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OVOIDS AND TRANSLATION PLANES

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1. Introduction. An *ovoid* in an orthogonal vector space V of type $\Omega^+(2n, q)$ or $\Omega(2n - 1, q)$ is a set Ω of $q^{n-1} + 1$ pairwise non-perpendicular singular points. Ovoids probably do not exist when $n > 4$ (cf. [**12**], [**6**]) and seem to be rare when $n = 4$. On the other hand, when $n = 3$ they correspond to affine translation planes of order q^2, via the Klein correspondence between $PG(3, q)$ and the $\Omega^+(6, q)$ quadric.

In this paper we will describe examples having $n = 3$ or 4. Those with $n = 4$ arise from $PG(2, q^2)$, $AG(2, q^3)$, or the Ree groups. Since each example with $n = 4$ produces at least one with $n = 3$, we are led to new translation planes of order q^2.

Some of the resulting translation planes are semifield planes; others seem to have somewhat small collineation groups. Some of the most interesting planes have the following properties:

If $q \equiv 2 \pmod 3$ and $q > 2$, there is a translation plane of order q^2 admitting an abelian collineation group $\mathbf{P}$ of order q^2 which fixes an affine point, has orbit lengths 1 and q^2 on the line at infinity, and contains exactly q elations; moreover, $\mathbf{P}$ is elementary abelian if q is odd, but is the direct product of cyclic groups of order 4 if q is even (cf. (4.5)). Another noteworthy example we will discuss is a nondesarguesian plane of order 8^2 admitting $\mathbf{Z}_7 \times SL(2, 4)$ as an irreducible collineation group (cf. (8.2)).

The ovoids with $n = 4$ are related, by triality, to orthogonal spreads. A number of such orthogonal spreads were discussed in [**4, 5**], and were used to construct translation planes of order q^3 when q is even. The latter planes arise from 6-dimensional symplectic spreads. Other characteristic 2 symplectic spreads occur in [**3, 4**]. Here, we will construct 4-dimensional symplectic nondesarguesian spreads over all fields of odd non-prime order (cf. (5.2)).

2. Background. A *spread* of a $2n$-dimensional $GF(q)$-space V is a family Σ of $q^n + 1$ subspaces of dimension n, any two of which span V. The corresponding translation plane $\mathbf{A}(\Sigma)$ of order q^n has V as its set of points and the cosets of the members of Σ as its lines (cf. [**9**]).

A *symplectic spread* is a spread Σ such that, for some symplectic geometry on V, Σ consists of totally isotropic n-spaces.

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An $\Omega^+(2n, q)$ space V is a $2n$-dimensional $GF(q)$-space equipped with a quadratic form such that totally singular n-spaces exist. (Thus, if V is $GF(q)^{2n}$ then the quadratic form is equivalent to the form $\sum_{i=1}^{n} x_i x_{n+i}$.) There are then two classes of totally singular n-spaces, two subspaces belonging to the same class if and only if the dimension of their intersection has the same parity as n.

Ovoids were defined in Section 1. Note that an ovoid in an $\Omega(2n-1, q)$ space is also an ovoid in an $\Omega^+(2n, q)$ space of which that space is a hyperplane. Also, an $\Omega^+(2n, q)$ space cannot contain more than $q^{n-1} + 1$ pairwise non-perpendicular singular points: ovoids are extremal with this property (see [**12**]).

If Ω is an ovoid of an $\Omega^+(2n, q)$ space, a count shows that every totally singular n-space contains a member of Ω. If x is any singular point not in Ω, then $x^\perp \cap \Omega$ projects onto an ovoid of $x^\perp/x$. Thus, $\Omega^+(8, q)$ ovoids produce $\Omega^+(6, q)$ ovoids. Similarly, $\Omega(7, q)$ ovoids produce $\Omega(5, q)$ ovoids in the same manner.

The Klein correspondence represents $PG(3, q)$ in an $\Omega^+(6, q)$ space, sending lines to singular points and sending points and planes to totally singular 3-spaces. The points of a line L of $PG(3, q)$ are sent to the 3-spaces of one class which contain the corresponding singular point x; the planes containing L are sent to the remaining totally singular 3-spaces containing x. A spread of a 4-dimensional $GF(q)$-space is sent to an ovoid of the $\Omega^+(6, q)$ space. Similarly, a 4-dimensional symplectic spread produces an $\Omega(5, q)$ ovoid. If Ω is an ovoid of an $\Omega(5, q)$ or $\Omega^+(6, q)$ space, let $\mathbf{A}(\Omega)$ denote the corresponding translation plane of order q^2. The plane $\mathbf{A}(\Omega)$ is desarguesian if and only if $\dim \langle\Omega\rangle = 4$; in this case, $\langle\Omega\rangle$ is an $\Omega^-(4, q)$ space, and Ω consists of all its singular points.

Under the Klein correspondence,

(2.1) $\langle 1, a, b, c, d, -ad - bc\rangle \leftrightarrow \langle (1, 0, c, -d), (0, 1, a, b)\rangle$.

Let Ω be an $\Omega^+(6, q)$ ovoid, and set $G = P\Gamma O^+(6, q)_\Omega$. If y is a singular point not in Ω, then G_y may not act on $\mathbf{A}(y^\perp \cap \Omega)$. For, G_y may induce both collineations and correlations of $PG(3, q)$. However, its subgroup of index at most 2 inducing collineations does, indeed, act on $\mathbf{A}(y^\perp \cap \Omega)$.

The triality principle in a sense generalizes the Klein correspondence. Let $\mathbf{P}$ denote the set of singular points of an $\Omega^+(8, q)$ space V, let $\mathbf{M}_1$ and $\mathbf{M}_2$ be the two classes of totally singular 4-spaces of V, and let $\mathbf{L}$ be the set of totally singular 2-spaces of V. A *triality map* is a mapping τ sending $\mathbf{L} \to \mathbf{L}$ and $\mathbf{P} \to \mathbf{M}_1 \to \mathbf{M}_2 \to \mathbf{P}$ which preserves incidence between members of $\mathbf{L}$ and members of $\mathbf{P} \cup \mathbf{M}_1 \cup \mathbf{M}_2$ ([**13**]). Here, τ induces an outer automorphism of the projective orthogonal group $P\Omega^+(8, q)$; this automorphism will also be called τ. If Ω is an ovoid of V then Ω^τ is an orthogonal spread: a family of $q^3 + 1$ totally singular 4-spaces partitioning the $(q^3 + 1)(q^4 - 1)/(q - 1)$ singular points of V. (Note that an orthogonal spread is not a spread as defined at the beginning of this section: any two members span V, but there are only $q^3 + 1$ members instead of $q^4 + 1$.) Conversely, if Σ is an orthogonal spread of V and $\Sigma \subset \mathbf{M}_1$, then $\Sigma^{\tau^{-1}}$ is an ovoid of V. Consequently, the orthogonal spreads described in [**4, 5**] can be used here. Moreover, if $x \in \mathbf{P} - \Omega$, the ovoid in $x^\perp/x$ produced by $x^\perp \cap \Omega$ corresponds, under τ, to the spread

$$\{x^\tau \cap M | M \in \Omega^\tau, x^\tau \cap M \neq 0\}$$

of the 4-space x^τ. We will call the resulting translation plane $\mathbf{A}(x^\perp \cap \Omega)$.

3. $\Omega^+(8, q)$ **ovoids when** $q \leqq 3$**.** There are unique $\Omega^+(8, q)$ ovoids when $q \leqq 3$ ([**11**], [**4**]). While they exhibit exceptional behavior, they also provide simple illustrative examples. Our discussion follows [**7**, § 2D].

Example 1. Let $e_1, \ldots, e_9$ be the standard basis for $V = GF(2)^9$. Define a quadratic form Q on V by requiring that $Q(e_i) = 0$ and $(e_i, e_j) = 1$ for $i \neq j$. The radical of V is $\langle r \rangle = \langle \Sigma e_i \rangle$. Set $\bar{e}_i = e_i + \langle r \rangle$. Then $\Omega = \{\langle \bar{e}_i \rangle | 1 \leqq i \leqq 9\}$ is an ovoid in the $\Omega^+(8, 2)$ space $V/\langle r \rangle$, whose stabilizer in $O^+(8, 2)$ is S_9. Moreover, S_9 has exactly two orbits of singular points. If $x = \langle \bar{e}_1 + \bar{e}_2 + \bar{e}_3 + \bar{e}_4 \rangle$ then $\mathbf{A}(x^\perp \cap \Omega)$ is the desarguesian plane of order 4, and S_5 is induced on the plane by $(S_9)_x$.

Example 2. Let $e_1, \ldots, e_8$ be the standard basis of $V = GF(3)^8$, and define Q by requiring that $Q(e_i) = 1$ and $(e_i, e_j) = 0$ for $i \neq j$. This turns V into an $\Omega^+(8, 3)$ space. Let Ω consist of the points $\langle e_i + e_7 + e_8 \rangle$ with $i \leqq 6$, $\langle -e_i + e_7 + e_8 \rangle$ with $i \leqq 6$,and $\langle \sum_{i=1}^6 \epsilon_i e_i \rangle$ with $\epsilon_i \in GF(3)$ and $\prod_{i=1}^6 \epsilon_i = 1$. Then Ω is an ovoid lying in $H = \langle e_7 - e_8 \rangle^\perp$, and the Weyl group W of type E_7 acts 2-transitively on Ω [**7**, § 2D]. Moreover, W has exactly 2 orbits of singular points x of H. If $v = e_1 + e_2 + e_3$ and $x = \langle v \rangle$, then $W_v = S_6 \times S_3$ induces $PSL(2, 9) \cdot \mathbf{Z}_2$ on $x^\perp \cap \Omega$. It is easy to check that dim $\langle x, x^\perp \cap \Omega \rangle / x = 4$, so that $\mathbf{A}(x^\perp \cap \Omega)$ is desarguesian.

Similarly, W is transitive on the singular points not in H. Each such point has the form $\langle n + e_7 - e_8 \rangle$ with $n \in H$ and $Q(n) = 1$. Thus, we must consider the ovoid $n^\perp \cap \Omega$ of $n^\perp \cap H$. If $n = e_6$ then $n^\perp \cap \Omega$ consists of the points $\langle e_i + e_7 + e_8 \rangle$ and $\langle -e_i + e_7 + e_8 \rangle$ with $i \leqq 5$, and hence spans $n^\perp \cap H$. Thus, $\mathbf{A}(n^\perp \cap \Omega)$ is the nearfield plane of order 9, and its canonical involution on L_∞ is evident (cf. [**2**, p. 232]). The group $\mathbf{Z}_2{}^4 \rtimes S_5$ acting on L_∞ is equally visible.

These ovoids will reappear in later sections.

4. Unitary ovoids. An $\Omega^+(8, q)$ ovoid associated with the unitary group $PGU(3, q)$ when $q \equiv 0$ or 2 (mod 3) was studied in [**4**, § 6]. In this section, we will describe an equivalent ovoid, obtained by changing coordinates in order to simplify calculations.

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Let q be a power of a prime p. Set $K = GF(q)$ and $L = GF(q^2)$. If $\alpha \in L$ set $\bar{\alpha} = \alpha^q$, $T(\alpha) = \alpha + \bar{\alpha}$ and $N(\alpha) = \alpha\bar{\alpha}$. If $p \neq 3$ let $\omega^3 = 1 \neq \omega$.

If $M = (\mu_{ij})$ is a 3×3 matrix over L, set $\operatorname{tr}(M) = \Sigma\, \mu_{ii}$, $\bar{M} = (\bar{\mu}_{ij})$ and $M^t = (\mu_{ji})$. Set

$$J = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

Let V be the K-space of those matrices M such that $\operatorname{tr}(M) = 0$ and $J^{-1}MJ = \bar{M}^t$. Then $\dim V = 8$. Write

$$Q(M) = -\sum_{i<j} \mu_{ii}\mu_{jj} + \sum_{i<j} \mu_{ij}\mu_{ji}.$$

Then Q is a quadratic form on V, with associated bilinear form

$$Q(M + N) - Q(M) - Q(N) = \operatorname{tr}(MN).$$

Explicitly, V consists of the matrices

$$(4.1) \quad M = \begin{pmatrix} \alpha & \beta & c \\ \gamma & a & \bar{\beta} \\ b & \bar{\gamma} & \bar{\alpha} \end{pmatrix} \text{ with } \alpha, \beta, \gamma \in L;\ a, b, c \in K; \text{ and } a + T(\alpha) = 0,$$

and Q is defined by

$$(4.2) \quad Q(M) = \alpha^2 + \alpha\bar{\alpha} + \bar{\alpha}^2 + T(\beta\gamma) + bc.$$

Thus, if $p = 3$ then $\operatorname{rad} V = \langle I \rangle$. Moreover, V is an $\Omega^+(8, q)$ space if and only if $q \equiv 2 \pmod 3$ ([**3**, (6)]). In this section, we will always assume that $q \equiv 0$ or $2 \pmod 3$.

Let G denote the unitary group $GU(3, q)$ of all invertible 3×3 matrices A over L such that $J^{-1}AJ = (\bar{A}^t)^{-1}$. Then G acts on V by conjugation, inducing $PGU(3, q)$ there. Moreover, G preserves Q [**4**, (6.2)]. Note that G preserves the form $(\rho, \sigma, \tau) \to T(\rho\bar{\tau}) + N(\sigma)$ on L^3.

Transvections in G have the form $I + Y$ with $Y^2 = 0$. Here,

$$I + J^{-1}YJ = (I + \bar{Y}^t)^{-1} = \overline{I - Y^t}.$$

Let $\bar{\theta} = -\theta$. Then $X = \theta Y \in V$. Thus,

$$\Omega = \{\langle X \rangle | 0 \neq X \in V, X^2 = 0\}$$

consists of $q^3 + 1$ singular points, permuted by G in its natural 2-transitive permutation representation. No two members of Ω are perpendicular: Ω is an ovoid if $p \neq 3$, and projects onto an ovoid of $V/\langle I \rangle$ if $p = 3$ [**4**, (6.12)].

This ovoid can be described explicitly, as follows. If $v = (\rho, \sigma, \tau) \neq 0$ and $T(\rho\bar{\tau}) + N(\sigma) = 0$, then $\bar{v}^t vJ$ lies in V and has square 0. This produces all $(q^3 + 1)(q - 1)$ nonzero matrices appearing in the definition of Ω.

Set $X_\infty = (1 \quad 0 \quad 0)^t(1 \quad 0 \quad 0)J$ and

$$X[\rho, \sigma] = \begin{pmatrix} \bar\rho \\ \bar\sigma \\ 1 \end{pmatrix} (\rho \quad \sigma \quad 1)J$$

$$= \begin{pmatrix} \bar\rho & \bar\rho\sigma & N(\rho) \\ \bar\sigma & N(\sigma) & \rho\bar\sigma \\ 1 & \sigma & \rho \end{pmatrix} \text{ whenever } T(\rho) + N(\sigma) = 0.$$

Then

(4.3) $\Omega = \{\langle X_\infty\rangle, \langle X[\rho, \sigma]\rangle | T(\rho) + N(\sigma) = 0\}.$

The stabilizer of $\langle X_\infty\rangle$ in G has a Sylow p-subgroup U of order q^3, consisting of the matrices

$$U[\lambda, \mu] = \begin{pmatrix} 1 & -\bar\mu & \lambda \\ 0 & 1 & \mu \\ 0 & 0 & 1 \end{pmatrix} \text{ with } T(\lambda) + N(\mu) = 0.$$

(Note that $U[\lambda, \mu]U[\sigma, \tau] = U[\lambda + \sigma - \bar\mu\tau, \mu + \tau]$.) Moreover U is transitive on $\Omega - \{\langle X_\infty\rangle\}$.

If $\phi \in L^*$ set $D(\phi) = \operatorname{diag}(\phi, 1, \bar\phi^{-1})$. Then $D(\phi) \in G$, $D(\phi)$ fixes $\langle x_\infty\rangle$ and $\langle X[0, 0]\rangle$, and

(4.4) $D(\phi)^{-1}X[\phi, \sigma]D(\phi) = X[\rho N(\phi)^{-1}, \sigma\phi^{-1}]N(\phi).$

We are now in a position to consider the translation planes determined by Ω.

Set

$$Y = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

Then $Y \in V$, $Q(Y) = 0$ and $X_\infty Y = YX_\infty = 0$. By (4.3),

$$Y^\perp \cap \Omega = \{\langle X_\infty\rangle, \langle X[\rho, \sigma]\rangle | T(\rho) + N(\sigma) = 0, T(\sigma) = 0\}.$$

Also,

$$U_Y = \{U[\lambda, \mu] | T(\lambda) + N(\mu) = 0, T(\mu) = 0\}.$$

THEOREM 4.5. *Let* $q \equiv 0$ *or* $2 \pmod 3$ *and* $q > 3$. *Set* $\mathbf{A} = \mathbf{A}(Y^\perp \cap \Omega)$. *Then the following hold.*

(i) $\mathbf{A}$ *is a nondesarguesian translation plane of order* q^2.

(ii) Aut $\mathbf{A}$ *fixes a point* x_∞ *at infinity.*

(iii) U_Y *induces an abelian collineation group* P *transitive on* $L_\infty - \{x_\infty\}$.

(iv) *If* $p \neq 3$ *then* P *contains exactly* q *elations. If* $p = 3$ *then* P *consists of elations.*

(v) *If* $p \neq 2$ *then* P *is elementary abelian. If* $p = 2$ *then* P *is the direct product of* $\log_2 q$ *cyclic groups of order* 4.

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(vi) *There is a cyclic collineation group of order* $q - 1$ *normalizing* P *and faithful on* L_∞.

(vii) *The normalizer of* P *in* $(\mathrm{Aut}\,\mathbf{A})_0$ *has a subgroup of order* $q^2(q-1)^2 \log_p q$.

(viii) *The kernel of* $\mathbf{A}$ *is* $GF(q)$.

(ix) *If* $p = 3$ *then* $\mathbf{A}$ *is defined by a symplectic spread.*

Proof. Since U_Y has the structure indicated in (v), both (iii) and (v) are clear. Let $1 \neq A = U[\lambda, \mu] \in U_Y$. Then A induces an elation on $\mathbf{A}$ if and only if $p \neq 3$ and it induces the identity on $\langle X_\infty, Y\rangle^\perp/\langle X_\infty, Y\rangle$, or $p = 3$ and it induces the identity on $\langle X_\infty, Y\rangle^\perp/\langle X_\infty, Y, I\rangle$. By (4.2), $\langle X_\infty, Y\rangle^\perp$ consists of all matrices (4.1) with $T(\gamma) = 0$ and $b = 0$. Since $\bar{\gamma} = -\gamma$ and $\bar{\mu} = -\mu$,

$$A^{-1}MA - M = \begin{pmatrix} -\mu\gamma & \beta' & c' \\ 0 & 2\mu\gamma & \bar{\beta}' \\ 0 & 0 & -\mu\gamma \end{pmatrix}$$

with $c' \in K$ and $\beta' = -\alpha\bar{\mu} - \bar{\mu}^2\gamma + \bar{\mu}a + \bar{\lambda}\gamma$. Thus, $A^{-1}MA - M \in \langle X_\infty, Y\rangle$ for all $M \in \langle X_\infty, Y\rangle^\perp$ if and only if $U = 0$. This proves (iv) when $p \neq 3$. If $p = 3$ then

$$\beta' = -\alpha\bar{\mu} + \gamma(-T(\lambda)) - \bar{\mu}T(\alpha) - \lambda\gamma$$
$$= -\bar{\mu}(-\alpha + \bar{\alpha}) - \gamma(\lambda - \bar{\lambda}) \in K;$$

since $\mu\gamma \in K$, (iv) holds.

By (4.4), $\{D(\phi) | \phi \in L^*\}$ induces the cyclic group in (vi), while (vii), (viii) and (ix) are obvious. (Note that the involutory field automorphism of $GF(q^2)$ induces a polarity of $PG(3, q)$, and hence does not act on $\mathbf{A}$.)

Moreover, if $p \neq 3$ then (iv) yields (i) and hence (ii). Thus, we must prove (vi) and show that (i) holds when $q > 3 = p$. Before doing this, we will provide a slightly more compact description for the ovoid produced by $Y^\perp \cap \Omega$.

By (4.2), $Y^\perp/\langle Y\rangle$ consists of the matrices (4.1) with $T(\gamma) = 0$ and β read mod K. Thus, $Y^\perp/\langle Y\rangle$ can be identified with

$$V^* = \{(\alpha, \beta + K, \gamma, b, c) | \alpha, \beta, \gamma \in L, b, c \in K \text{ and } T(\gamma) = 0\},$$

with Q inducing

$$Q^*(\alpha, \beta + K, \gamma, b, c) = \alpha^2 + \alpha\bar{\alpha} + \bar{\alpha}^2 + T(\beta\gamma) + bc.$$

In this notation, $Y^\perp \cap \Omega$ produces the set Ω^* consisting of the points $\langle 0, 0, 0, 0, 1\rangle$ and

$$\langle \rho, \rho\sigma + K, \bar{\sigma}, 1, \rho\bar{\rho}\rangle \text{ with } T(\sigma) = 0 = T(\rho) + N(\sigma).$$

Now let $p = 3$. We must show that $W = \langle \Omega^*, (1, 0, 0, 0, 0)\rangle$ coincides with V^*. Clearly, W contains $(1,0,0,0,0)$, $(0,0,0,0,1)$, $(0, 0, 0, 1, 0)$, and

$(\rho, \rho\sigma + K, \bar{\sigma}, 0, 0)$ whenever $T(\sigma) = 0 = T(\rho) + N(\sigma)$. Set $\sigma = 0$ and $\rho \neq 0$, and deduce that $(\alpha, 0, 0, 0, 0) \in W$ for all α. Hence, so is $(0, \rho\sigma + K, \bar{\sigma}, 0, 0)$. Fix $\rho, \sigma \neq 0$ with $T(\sigma) = 0 = T(\rho) + N(\sigma)$, and let $k \in K - GF(3)$. Then

$$(0, k^3\rho\sigma + K, k\bar{\sigma}, 0, 0) - k^3(0, \rho\sigma + K, \bar{\sigma}, 0, 0) = (0, 0, (k - k^3)\sigma, 0, 0) \in W.$$

Consequently $W = V^*$. This completes the proof of (4.5).

Remark. The planes in (4.5) are not the only planes behaving as in (4.5i–vi). Others exist for at least some odd prime powers q. The planes in (4.5) with $q \equiv 5 \pmod 6$ can be shown to coincide with those found by Walker [**15**]; those with $q \equiv 2$ or $3 \pmod 6$ appear to be new.

We now turn to other planes produced by Ω.

THEOREM 4.6. *Let $q \equiv 2 \pmod 3$ and $q > 2$. Set $Y' = \text{diag}(\omega, 1, \bar{\omega})$ and $\mathbf{A}' = \mathbf{A}(Y'^{\perp} \cap \Omega)$. Then $\mathbf{A}'$ is a nondesarguesian plane. It has a collineation of order $q^2 - 1$ fixing two points at infinity and transitively permuting the remaining points at infinity.*

Proof. By (4.2), Y' is singular and $Y'^{\perp}$ consists of those matrices (4.1) for which

$$a + T(\alpha) = 0 = a + T(w\alpha).$$

Since $\dim_K L = 2$ and $T(\omega) = T(\omega\omega)$, we can write $\alpha = k\omega$ with $k \in K$. By (4.3),

$$Y'^{\perp} \cap \Omega = \{\langle X_\infty \rangle, \langle X[k\bar{\omega}, \sigma]\rangle | k = N(\sigma)\}.$$

By (4.4), $\{D(\phi) | \phi \in L^*\}$ has the desired transitivity properties. That $\dim \langle Y', Y' \cap \Omega \rangle > 5$ is proved as in the preceding theorem.

Remarks. Since $\alpha \in K\omega$, $Y'^{\perp}/\langle Y' \rangle$ can be identified with $K \oplus L \oplus L \oplus K$, with Q inducing $Q^*(b, \beta, \gamma, c) = T(\beta\gamma) + bc$. The corresponding ovoid is

(4.7) $\{\langle 0, 0, 0, 1\rangle, \langle 1, N(\sigma)\sigma\omega, \bar{\sigma}, N(\sigma)^2\rangle | \sigma \in L\}$.

If $q \equiv 2 \pmod 3$, the group G has exactly 3 orbits of singular points of V with orbit representatives $\langle X_\infty \rangle$, $\langle Y \rangle$ and $\langle Y' \rangle$. Similarly, if $p = 3$ there are just 2 orbits of singular points, along with 1 orbit of non-singular points $\langle N \rangle$ for which $N^{\perp}/\langle I \rangle$ is an $\Omega^+(6, q)$ space. One such N is $N = \text{diag}(\lambda, 0, \bar{\lambda})$, where $\lambda \in L^*$ and $T(\lambda) = 0$.

THEOREM 4.8. *If $q \equiv 0 \pmod 3$ then $\mathbf{A}(N^{\perp} \cap \Omega)$ is a nondesarguesian plane, and admits a collineation of order $q^2 - 1$ behaving as in* (4.6).

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The proof is similar to the preceding ones. In fact, the matrix (4.1) is in $N^{\perp}$ if and only if $T(\alpha\lambda) = 0 = T(\lambda)$; that is, if and only if $\alpha \in K$. Thus, the required ovoid can be described precisely as in (4.7), with ω replaced by 1.

For $q \equiv 0$ or 2 (mod 3), a spread of $L \oplus L$ corresponding to the ovoid (4.7) can be described as follows. Fix $\pi, \theta \in L$ with $\pi \notin K$ and $\bar{\theta} = -\theta$. Then the spread consists of $0 \times L$ together with the K-subspaces

$$\langle (1, \theta), (\pi, N(\sigma)\sigma\omega\theta) \rangle \quad \text{for } \sigma \in L.$$

5. Some 5- and 6-dimensional ovoids. Let $K = GF(q)$, where q is odd and not a prime. Fix a nonsquare n of K, and automorphisms σ and τ of K at least one of which is nontrivial.

Equip $V = K^6$ with the quadratic form $Q(x, y, z, u, v, w) = xw + yv + zu$. Let Ω consist of the points

$$(5.1) \quad \begin{array}{l} \langle 0, 0, 0, 0, 0, 1 \rangle \\ \langle 1, y, z, z^{\tau}, -ny^{\sigma}, -z^{\tau+1} + ny^{\sigma+1} \rangle, \quad y, z \in K. \end{array}$$

Then Ω consists of $q^2 + 1$ pairwise non-perpendicular singular points.

If $\tau = 1$ or $\sigma = 1$ then $\langle \Omega \rangle$ is a nonsingular hyperplane of V. In all other cases, $\langle \Omega \rangle = V$. This proves the following result.

PROPOSITION 5.2. (i) $\mathbf{A}(\Omega)$ *is nondesarguesian.* (ii) *If* $\tau = 1 \neq \sigma$ *or* $\sigma = 1 \neq \tau$ *then* $\mathbf{A}(\Omega)$ *arises from a symplectic spread.*

The plane $\mathbf{A}(\Omega)$ is a semifield plane: the orthogonal transformations

$$(x, y, z, v, w) \rightarrow (x, y + ax, z + bx, u + b^{\tau}x, v - na^{\sigma}x,$$
$$w + na^{\sigma}y - av - b^{\tau}z - bu - b^{\tau+1}x + na^{\sigma+1}x)$$

all preserve Ω, send $p = \langle 0, 0, 0, 0, 0, 1 \rangle$ to itself, and induce the identity on $p^{\perp}/p$.

In fact, $\mathbf{A}(\Omega)$ is a known plane. By (2.1), $\langle 1, a, b, c, d, -ad - bc \rangle$ corresponds to the 2-space

$$\{(X, XM) | X \in K^2\} \text{ of } K^2 \oplus K^2, \quad \text{where } M = \begin{pmatrix} c & -d \\ a & b \end{pmatrix}.$$

Replacing M by its transpose and using (5.1), we obtain a plane coordinatized by one of the semifields discovered by Knuth [**8**] (cf. [**2**, 5.3.6]).

Remark. By [**1**], if an ovoid Ω of V consists of the points $\langle 0, 0, 0, 0, 0, 1 \rangle$ and $\langle 1, y, z, z, f(y), -z^2 - yf(y) \rangle$ for $y, z \in K$, then Ω is equivalent to (5.1) for some n and σ. Presumably, the ovoids in (5.1) can all be characterized in an analogous manner.

6. Ree-Tits ovoids. Let $K = GF(q)$ and $V = K^7$, where $q = 3^{2e-1}$. If $a \in K$ set $a^\sigma = a^{3^e}$, so that $a^{\sigma^2} = a^3$. Equip V with the quadratic form $Q(x_i) = x_4{}^2 + x_1x_7 + x_2x_6 + x_3x_5$. The Ree-Tits ovoid Ω consists of the $q^3 + 1$ singular points

$$\langle 0, 0, 0, 0, 0, 0, 1\rangle$$
$$\langle 1, x, y, z, u, v, w\rangle \quad \text{with } x, y, z \in K,$$

where

$$u = x^2y - xz + y^\sigma - x^{\sigma+3}$$
$$v = x^\sigma y^\sigma - z^\sigma + xy^2 + yz - x^{2\sigma+3}$$
$$w = xz^\sigma - x^{\sigma+1}y^\sigma - x^{\sigma+3}y + x^2y^2 - y^{\sigma+1} - z^2 + x^{2\sigma+4}$$

([**14**]). The Ree group $R(q)$ acts 2-transitively on Ω, and has exactly 3 orbits of singular points of V; orbit representatives are $\langle 0, 0, 0, 0, 0, 0, 1\rangle$, $\langle 0, 0, 0, 0, 0, 1, 0\rangle$ and $\langle 0, 0, 0, 0, 1, 0, 0\rangle$. The second and third of these produce the following 5-dimensional ovoids:

$$\begin{aligned}&\langle 0, 0, 0, 0, 1\rangle\\ &\langle 1, y, z, y^\sigma, -y^{\sigma+1} - z^2\rangle \quad \text{with } y, z \in K;\end{aligned} \tag{6.1}$$

and

$$\begin{aligned}&\langle 0, 0, 0, 0, 1\rangle\\ &\langle 1, x, z, -z^\sigma - x^{2\sigma+3}, xz^\sigma - z^2 + x^{2\sigma+4}\rangle \quad \text{with } x, z \in K.\end{aligned} \tag{6.2}$$

Ovoid (6.1) appears in Section 5 (with $n = -1$ and $\tau = 1$).

Ovoid (6.2) gives rise to 4-dimensional symplectic spread. If $q = 3$, the resulting plane is desarguesian; if $q > 3$ it is not. A Frobenius group of order $q(q - 1)$ acts on the ovoid, with orbits of length 1, q and $q(q - 1)$. This group is generated by the following orthogonal transformations (where $b \in K$ and $k \in K^*$):

$$(t, x, z, v, w) \rightarrow (t, x, y + bt, v - b^\sigma t, w + b^\sigma x + bz + b^2t)$$

and

$$(t, x, z, v, w) \rightarrow (t, kx, k^{\sigma+2}, k^{2\sigma+3}v, k^{2\sigma+4}w).$$

Its Sylow 3-subgroup contains no elations.

A further class of planes arises from Ω using nonsingular points of V. There is just one $R(q)$-orbit of nonsingular points n of V such that $n^\perp \cap V$ is an $\Omega^+(6, q)$ space. One such point is $n = \langle 0, 0, 0, 1, 0, 0, 0\rangle$ (which is perpendicular to the totally singular 3-space $\langle (0, 0, 0, 0, 0, 0, 1)$, $(0, 0, 0, 0, 0, 1, 0)$, $(0, 0, 0, 0, 1, 0, 0)\rangle$). This produces an ovoid $n^\perp \cap \Omega$. Projecting into six dimensions, we obtain the ovoid

$$\begin{aligned}&\langle 0, 0, 0, 0, 0, 1\rangle\\ &\langle 1, x, y, x^2y + y^\sigma - x^{\sigma+3}, x^\sigma y^\sigma + xy^2 - x^{2\sigma+3},\\ &\qquad -x^{\sigma+1}y^\sigma + x^{\sigma+3}y + x^2y^2 - y^{\sigma+1} + x^{2\sigma+4}\rangle \quad \text{with } x, y \in K.\end{aligned} \tag{6.3}$$

Even when $q = 3$, this ovoid spans the 6-space (compare Section 3), so that we obtain a nondesarguesian plane for each q. The Ree group only provides a collineation group of order $q - 1$, consisting of the orthogonal transformations

$$(t, x, y, u, v, w) \to (t, kx, k^{\sigma+1}y, k^{\sigma+3}u, k^{2\sigma+3}v, k^{2\sigma+4}w).$$

7. Desarguesian ovoids. Let q be a power of 2. Set $K = GF(q)$, $F = GF(q^3)$, and $V = K \oplus F \oplus F \oplus K$. Equip V with the quadratic form $Q(a, \beta, \alpha, d) = ad + T(\beta\gamma)$, where $T : F \to K$ is the trace map.

The following set of points is an ovoid Ω (compare [**4**, (8.1)]):

$$\langle 0, 0, 0, 1 \rangle$$
$$\langle 1, t, t^{q+q^2}, N(t) \rangle \quad \text{for } t \in F,$$

where $N(t) = t^{1+q+q^2}$. There is a group $G = PSL(2, q^3)$ of orthogonal transformations acting 3-transitively on Ω. This group has exactly one further orbit of singular points, of which $x = \langle 0, 0, 1, 0 \rangle$ is a representative. Note that $\Omega' = x^\perp \cap \Omega$ consists of the points

$$\langle 0, 0, 0, 1 \rangle$$
$$\langle 1, t, t^{q+q^2}, N(t) \rangle \quad \text{where } T(t) = 0.$$

The stabilizer of x in G has order $q^2(q - 1)$. Its subgroup of order q^2 consists of all transformations

$$(a, \beta, \gamma, d) \to (a, as + \beta, as^{q+q^2} + \beta^q s^{q^2} + \beta^{q^2} s^q + \gamma,$$
$$aN(s) + T(\beta s^{q+q^2}) + T(\gamma s) + d)$$

with $T(s) = 0$.

THEOREM 7.1. *If $q > 2$ then $\mathbf{A}(\Omega')$ is a nondesarguesian semifield plane of order q^2.*

Proof. The plane is nondesarguesian since $\dim \langle \Omega' \rangle = 7$. In order to prove that it is a semifield plane, it suffices to show that P induces the identity on $\langle x, y \rangle^\perp / \langle x, y \rangle$, where $y = \langle 0, 0, 0, 1 \rangle$. Here, $\langle x, y \rangle^\perp$ consists of all vectors $(0, \beta, \gamma, d)$ such that $T(\beta) = 0$. It then suffices to note that $\beta^{q^2} s^q + \beta^q s^{q^2} \in K$ whenever $T(\beta) = 0 = T(s)$. (Namely,

$$(\beta s^q + \beta^q s)^q = \beta^q s^{q^2} + \beta^{q^2} s^q = \beta^q (s + s^q) + (\beta + \beta^q) s^q$$
$$= \beta s^q + \beta^q s.)$$

Remark 1. The plane $\mathbf{A}(\Omega')$ of order q^2 has been constructed using $GF(q^3)$. This unusual means of describing a plane of order q^2 is remarkable, in view of the following relationship between Ω and $AG(2, q^3)$.

If τ is a suitable triality map, then Ω^τ is the orthogonal spread which is called *desarguesian* in [**4**, **5**]; one of its intersections with a nondegenerate hyperplane arises from the usual $AG(2, q^3)$ spread. For this reason, Ω deserves to be called the *desarguesian ovoid* in V.

Remark 2. A presemifield for this plane can be described as follows. Let $W = \text{Ker } T$. Then $F = K \oplus W$; let π denote the corresponding projection onto W. Fix a basis σ, τ of W. Then

$$(a\sigma + b\tau) \cdot r = (ar + br^{q+q^2})\pi$$

defines the desired presemifield on W (where $a, b \in K, r \in W$).

8. Dye's ovoid. Exactly one further $\Omega^+(8, q)$ ovoid is presently known. It is an $\Omega^+(8, 8)$ ovoid Ω, discovered by Dye [**3**, § 4].

Let $\{\langle e_i\rangle | 1 \leqq i \leqq 9\}$ be an $\Omega^+(8, 2)$ ovoid; then $\sum_{i=1}^{9} e_i = 0$ (cf. Section 3). Embed the $\Omega^+(8, 2)$ space into an $\Omega^+(8, 8)$ space. If $\phi \in GF(8)$ and $\phi^3 + \phi^2 + 1 = 0$, then Ω consists of the points

$$\langle e_i\rangle, \quad 1 \leqq i \leqq 9,$$
$$\langle \phi e_i + \phi^2 e_j + \phi^4 e_k\rangle \quad \text{with } i, j, k \text{ distinct.}$$

Clearly, $P\Gamma O^+(8, 8)_\Omega \geqq S_9 \times \mathbf{Z}_3$ (with $\mathbf{Z}_3$ fixing each e_i); in fact, these groups coincide (cf. [**4**, § 9]). Set $G = A_9 \times \mathbf{Z}_3$. If y is a singular point not in Ω, then G_y acts on $\mathbf{A}(y^\perp \cap \Omega)$. We will mention properties of $\mathbf{A}(y^\perp \cap \Omega)$ for four choices of y.

Example 8.1. $y = \langle e_6 + e_7 + e_8 + e_9\rangle$. Here, $\langle y^\perp \cap \Omega\rangle = \langle e_1, e_2, e_3, e_4, e_5\rangle$, $\mathbf{A}(y^\perp \cap \Omega)$ is desarguesian, and G_y induces S_5 on $\mathbf{A}(y^\perp \cap \Omega)$.

Example 8.2. $y = \langle e_6 + e_7 + \phi e_8 + \phi^{-1} e_9\rangle$. If $\Omega' = y^\perp \cap \Omega$, then $\mathbf{A}(\Omega')$ has the following properties.

(i) $\mathbf{A}(\Omega')$ is a nondesarguesian plane of order 8^2.

(ii) There is a collineation group $SL(2, 4)$ fixing 7 subplanes of order 4 containing 0 which are permuted transitively by the homologies of $\mathbf{A}(\Omega')$ with center 0.

(iii) $\mathbf{Z}_7 \times SL(2, 4)$ acts irreducibly on the 4-dimensional $GF(8)$-space underlying $\mathbf{A}(\Omega')$; the representation is exactly the same as for $AG(2, 4^3)$.

(iv) All involutions in $SL(2, 4)$ are elations.

(v) $SL(2, 4)$ has orbit lengths 5, 20, 20, 20 on L_∞.

(vi) There is a collineation group S_5 whose transpositions are Baer involutions and whose orbit lengths on L_∞ are 5, 20, 40.

(vii) Elements of order 3 of $SL(2, 4)$ fix exactly 8 points on L_∞.

Proof. Here Ω' consists of the 65 points spanned by the following vectors (where $i, j \leqq 5, i \neq j$)

$$\phi^4 e_i + \phi^2 e_8 + \phi e_9$$
$$\phi^2 e_i + \phi^4 e_j + \phi e_8$$
$$\phi e_i + \phi^4 e_j + \phi^2 e_6$$
$$\phi e_i + \phi^4 e_j + \phi^2 e_7.$$

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The first 5 of these vectors have sum $\phi^4(e_6 + e_7 + \phi e_8 + \phi^{-1}e_9)$, and hence determine the subplanes appearing in (ii). Since G_y induces S_5 on $\mathbf{A}(\Omega')$, all remaining assertions also follow easily from the above list of vectors.

Remarks. 1. There are many other subplanes of order 4. Since

$$\phi^4 e_5 + \phi^2 e_8 + \phi e_9 = \phi^4(e_1 + e_2 + e_3 + e_4) + \phi^4(e_6 + e_7 + \phi e_8 + \phi^{-1}e_9),$$

these can be obtained, for example, by using $\langle u_1, u_2, v_3, v_4, \phi^4 e_5 + \phi^2 e_8 + \phi e_9\rangle$ whenever u_1, u_2, v_3, v_4 are among the above 65 vectors and

$$u_1 + u_2 \in \langle e_1 + e_2 + \alpha(e_6 + e_7)\rangle \quad \text{and} \quad v_3 + v_4 \in \langle e_3 + e_4 + \alpha(e_6 + e_7)\rangle$$

for some $\alpha \in GF(8)$. There are several different ways to choose the pairs $\{u_1, u_2\}$ and $\{v_3, v_4\}$.

2. A more compact description of $\mathbf{A}(\Omega')$ can be obtained as follows. Set

$$s = e_1 + e_2 + e_3 + e_4 + e_5, f_i = e_i + s \quad \text{for } 1 \leqq i \leqq 5, \quad \text{and} \quad g_k = e_k + \phi s \quad \text{for } k = 6, 7.$$

Then

$$y^\perp = y \perp \langle f_1, f_2, f_3, f_4, f_5\rangle \perp \langle g_6, g_7\rangle$$

with

$$Q(f_i) = 0 = (f_i, g_k), (f_i, f_j) = 1 = (g_6, g_7) \quad \text{for } i \neq j,$$
$$Q(g_k) = \phi \quad \text{and} \quad f_1 + f_2 + f_3 + f_4 + f_5 = 0.$$

The ovoid of $\langle f_1, f_2, f_3, f_4, f_5, g_6, g_7\rangle$ upon which Ω' projects consists of the points

$$\langle f_i\rangle, \langle \phi f_i + \phi^4 f_j + \phi^2 g_k\rangle, \langle \phi^2 f_i + \phi^4 f_j + \phi^3(g_6 + g_7)\rangle$$

with $i, j \leqq 5$, $i \neq j$, and $k = 6, 7$.

3. It follows readily from the preceding remark that Aut $\mathbf{A}(\Omega') = \mathbf{Z}_7 \times S_5$.

Example 8.3. $y = \langle e_5 + e_6 + \phi^{-1}e_7 + \phi^{-2}e_8 + \phi^{-4}e_9\rangle$. Here, $G_y \cong S_4 \times \mathbf{Z}_3$, where the $\mathbf{Z}_3$ is nonlinear, induces (7, 8, 9), and fixes exactly 5 points of $y^\perp \cap \Omega$: $\langle e_i\rangle$, $1 \leqq i \leqq 4$, and $\langle \phi^4 e_7 + \phi e_8 + \phi^2 e_9\rangle$. Moreover, G_y induces S_4 on each of the resulting 7 subplanes $AG(2, 4)$.

Example 8.4. $y = \langle (e_4 + e_5) + \phi(e_6 + e_7) + (\phi + 1)(e_8 + e_9)\rangle$. Once again $\langle y^\perp \cap \Omega\rangle = y^\perp$. This time, $G_y \cong \mathbf{Z}_2{}^2 \times S_3$; its Sylow 2-subgroups induce exactly 6 Baer involutions and 1 nontrivial elation.

9. Concluding remarks. 1. Most of the automorphism group of each of the planes studied in [**4**, **5**] could be obtained using the associated orthogonal spread. However, for the planes discussed here the groups induced by Aut **A** and $\Gamma O^+(8, q)_\Omega$ on L_∞ need not coincide (cf. (3.2) and (8.1)). It would be desirable to know how close they are in each case we have discussed.

2. We have surveyed all the known $\Omega^+(8, q)$ ovoids. Are there further examples?

3. Presumably, planes of the form $\mathbf{A}(x^\perp \cap \Omega)$ have intrinsic properties not shared by most translation planes. However, I know no such property.

4. The duals of the planes (4.5) with $q \equiv 2 \pmod 3$ can be derived so as to obtain planes of type II.1, as in [**10**].

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