{D}$ may not be connected. Note that $\bar{D} \geqq H + \psi^{-1}(0) + \psi^{-1}(\infty)$. Since $H + \psi^{-1}(0) + \psi^{-1}(\infty)$ is connected, we denote by $\mathscr{D}_1$ the connected component of $\bar{D}$ that contains $H + \psi^{-1}(0) + \psi^{-1}(\infty)$. Note that $\kappa(H + \psi^{-1}(0) + \psi^{-1}(\infty), \bar{S}) = 2$ and so $\kappa(\mathscr{D}_1, \bar{S}) = 2$. We write $\bar{D}$ as a sum of connected divisors $\mathscr{D}_1, \mathscr{D}_2, \cdots, \mathscr{D}_s$. By the remark at the end of § 2, each intersection matrix of $\mathscr{D}_j$ ($j \geqq 2$) is negative definite. Hence $\bar{q}(\bar{S} - \mathscr{D}_1) = \bar{q}(\bar{S} - \bar{D}) = 2$. The graph $G(\mathscr{D}_1)$ contains $G(H + \psi^{-1}(0) + \psi^{-1}(\infty))$ which has one loop. Hence $\bar{p}_g(\bar{S} - \mathscr{D}_1) \geqq 1$. By the fact that $\bar{\kappa}(\bar{S} - \mathscr{D}_1) \leqq \bar{\kappa}(S) = 0$, we have $\bar{\kappa}(\bar{S} - \mathscr{D}_1) = 0$. Hence applying Theorem 2, we conclude that $\bar{S} - \mathscr{D}_1$ is obtained from G_m^2 by successive blowing ups. Since each $\mathscr{D}_j$ ($j \geqq 2$) consists of $\boldsymbol{P}^1$ in $\bar{S} - \mathscr{D}_1$, it follows that $\alpha(\mathscr{D}_j) = p_j$ a point for each $j \geqq 2$, where α is the quasi-Albanese map of $\bar{S} - \mathscr{D}_1$. Hence we have

$$S^0 = S - \bigcup \alpha^{-1}(p_j) \xrightarrow{\alpha} G_m^2 - \{p_2, \cdots, p_s\}$$

and $S^0 : S^0 \to G_m^2 - \{p_2, \cdots, p_s\}$ is a proper birational morphism.

Case 3: $q(S) = 1$. Then the Albanese map of the quasi-Albanese variety $\tilde{\mathscr{A}}_s$ is a surjective morphism $\pi : \tilde{\mathscr{A}}_s \to E, E$ being the Albanese variety of S, which is an elliptic curve. Any fiber of π is G_m and so $\varphi = \pi \cdot \alpha : S \to E$ is an algebraic fibered surface whose fibers are G_m. In fact, by the same reasoning as in the case 2, we can conclude that

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φ has connected fibers. Indicate by $\bar{Z}$ the completion of $Z = \tilde{\mathscr{A}}$ with smooth boundary Δ which was constructed in § 10 [2]. Since $\bar{Z} \to E$ is the G_m-bundle whose fibers are $\boldsymbol{P}^1$, Δ is a sum of two sections Δ_1 and Δ_2. $\bar{q}(Z) = q(Z) + 1 = 2$ implies that Δ_1 and Δ_2 have the same class in $H^2(\bar{Z}, \boldsymbol{Z})$ by Theorem 1 in [2]. We choose a completion $\bar{S}$ of S with smooth boundary $\bar{D}$ such that a rational map $\psi : \bar{S} \to E$ defined by φ and a rational map $\bar{\alpha} : \bar{S} \to \bar{Z}$ defined by α are both morphisms. Using the same argument as in the case 2, we conclude that α is birational. Moreover, letting $\mathscr{D}_i$ be the connected components of $\bar{D}$ containing D_i, we know that $\bar{D} = (\mathscr{D}_1 + \mathscr{D}_2) = \mathscr{D}_1 \cup \mathscr{D}_2$ if and only if α is proper. Therefore, if S is a non-singular surface with $\bar{\kappa}(S) = 0$, $q(S) = 1$ and $\bar{q}(S) = 2$, then the quasi-Albanese map $\alpha : S \to Z$ is dominant and satisfies the property to the effect that the composition:

$$S - \bigcup \alpha^{-1}(p_j) \hookrightarrow S \to Z - \{p_1, \cdots, p_r\}$$

is proper. Hence S is $WWPB$-equivalent to Z.

Remark. The proof of the case $q(S) = 0$ could be replaced by the much easier argument in the proof of Theorem 12 [3]. However, our proof will do for the case $q(S) = 1$.

§ 4. Proof of Theorem II

In this section by S we denote an affine normal algebraic surface with $\bar{\kappa}(S) = 0$ and $\bar{q}(S) = 2$. We use the following

LEMMA 3. *Let V be an affine normal variety and consider a completion $\bar{V}$ of $\bar{V}$. Then the algebraic boundary $\bar{D} = \bar{V} - V$ is connected, provided that* $\dim V \geqq 2$. *When $\bar{V}$ is normal and $\bar{D}$ is a reduced divisor, $\kappa(\bar{D}, \bar{V})$ is equal to* $\dim V$.

The proof follows from the connectedness principle. Q.E.D.

Let $\mu : S^* \to S$ be a non-singular model and let S^* be a completion of S^* with smooth boundary D^*. Then D^* is connected and $\kappa(D^*, \bar{S}^*) = 2$. Hence $q(S) \leqq 1$, and so the quasi-Albanese map $\alpha^* : S^* \to \tilde{\mathscr{A}}_S$ is proper and birational. Hence $\alpha = \alpha_S : S \xrightarrow{\mu^{-1}} S^* \to \tilde{\mathscr{A}}_S$ is also a proper birational map. If $q(S) = 0$, then $\tilde{\mathscr{A}}_S = G_m^2$ is affine. By Lemma 1 [3], α_S turns out to be an isomorphism. Hence $S \rightrightarrows G_m^2$. If $q(S) = 1$, $\mathscr{A}_S = Z$ is a G_m-bundle over E. From $\kappa(D^*, \bar{S}^*) = 2$, it follows that

$\kappa(\Delta_1 + \Delta_2, \bar{Z}) = 2$. Since Δ_1 is cohomologous to Δ_2, we have $\Delta_1^2 = (\Delta_1, \Delta_2) > 0$ for $\kappa(\Delta_1 + \Delta_2, \bar{Z}) = 2$. Hence Δ_1 and Δ_2 are both ample and so $\Delta_1 + \Delta_2$ is ample. This implies that $Z = \bar{Z} - (\Delta_1 + \Delta_2)$ is an affine surface. Thus Z is a quasi-abelian surface which is an affine algebraic group. This is a contradiction.

EXAMPLE. Let $\bar{Z} = \boldsymbol{P}^1 \times E$ and $\varphi: E \to \boldsymbol{P}^1$ a rational function. Then the graph Γ_φ has the following property:

$\Gamma_\varphi^2 = 2\cdot\deg\varphi$, $\deg\varphi = [k(E): k(\boldsymbol{P}^1)]$ and if $\deg\varphi > 0$, then $Z = \bar{Z} - \Gamma_\varphi$ is affine and $\bar{\kappa}(Z) = -\infty$, $\bar{q}(Z) = 0$. Put $S = \bar{Z} - (\Gamma_\varphi + \Gamma_\psi)$, $\varphi \neq \psi$. Then S is affine and $\bar{q}(S) \geqq 1$ and $\bar{\kappa}(S) \geqq 0$. Moreover

$\bar{q}(S) = 2$ if and only if $\deg\varphi = \deg\psi$,

$\bar{\kappa}(S) = 0$ if and only if φ and ψ are constants and hence, $S = E \times G_m$.

§ 5. Surfaces with $\bar{\kappa}(S) = 0$, $\bar{q}(S) = 1$

Let S be a non-singular surface with $\bar{\kappa}(S) = 0$ and $\bar{q}(S) = 1$. The quasi-Albanese variety $Y = \tilde{\mathscr{A}}_S$ is an elliptic curve or G_m according to $q(S) = 1$ or 0. Then quasi-Albanese map $\alpha: S \to Y$ has connected fibers. Let $C_u = \alpha^{-1}(u)$ be a general fiber. Then by Kawamata's theorem,

$$0 = \bar{\kappa}(S) \geqq \bar{\kappa}(C_u) + \bar{\kappa}(Y) \geqq \bar{\kappa}(C_u) .$$

Hence $\bar{\kappa}(C_u) = 0$. However, $\kappa(Y) = 0$, $\kappa(C_u) = -\infty$ do not hold at the same time. Moreover, if S is affine, then $Y = G_m$ and $C_u = G_m$.

EXAMPLE. Let $S = \operatorname{Spec} \boldsymbol{C}[x, y, 1/F]$, $F = x^m y - 1$. Then $\bar{P}_1(S) = \bar{P}_2(S) = \cdots = 1$, $\bar{\kappa}(S) = 0$ and $\bar{q}(F) = 1$.

§ 6. Surfaces with $\bar{\kappa}(S) = -\infty$ and $\bar{q}(S) \geqq 1$

Let S be a non-singular surface with $\bar{\kappa}(S) = -\infty$ and $\bar{q}(S) \geqq 1$. Consider the quasi-Albanese map $\alpha: S \to Y = \tilde{\mathscr{A}}_S$. By Kawamata's theorem, a general fiber C_u is of elliptic type, that is, $C_u = \boldsymbol{P}^1$ or G_a.

THEOREM 3. *Let* $F \in \boldsymbol{C}[x, y]$ *and* $S = \operatorname{Spec} \boldsymbol{C}[x, y, 1/F]$. *Assume that* $\bar{\kappa}(S) = -\infty$. *Then there are new variables* $u, v \in \boldsymbol{C}[x, y]$ *such that* $\boldsymbol{C}[x, y] = \boldsymbol{C}[u, v]$, $F = F_0(u) \in \boldsymbol{C}[u]$.

Proof. Let R be the integral closure of $\boldsymbol{C}[F]$ in $\boldsymbol{C}[x, y]$. Then R is normal and $\bar{g}(\operatorname{Spec} R) \leqq \bar{q}(A^2) = 0$. Hence $\operatorname{Spec}(R) = G_a$, in other words, $R = \boldsymbol{C}[f]$ such that $f - \lambda$ is irreducible for a general λ. Since

$F \in \boldsymbol{C}[F] \subset R = \boldsymbol{C}[f]$, F is a polynomial of f and $f: \boldsymbol{A}^2 \to \boldsymbol{A}^1$ is the Stein factorization of $F: \boldsymbol{A}^2 \to \boldsymbol{A}^1$. Write $F = a_0 \prod (f - a_j)^{e_j}$, $e_j > 0$. Then $V(F) = V(f - a_1) \cup \cdots \cup V(f - a_S)$. Hence $\bar{\kappa}(\boldsymbol{A}^2 - V(f - a_1)) \leqq \bar{\kappa}(\boldsymbol{A}^2 - V(F)) = -\infty$. Applying Kawamata's theorem to $f - a_1: \boldsymbol{A}^2 - V(f - a_1) \to \boldsymbol{C}^*$, we have for general λ, $V(f - \lambda) \cong \boldsymbol{G}_a$. Hence by Jung-Gutwirth-Nagata's pencil theorem, there are new variables $u, v \in \boldsymbol{C}[x, y]$ such that $\boldsymbol{C}[x, y] = \boldsymbol{C}[u, v]$ and $f - a_1 = u$. Q.E.D.

COROLLARY 1. *If* $\dim \operatorname{Aut} \boldsymbol{C}[x, y, 1/F] \geqq 3$, *then* $F = F_0(u)$ *as in the theorem above. If* $\dim \operatorname{Aut} \boldsymbol{C}[x, y, 1/F] = 2$, *then* $\boldsymbol{C}[x, y, 1/F] = \boldsymbol{C}[u, v, u^{-1}, v^{-1}]$.

Proof. If $\dim \operatorname{Aut} \boldsymbol{C}[x, y, 1/F] \geqq 3$, then by Theorem 7 [1], we conclude that $\bar{\kappa}(\boldsymbol{A}^2 - V(F)) = -\infty$. Then, apply Theorem 3. Note that $\operatorname{Aut} \boldsymbol{C}[x, y, 1/(\Pi(x - a_j))]$ contains T such that $Tx = x$, $Ty = y + \alpha_0 + \alpha_1 x + \cdots + \alpha_d x^d$, α_i belonging to $\boldsymbol{C}$. Hence $\dim \operatorname{Aut} \boldsymbol{C}[x, y, 1/\Pi(x - a_j)] = \infty$. The assumption $\dim \operatorname{Aut} \boldsymbol{C}[x, y, 1/F] = 2$ implies that $\bar{\kappa}(\operatorname{Spec} \boldsymbol{C}[x, y, 1/F]) \geqq 0$. Hence by Theorem 6 [1], we conclude that $\operatorname{Spec} \boldsymbol{C}[x, y, 1/F] = G_m^2$.

COROLLARY 2. *Let* $R_0 = \boldsymbol{C}[x, y, 1/F]$ *and* R_1, R_2 *be integral domains which are finitely generated over* $\boldsymbol{C}$. *Then we have two cases: Case* 1. *Any* $\boldsymbol{C}$*-isomorphism* $\Phi: R_0 \otimes R_2 \cong R_1 \otimes R_2$ *induces the isomorphism* $\varphi: R_0 \cong R_1$ *such that* $\Phi = \varphi \otimes 1$. *Case* 2. $R_0 \cong \boldsymbol{C}[u, 1/f(u)]\,[v]$. *In this case, let* $R_1 = R_0$ *and* $R_2 = \boldsymbol{C}[w]$. *Define* Φ *by* $\Phi(v) = v + w$, $\Phi(u) = u$, $\Phi(w) = w$. *Then* Φ *does not induce* φ *as in case* 1.

Proof. Combining Theorem 1 in [6] with Theorem 3, we are through.

Note that the corollary is an affirmative solution of the conjecture in [6].

THEOREM 4. *Let* $R_0 = \boldsymbol{C}[x, y, x^{-1}, y^{-1}]$ *which is* $\Gamma(G_m^2, \mathcal{O})$ *and let* R_1 *and* R_2 *be integral domains that are finitely generated over* $\boldsymbol{C}$. *Assume that* $\Phi: R_0 \otimes R_2 \cong R_1 \otimes R_2$. *Then* $R_0 \cong R_1$.

Proof. Let $V_1 = \operatorname{Spec} R_1$. Then by the isomorphism Φ, we have $\bar{\kappa}(V_1) = 0$ and $\bar{q}(V_1) = 2$. Hence the normalization of V_1 is G_m^2 by Theorem II. Counting the irreducible components of the singular set:

$$\operatorname{Sing}(V_0 \times \operatorname{Spec} R_2) = V_0 \times \operatorname{Sing}(\operatorname{Spec} R_2)$$
$$\cong \operatorname{Sing}(V_1 \times \operatorname{Spec} R_2) = V_1 \times \operatorname{Sing}(\operatorname{Spec} R_2) \cup \operatorname{Sing}(V_1) \times \operatorname{Spec} R_2$$

we have $\operatorname{Sing}(V_1) = \phi$. Hence $V_1 = G_m^2$. Q.E.D.

§ 7. Polynomials $\varphi(x, y)$

Let $\varphi \in \boldsymbol{C}[x, y] - \boldsymbol{C}$ and let $S = D(\varphi) = \boldsymbol{A}^2 - V(\varphi)$. If $\bar{\kappa}(S) = 1$, then there is a surjective morphism $f: S \to \Delta, \Delta$ being a rational curve, for $g(\Delta) \leqq q(S) = 0$. Hence $f = \psi/\varphi^d$ for some $\psi \in \boldsymbol{C}[x, y]$. Moreover, for a general λ, $V(\psi - \lambda\varphi^d) - V(\varphi) \simeq G_m$. Such φ is called a G_m-polynomial, which will be studied in a forthcoming paper. We have the following table:

TABLE

$\bar{\kappa}(D(\varphi))$	$\bar{q}(D(\varphi))$	φ	$S=D(\varphi)=\boldsymbol{A}^2-V(\varphi)$
$-\infty$	$\geqq 1$	$\varphi=\varphi_0(u)$	$S=\boldsymbol{A}^1\times C$
0	1	for example $\varphi=xy^m-1$	$f: S\to G_m$, general fiber being G_m
	2	$\varphi=u^r v^s$	$S=G_m^2$
1	$\geqq 1$	G_m-polynomial	$f: S\to\Delta$, general fiber being G_m
2	$\geqq 1$	polynomial of hyperbolic type	hyperbolic type

Referring to the following result by Sakai:
Theorem (Sakai [8]). If $\bar{\kappa}(V) = \dim V$, then V is measure-hyperbolic, we obtain the Brody-type Theorem:

THEOREM 5. *$D(\varphi)$ is measure-hyperbolic if and only if $\bar{\kappa}(D(\varphi)) = 2$, that is, $D(\varphi)$ is of hyperbolic type.*

Remark. In order to generalize the theorem above, we have to study the following surfaces.

A. Surfaces with $\bar{\kappa}(S) = -\infty$, $\bar{q}(S) = 0$. These might be called logarithmic rational surfaces.

B. Surfaces with $\bar{\kappa}(S) = 0$, $\bar{q}(S) = 0$. These might be called logarithmic $K3$ surfaces.

After the completion of this paper, Kawamata succeeded in generalizing our Theorem I* and obtained Theorem 5 ([7]). His proof is quite different from ours.

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