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Generic 3-parameter families of vector fields on the plane, unfolding a singularity with nilpotent linear part. The cusp case of codimension 3

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Abstract. A cusp type germ of vector fields is a C^∞ germ at $0\in\mathbb{R}^2$, whose 2-jet is C^∞ conjugate to

$$y\frac{\partial}{\partial x}+(\alpha x^2+\beta xy)\frac{\partial}{\partial y}\qquad \text{with } \alpha\neq 0.$$

We define a submanifold of codimension 5 in the space of germs $\sum^3_{C\pm}$, consisting of germs of cusp type whose 4-jet is C^0 equivalent to

$$y\frac{\partial}{\partial x}+(x^2\pm x^3y)\frac{\partial}{\partial y}.$$

Our main result can be stated as follows: any local 3-parameter family in $(0,0)\in\mathbb{R}^2\times\mathbb{R}^3$, cutting $\sum^3_{C\pm}$ transversally in $(0,0)$ is fibre-C^0 equivalent to

$$\tilde{X}^{\pm}_{\lambda}=y\frac{\partial}{\partial x}+(x^2+\mu+y(\nu_0+\nu_1x\pm x^3))\frac{\partial}{\partial y}$$

1. *Introduction and acknowledgments*

Generically, C^∞ vector fields on 2-dimensional manifolds only have hyperbolic singularities (zeros) and these hyperbolic singularities are stable for topological equivalence.

Bifurcations occur if one considers 1-parameter families of such vector fields. Generically, the only *local* bifurcations are the (codimension 1) saddle-node bifurcation and the (codimension 1) Hopf bifurcation (see e.g. [S]). The saddle-node bifurcation is an unfolding of a singularity whose linear part has exactly one zero eigenvalue, and whose restriction to a centre manifold starts with non-zero quadratic terms. The Hopf bifurcation is an unfolding of a singularity whose linear part has a pair of imaginary eigenvalues and whose radial component of the normal form in polar coordinates starts with non-zero cubic terms.

In generic 2-parameter families of 2-dimensional vector fields one encounters some extra bifurcations like, for example, a (generalized) saddle-node bifurcation of codimension 2, which is an unfolding of a singularity whose linear part has codimension 2' is

$$y\frac{\partial}{\partial x}+(x^2+\mu+\nu y\pm xy)\frac{\partial}{\partial y}.$$

A more detailed introduction to these unfoldings of singularities of codimension ≤ 2 can be found in [**D**] and [**D, R**]. See also [**A1**].

In generic 3-parameter families one locally encounters generalized saddle-node bifurcations and generalized Hopf bifurcations of codimension at most three, one encounters the unfolding of the cusp-singularity of codimension 2, but one also finds unfoldings of singularities with nilpotent 1-jet which are however more degenerate than the cusp of codimension 2.

Our aim in this paper is to study the generic 3-parameter unfoldings of a singularity whose 2-jet is C^∞ equivalent to $y\,\partial/\partial x+(\alpha x^2+\beta xy)\,\partial/\partial y$ with $\alpha\neq 0$ and $\beta=0$. We call it the cusp singularity of codimension 3. The set of germs of such vector field constitute a semi-algebraic subset of codimension 5, which we denote by Σ_C^3 (Σ_C^3 is a semi-algebraic subset of codimension 1 in Σ_C^2, manifold defined by the condition $\alpha\neq 0$; one has $\Sigma_C^2=\Sigma_{C^+}^2\cup\sum\Sigma_{C^-}^2\cup\Sigma_C^3$).

We will define a generic condition in Σ_C^3, by showing first that each $X_0\in\Sigma_C^3$ has a 4-jet C^∞ equivalent to

$$y\frac{\partial}{\partial x}+(x^2+\gamma x^3y)\frac{\partial}{\partial y} \tag{2}$$

and by imposing $\gamma\neq 0$.

One defines Σ_C^4 by the condition $\gamma=0$; Σ_C^4 is a semi-algebraic subset of codimension 1 in Σ_C^3 and $\Sigma_C^3=\Sigma_{C^+}^3\cup\Sigma_{C^-}^3\cup\Sigma_C^4$, where $\Sigma_{C^\pm}^3$ is the submanifold of codimension 5 consisting of germs of vector fields whose 4-jet is C^∞-equivalent to $y\,\partial/\partial x+(x^2\pm x^3y)\,\partial/\partial y$.

We study the generic 3-parameter families X_λ with $X_0\in\Sigma_{C^\pm}^3$. The genericity condition consists in the transversality of the mapping $(x,\lambda)\to j^4X_\lambda(x)$ with respect to $\Sigma_{C^\pm}^3$. An example of such a family is given by:

$$\tilde{X}_\lambda^\pm=y\frac{\partial}{\partial x}+(x^2+\mu+y(\nu_0+\nu_1x\pm x^3))\frac{\partial}{\partial y}, \tag{3}$$

with $\lambda=(\mu,\nu_0,\nu_1)$; $\tilde{X}_0$ belongs to $\Sigma_{C^+}^3$ (resp. $\Sigma_{C^-}^3$) depending on the sign $\pm$.

Our main result can be stated as follows:

THEOREM. *A local* 3-*parameter family in* $(0,0)\in\mathbb{R}^2\times\mathbb{R}^3$, *cutting* $\Sigma_{C^+}^3$ (*resp.* $\Sigma_{C^-}^3$) *transversally in* $(0,0)$ *is fibre-*C^0 *equivalent to* $\tilde{X}_\lambda^+$ (*resp.* $\tilde{X}_\lambda^-$).

For a definition of fibre-C^0-equivalence see the beginning of § 2 (the equivalence between the vector fields does not necessarily depend in a C^0 way on the parameter). To prove this result, we show that each generic family X_λ with $X_0\in\Sigma_{C^+}^3$ (resp. $\Sigma_{C^-}^3$) has the same bifurcation set as $\tilde{X}_\lambda^+$ (resp. $\tilde{X}_\lambda^-$), at least up to a homeomorphism in the parameter space. This bifurcation set is a cone with vertex in 0.

Let us analyse the phenomena happening in $\tilde{X}_\lambda^+$. We first remark that the equation for the critical points of $\tilde{X}_\lambda^+$ is given by

$$y=0,\qquad x^2+\mu=0.$$

Hence $\tilde{X}_\lambda^+$ has no critical points for $\mu > 0$. The $\{\mu = 0\}$-plane outside the origin is a bifurcation surface of saddle-node type: crossing it in the direction of decreasing μ, one observes the creation of two singularities: a saddle and a node. The other surfaces of bifurcation are situated in the half space $\{\mu < 0\}$. They can best be visualized by drawing their trace on the half-sphere

$$S = \{(\mu, \nu_0, v_1) \mid \mu < 0, \mu^2 + \nu_0^2 + \nu_1^2 = \varepsilon^2\}$$

for $\varepsilon > 0$ sufficiently small. We recall that the bifurcation set is a 'cone' based on its trace with S.

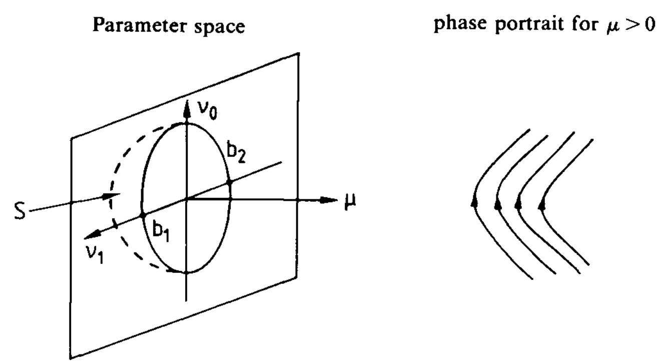

FIGURE 2

Trace of the bifurcation set of $\tilde{X}_\lambda^+$ on S and codimension 0 phase portraits for $\mu < 0$

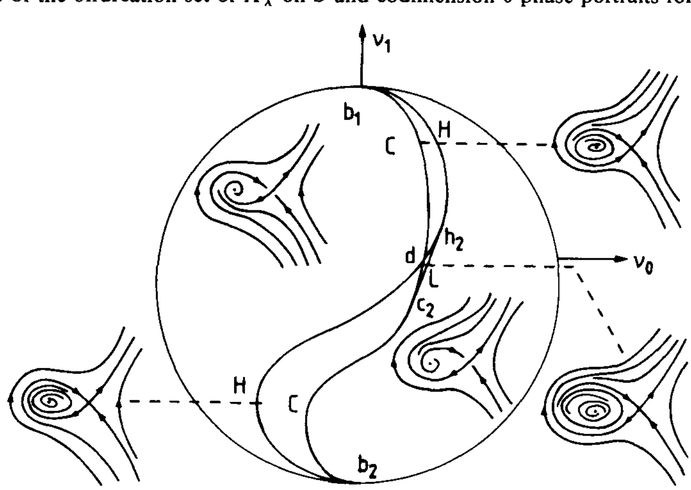

FIGURE 3

This trace on S consists of 3 curves: a curve H of Hopf bifurcation, a curve C of saddle connexion and a curve L of generic coalescence of closed orbits. The curve L joins a point h_2 on H to a point c_2 on C, and in these points L is tangent to (resp.) H and C. On the other hand, the curves H and C both touch ∂S with a first order contact in the points b_1 and b_2. In the neighbourhood of b_1 and b_2 one finds back the unfolding of the cusp-singularity of codimension 2, studied by Bogdanov;

for instance, there exists a unique repelling closed orbit in between H and C in the neighbourhood of b_1 and an unique attracting closed orbit in between H and C in the neighbourhood of b_2. Along the curve H, outside h_2 occurs a Hopf bifurcation with the appearance of a repelling closed orbit when crossing the arc $]b_1, h_2[$ of H from right to left and the appearance of an attracting closed orbit when crossing the arc $]h_2, b_2[$ of H from left to right. The point h_2 corresponds to a Hopf bifurcation of codimension 2.

Along the curve C, outside c_2, occurs a saddle connection of codimension 1. When crossing the arc $]b_1, c_2[$ of C from left to right, 2 separatrices of the saddle point at a certain moment coincide and a repelling closed orbit appears; the same phenomenon happens giving rise to an attracting closed orbit, when crossing the arc$]c_2, b_2[$ of C from right to left. The point c_2 corresponds to a saddle connection of codimension 2.

codimension 1 phase portraits ($\mu \leq 0$)

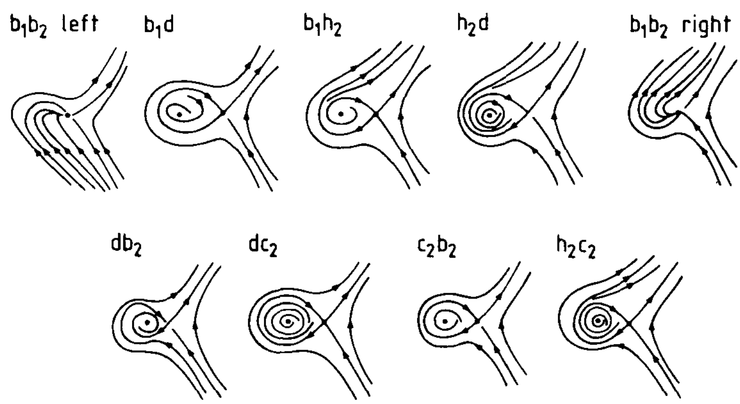

FIGURE 4

codimension 2 phase portraits ($\mu \leq 0$)

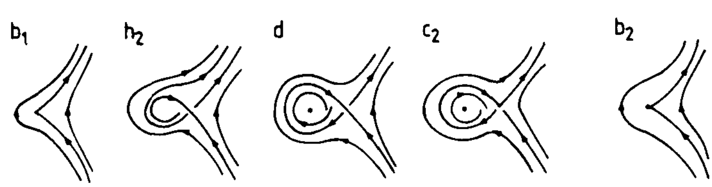

FIGURE 5

The curves H and C intersect transversally in a unique point d representing a parameter value of simultaneous Hopf bifurcation and saddle connection.

For parameter values in the curved triangle (d, h_2, c_2) there exist exactly two closed orbits, of which the inner one is attracting and the outer one is repelling. These two limit cycles coalesce in a generic way when crossing the curve L from left to right. On L itself we have a unique semi-stable closed orbit. In terms of the subset $\Sigma = \partial S \cup H \cup C \cup L$ of S, the bifurcation set of $\tilde{X}^+_\lambda$ inside the ball $B_\varepsilon = \{\mu^2 + \nu_0^2 + \nu_1^2 \le \varepsilon\}$ is a cone homeomorphic to

$$\{(t^4\bar{\mu}, t^6\bar{\nu}_0, t^4\bar{\nu}_1) \mid t \in [0, \varepsilon] \text{ and } (\bar{\mu}, \bar{\nu}_0, \bar{\nu}_1) \in \Sigma\}$$

The bifurcation set of any generic family X_λ with $X_0 \in \Sigma^3_{C^+}$, is homeomorphic to the bifurcation set of $\tilde{X}^+_\lambda$. In both cases the homeomorphism blows up to a C^∞ diffeomorphism in the coordinates $((\bar{\mu}, \bar{\nu}_0, \bar{\nu}_1), t)$ on $S \times [0, \varepsilon]$, at least outside the line corresponding to the point c_2 (for more explanation concerning this line, see the final remark in appendix 3). The topological type of X_λ in a fixed neighbourhood of $0 \in \mathbb{R}^2$ is constant in each connected component of the complement of the bifurcation set (6 components) and is constant in each part of the bifurcation set (9 surfaces and 5 curves). It is then trivial to deduce that the families X_λ and $\tilde{X}^+_\lambda$ are topologically equivalent, if one does not demand the equivalence to depend continuously on the parameter; such an equivalence is called a fiber-C^0-equivalence (see [**D**]). The analogous result for families X_λ with $X_0 \in \Sigma^3_{C^-}$ is obtained by merely observing that the 2 types of families can be obtained one from another by the coordinate-, parameter-, and time change $(x, y, \mu, \nu_0, \nu_1, t) \to (x, -y, \mu, -\nu_0, -\nu_1, -t)$.

The necessary definitions and useful notations can be found in § 2. In § 3, we calculate a usable normal form for the generic families X_λ. Finally, in § 4 we prove the theorem essentially by showing that all generic families X_λ have a bifurcation set as we come to describe.

This paper grew out of different bilateral contacts, spread over a period of about 10 years. Each step has its significance in the present result and for this the authors want to thank different institutes for their repeated hospitality and financial support: the Instituto de Matematica Pura e Aplicada in Rio de Janeiro, the University of Dijon and the Limburgs Universitair Centrum in Diepenbeek. We also thank the Banach Centre in Warsaw for the opportunity given to R. Roussarie to have interesting discussions. Especially some suggestions of H. Zoładek were helpful. We also learned from conversations with A. Chenciner, and surely too from the translation that J. C. Yoccoz made of some papers of R. Bogdanov.

2. *Definitions and notations*

A k-parameter family of vector fields on $\mathbb{R}^2$, X_λ, where $\lambda \in \mathbb{R}^k$ denotes the parameter, is defined to be a vector field

$$X_\lambda = a(m, \lambda)\frac{\partial}{\partial x} + b(m, \lambda)\frac{\partial}{\partial y}, \qquad m = (x, y) \in \mathbb{R}^2 \tag{1}$$

where the coefficient functions a and b are C^∞ with respect to $(m, \lambda) \in \mathbb{R}^2 \times \mathbb{R}^k$.

We will study *local families* around $(0, 0) \in \mathbb{R}^2 \times \mathbb{R}^k$, this means families defined on some neighbourhood of $(0, 0)$, or better *germs* of families in $(0, 0)$, since the neighbourhood itself does not matter. Such a (local) family X_λ will be called a k-parameter *unfolding* (or deformation) of X_0.

Between the vector fields on $\mathbb{R}^2$, we introduce the notion of *topological* (or C^0) *equivalence*: 2 vector fields X and Y are C^0 equivalent if there exists a homeomorphism h on $\mathbb{R}^2$ sending X-orbits to Y-orbits in a sense-preserving way. This notion extends to germs of vector fields in $0 \in \mathbb{R}^2$. Related to this is the notion of (fibre) *C^0-equivalence for families of vector fields*: 2 families X_λ and Y_μ are called (fibre-) C^0-equivalent if there exists a homeomorphism $\mu = \phi(\lambda)$ between the parameter spaces (of the same dimension k) and a family of homeomorphisms of $\mathbb{R}^2$ depending on the parameters λ: $h_\lambda(m)$ such that for all $\lambda \in \mathbb{R}^k$, h_λ is a topological equivalence between X_λ and $Y_{\phi(\lambda)}$.

We remark that we do not ask the equivalence to depend continuously on λ. Although we believe this to be the case in the problem considered here, we do not want to include this in our study.

We remark also that this relation induces an equivalence relation for local families around $(0, 0) \in \mathbb{R}^2 \times \mathbb{R}^k$. It is a relation on the level of germs of families, and not of families of germs.

Suppose now that a certain family X_λ is given. The *bifurcation set* of X_λ is the smallest closed subset $\Sigma \subset \mathbb{R}^k$ such that the topological type of the vector field X_λ for $\mathbb{R}^k \backslash \Sigma$ is locally constant (for the notion of C^0-equivalence). Clearly: if 2 families are C^0-equivalent, the change in parameters ϕ exchanges the respective bifurcation sets. We denote by V_0 the space of germs at 0 of vector fields on $\mathbb{R}^2$, by $]_0^N V$ the vector space of their N-jets at 0, by $\pi_{PN}:]_0^P V \to]_0^N V$ (for $P \geq N \geq 0$) the natural restriction mapping and by $\pi_N: V_0 \to]_0^N V$ the mapping sending a germ to its N-jet.

The natural algebraic structure of $]_0^N V$ permits us to define the notion of submanifold or (semi-) algebraic subset in $]_0^N V$; for each $\Sigma \subset]_0^N V$ we will identify Σ with its contra-images by π_{PN} and π_N in resp. $]_0^P V$ and V_0, denoting these contra-images by the same symbol Σ. Conversely, a *submanifold* or a *(semi-)algebraic subset* Σ of *codimension q in V_0* is by definition the contra-image of a submanifold or a (semi-) algebraic subset of codimension q contained in some $]_0^N V$ and which we also denote Σ.

In the space of germs V_0, we consider the action of the group of germs of diffeomorphisms fixing 0 in $\mathbb{R}^2$ (C^∞ conjugacy) defined by $g^*X(x) = (dg_x)^{-1}X(g(x))$ as well as the action of the group of pairs (f, g) consisting of the germ of a strictly positive function and the germ of a diffeomorphism fixing 0 (*C^∞ equivalence*). This last action is defined by $((f, g) \cdot X)(x) = f(x)g^*X(x)$, and the group operation by $(f, g) \cdot (f', g') = (f \cdot (f' \circ g), g' \circ g)$.

These differentiable actions on the germs induce algebraic actions on each space of jets $]_0^N V$. Precisely these actions will be used in § 3 to obtain simpler expressions (normal forms). We hereby need the following observations: in a fixed $]_0^N V$ the subset of jets conjugate or equivalent to a certain given jet (this means an orbit of one of the given group-actions) form a submanifold, the set of jets conjugate or equivalent to the jets belonging to a given semi-algebraic subset form a semi-algebraic subset (theorem of Tarski-Seidenberg) [**Se**].

We may also define the action of C^∞ conjugacy or C^∞ equivalence on the (local) families, asking that ϕ be a (local) diffeomorphism and that $h_\lambda(m)$ be a C^∞ family of C^∞ diffeomorphisms (i.e. $h_\lambda(m)$ depends in a C^∞ way on m and λ). We'll use these relations to obtain 'normal forms' for the families X_λ.

At each point $m \in \mathbb{R}^2$ we identify the space of N-jets in m of vector fields on $\mathbb{R}^2$ to the space $]_0^N V$. If X is a vector field on $\mathbb{R}^2$ we hence obtain the N-jet mapping:

$$J^N X : \mathbb{R}^2 \to]_0^N V, \qquad m \to J^N X(m).$$

If $X_\lambda(m)$ is a k-parameter family, we also consider the mapping

$$\mathbb{R}^2 \times \mathbb{R}^k \to]_0^N V, \qquad (m, \lambda) \to J^N X_\lambda(m).$$

This mapping will permit us to define the genericity conditions needed on the family X_λ.

3. *Putting in normal form*

In this section, we will define the submanifolds $\Sigma^3_{C^\pm} \subset]_0^4 V$ and we'll show that 3-parameter families cutting $\Sigma^3_{C\pm}$ transversally can be brought - up to C^∞ equivalence - in a simplified form called a normal form. A large part of this reduction is valid under rather general conditions. Therefore, we'll present the reduction to the normal form in *successive steps*, making precise each time the supplementary conditions which are required. To simplify calculations and presentation we associate to the family X_λ a *dual family of* 1-*forms* defined by

$$\omega_\lambda = X_\lambda \lrcorner (dx \wedge dy) \tag{1}$$

($\lrcorner$ denoting the interior product). For $X_\lambda = a_\lambda\, \partial/\partial x + b_\lambda\, \partial/\partial y$ we have $\omega_\lambda = -b_\lambda\, dx + a_\lambda\, dy$.

On the 1-forms one can transpose the notion of C^∞ equivalence (conjugacy by a diffeomorphism and multiplication by a non-zero function having a same sign as the determinant of the diffeomorphism) and the corresponding notion on the germ level. Two families of vector fields are C^∞ equivalent if and only if the dual families of 1-forms are C^∞ equivalent.

Let us now start with a k-parameter family X_λ with, as unique hypothesis:

(*Hyp* 1) $J^1X_0(0)$ is linearly conjugate to $y\, \partial/\partial x$.

So, up to linear conjugacy, we may suppose $j^1X_0(0) = y\, \partial/\partial x$. This condition defines an algebraic subset of codimension 4 in $]_0^1 V$ and as we know from **[A2]**, **[B1]**, **[T2]** the family X_λ can be put in the following normal form by C^∞ equivalence (even C^∞ conjugacy).

$$X_\lambda \sim y\frac{\partial}{\partial x} + [F(x,\lambda) + yG(x,\lambda)]\frac{\partial}{\partial y} + Q_1\frac{\partial}{\partial x} + Q_2\frac{\partial}{\partial y} \tag{2}$$

where $\sim$ is C^∞ equivalence, Q_1 and Q_2 are of order $O((\|m\| + \|\lambda\|)^N)$ for a certain N that one can choose arbitrarily big, $m = (x, y)$; $\|\ \|$ are any norms on $\mathbb{R}^2$ and $\mathbb{R}^k$.

$F(x, \lambda)$ and $G(x, \lambda)$ are C^∞ functions in (x, λ) and we may suppose that they are polynomials of degree N. The equation for the orbits of (2) is

$$\begin{cases} \dot{x} = y & + Q_1 \\ \dot{y} = F(x,\lambda) + yG(x,\lambda) + Q_2 \end{cases} \tag{2'}$$

1*st step*: *Reduction to a differential equation of* 2*nd order.*

Let us consider the λ-dependent coordinate change

$$Y = y + Q_1, \qquad X = x \tag{3}$$

Equation (2′) transforms into:

$$\begin{cases} \dot{X}=Y \\ \dot{Y}=\dot{y}+\dot{Q}_1=F(X,\lambda)+(Y-Q_1)G(X,\lambda)+\dfrac{\partial Q_1}{\partial x}Y \\ \qquad +\dfrac{\partial Q_1}{\partial y}(F+(Y-Q_1)G+Q_2)+Q_2 \end{cases} \tag{4}$$

This gives

$$\begin{aligned} &\dot{X}=Y \\ &\dot{Y}=F(X,\lambda)+YG(X,\lambda)+Q_2'(X,Y,\lambda) \end{aligned} \tag{5}$$

where $Q_2'(X,Y,\lambda)=O((\|M\|+\|\lambda\|)^{N-1})$, $M=(X,Y)$. Changing $N-1$ to N and (X,Y) to (x,y) we regain the expression (2′) with $Q_1\equiv 0$.

Using C^∞ equivalence we changed the original family of differential equations into a parameter dependent differential equation of second order.

$$\ddot{x}=F(x,\lambda)+\dot{x}G(x,\lambda)+Q(x,\dot{x},\lambda) \tag{6}$$

where Q is of order N, $F(0,0)=0$, $G(0,0)=0$, $\partial F/\partial x(0,0)=0$.

2*nd step*: *Division of the term Q by y^2.*

We develop the function Q in powers of y:

$$Q(x,y,\lambda)=\tilde{F}(x,\lambda)+y\tilde{G}(x,\lambda)+y^2\tilde{Q}(x,y,\lambda),$$

where $\tilde{F}$ is of order N, $\tilde{G}$ of order $N-1$ and $\tilde{Q}$ of order $N-2$. Changing $N-2$ to N, and $F+\tilde{F}$, $G+\tilde{G}$, $Q+\tilde{Q}$ to resp. F, G and Q we obtain

$$\ddot{x}=F(x,\lambda)+\dot{x}G(x,\lambda)+(\dot{x})^2Q(x,\dot{x},\lambda) \tag{7}$$

where Q is of order N and $F(0,0)=G(0,0)=(\partial F/\partial x)(0,0)=0$.

We now introduce the second hypothesis:

(*Hyp* 2) $(\partial^2F/\partial x^2)(0,0)\neq 0$

The hypotheses (1) and (2) define the submanifold Σ_C^2 in the space of 2-jets.

In the next two steps we only pay attention to the vector field X_0 which up to now has the expression

$$X_0=y\frac{\partial}{\partial x}+(F(x,0)+yG(x,0)+y^2Q(x,y,0))\frac{\partial}{\partial y} \tag{8}$$

3*rd step*: *Reduction of $F(x,0)$ to x^2.*

The dual form ω_0 of X_0 is

$$\omega_0=y\,dy-(F(x,0)+yG(x,0)+y^2Q(x,y,0))\,dx. \tag{9}$$

As $\partial^2F(0,0)/\partial x^2\neq 0$ there exists a local diffeomorphism around the origin in the x-axis: $X=u(x)$, with $u(0)=0$, so that

$$X^2\,dX=F(x,0)\,dx. \tag{10}$$

If $\partial u(0)/\partial x<0$, we take simultaneously the change $Y=-y$ otherwise we take $Y=y$. The change $(x,y)\to(X,Y)$ does not affect the form of the expression (7) but makes $F(x,0)$ equal to x^2. (We again change X, Y into x, y.)

We now introduce the third hypothesis:

$(Hyp\ 3)$: $\dfrac{\partial G}{\partial x}(0,0)=0$

The hypotheses (1), (2) and (3) define the semi-algebraic set Σ_C^3, having codimension 5. We shall finally impose a last genericity condition on Σ_C^3, defining $\Sigma_{C^+}^3 \cup \Sigma_{C^-}^3$.

Up to now the 4-jet of X_0 has the expression

$$j^4X_0(0)=y\frac{\partial}{\partial x}+(x^2+y(\alpha x^2+\beta x^3))\frac{\partial}{\partial y}. \tag{11}$$

4th step: Reduction of $j^4X_0(0)$ *to* $y\dfrac{\partial}{\partial x}+(x^2+yx^3)\dfrac{\partial}{\partial y}$.

LEMMA 1. *Let X_0 be a germ of a vector field with*

$$j^4X_0(0)=y\frac{\partial}{\partial x}+(x^2+y(\alpha x^2+\beta x^3))\frac{\partial}{\partial y}.$$

Then X_0 is C^∞ equivalent to a germ of a vector field having as 4-jet: $y\,\partial/\partial x+(x^2+\beta x^3y)\,\partial/\partial y$ (with the same value of β).

Proof: We'll work with the dual form ω_0 of X_0:

$$j^4\omega_0(0)=y\,dy-(x^2+y(\alpha x^2+\beta x^3))\,dx \tag{12}$$

and we put $H(x,y)=\frac{1}{2}y^2-\frac{1}{3}x^3$; $dH=y\,dy-x^2\,dx$ so

$$yx^2\,dx=y^2\,dy-y\,dH \tag{13}$$

Substitute (13) into (12):

$$j^4\omega_0(0)=(1+\alpha y)\,dH-\alpha y^2\,dy-\beta y^3\,dx; \tag{14}$$

it follows that:

$$\frac{\omega_0}{1+\alpha y}=dH-\frac{\alpha y^2}{1+\alpha y}\,dy-\frac{\beta y}{1+\alpha y}x^3\,dx+O(\|m\|^5) \tag{15}$$

A coordinate change of the form $X=x$, $Y=y+\cdots$ can transform the exact 1-form $dH-(\alpha y^2/1+\alpha y)\,dy$ into dH while the 1-form $yx^3\,dx$ remains unchanged up to terms of order 5. This proves the result.

We can now define the subset $\Sigma_{C^+}^3$ as the set of germs X_0 whose 4-jet in 0 is C^∞ equivalent to

$$y\frac{\partial}{\partial x}+(x^2+y(\alpha x^2+\beta x^3))\frac{\partial}{\partial y} \qquad \text{with } \beta>0,$$

and $\Sigma_{C^-}^3$ as the set of germs X_0 whose 4-jet in 0 is C^∞ equivalent to the same jet, with $\beta<0$. (One can also introduce the subset Σ_C^4 of codimension 6 determined by the condition $\beta=0$; $\Sigma_C^3=\Sigma_{C^+}^3\cup\Sigma_{C^-}^3\cup\Sigma_C^4$.)

By a linear change of coordinates, we can now reduce, up to C^∞ equivalence, a 4-jet of the form

$$y\frac{\partial}{\partial x}+(x^2+\beta yx^3)\frac{\partial}{\partial y} \qquad \text{with } \beta\neq 0,$$

to an expression

$$y\frac{\partial}{\partial x}+(x^2\pm yx^3)\frac{\partial}{\partial y}$$

(where $\pm$ depends on the sign of β). Lemma 1 shows that $X_0\in\Sigma^3_{C^+}$ iff $j^4X_0(0)\sim y\,\partial/\partial x+(x^2+yx^3)\,\partial/\partial y$ while $X_0\in\Sigma^3_{C^-}$ iff $j^4X_0(0)\sim y\,\partial/\partial x+(x^2-yx^3)\,\partial/\partial y$.
Our last hypothesis on X_0 is precisely:

(*Hyp* 4): $X_0\in\Sigma^3_{C^\pm}$.

Performing on the family X_λ the changes made in lemma 1, we obtain, up to C^∞ equivalence, the following expression for the family X_λ:

$$X_\lambda=y\frac{\partial}{\partial x}+(F(x,\lambda)+yK(y)G(x,\lambda)+y^2Q(x,y,\lambda))\frac{\partial}{\partial y}, \tag{16}$$

with the following conditions:

$$\begin{aligned}&F(x,0)=x^2,\quad K(y)\text{ is a }C^\infty\text{ function in }y\text{ with K}(0)=1,\\&G(x,0)=\pm x^3+O(x^4)\quad\text{and }Q(x,y,\lambda)\text{ has order }N\text{ in }(x,y,\lambda).\end{aligned}\tag{17}$$

5*th step*: *Reduction of* $F(x,\lambda)$ *to* $x^2+\mu(\lambda)$.

We already have $F(x,0)\,dx=d(x^3/3)$ and the germ of the function $x^3/3$ admits as universal unfolding $(x^3/3)+\mu x$ (fold). There hence exists a differentiable mapping $\mu(\lambda)$ and a family of diffeomorphisms depending on the parameter λ:

$$u_\lambda(x)=x+O(x^2)+O(\|\lambda\|)$$

such that

$$u_\lambda^*(F(x,\lambda)\,dx)=(x^2+\mu(\lambda))\,dx\qquad\text{with }\mu(0)=0.$$

Performing this same C^∞ equivalence $(\mu(\lambda),(x,y)\to(u_\lambda(x),y))$ to the dual family ω_λ, we get:

$$\omega_\lambda\sim y\,dy-[(\mu(\lambda)+x^2)+yK(y)\tilde G(x,\lambda)+y^2\tilde Q(x,y,\lambda)]\,dx \tag{18}$$

As $\tilde G\,dx=u_\lambda^*(G\,dx)$, $\tilde Q\,dx=u_\lambda^*(Q\,dx)$, $u(x,0)=x+O(x^2)$ and $u(x,\lambda)=O((\|x\|+\|\lambda\|))$, the functions $\tilde G$, $\tilde Q$ have the same properties as G and Q in (16): $\tilde G(x,0)=\pm x^3+O(x^4)$ and $\tilde Q(x,y,\lambda)=O((\|m\|+\|y\|)^N)$. So we obtain (simplifying the notation).

$$X_\lambda\sim y\frac{\partial}{\partial x}+((\mu(\lambda)+x^2)+yK(y)G(x,\lambda)+y^2Q(m,\lambda))\frac{\partial}{\partial y}, \tag{19}$$

with $G(x,0)=\pm x^3+O(x^4)$, $\mu(0)=0$ and Q of order N.

6*th step*: *Genericity condition on the family* X_λ.

Let us consider the expression (19). The function $G(x,\lambda)$ can be developed in powers of x:

$$G(x,\lambda)=\nu_0(\lambda)+\nu_1(\lambda)x+\nu_2(\lambda)x^2+\nu_3(\lambda)x^3+x^4h(x,\lambda), \tag{20}$$

where ν_0,ν_1,ν_2,ν_3 are C^∞ functions in λ and h is C^∞ in (x,λ). Because of our hypotheses: $\nu_0(0)=\nu_1(0)=\nu_2(0)=0$ and $\nu_3(0)=\pm1$. Using a transformation of the form $X=u(\lambda)x$, $Y=v(\lambda)y$ it is easy to reduce $\nu_3(\lambda)$ to ±1; from now on we take this for granted.

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Let us now suppose that $\lambda \in \mathbb{R}^3$: $\lambda = (\lambda_1, \lambda_2, \lambda_3)$. The transversality of j^4X_λ with respect to $\Sigma^3_{C^\pm}$ expresses itself as

(*Hyp* 5) $\dfrac{D(\mu, \nu_0, \nu_1)}{D(\lambda_1, \lambda_2, \lambda_3)}(0) \neq 0.$

Under this condition (being our genericity condition) we may choose $\lambda = (\mu, \nu_0, \nu_1)$, after a change in the parameter space.

We finally obtain the definitive normal form: (obtained using C^∞ equivalence):

$$X_\lambda = y\frac{\partial}{\partial x} + [x^2 + \mu + yK(y, \lambda)(\nu_0 + \nu_1 x + \alpha(\lambda)x^2 \pm x^3 + x^4 h(x, \lambda)) + y^2 Q(x, y, \lambda)]\frac{\partial}{\partial y}, \tag{21}$$

where $\lambda = (\mu, \nu_0, \nu_1)$ is the parameter, $K(y, \lambda)$ is a C^∞ function in (y, λ) with $K(0, \lambda) = 1$, $\alpha(\lambda)$ is a C^∞ function in λ with $\alpha(0) = 0$, $h(x, \lambda)$ and $Q(x, y, \lambda)$ are C^∞ functions and Q is of order N in (c, y, λ), where N is arbitrarily high.

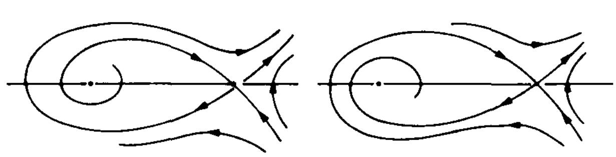
4. *Study of the generic family* X_λ

Consider a generic family of vector fields X_λ in the normal form (21) obtained in § 3. As we remarked in the introduction, it suffices to study the case: $X_0 \in \Sigma^3_{C^+}$ (i.e. the case where the coefficient of $yx^3\,\partial/\partial y$ is equal to $+1$).

First, it is very easy to study the bifurcation of the critical points (the zeros of the vector fields X_λ). Indeed, a point $m = (x, y)$ will be critical iff

$$y = 0, \qquad \mu + x^2 = 0 \tag{1}$$

Hence, the vector field X_λ has (locally) no critical points for $\mu > 0$ and two critical points for $\mu < 0$: $e_\lambda = (-\sqrt{-\mu}, 0)$ and $s_\lambda = (\sqrt{-\mu}, 0)$. It is easy to verify that e_λ is a node or a focus, while s_λ is a saddle point. We note also that the segment $]e_\lambda, s_\lambda[$ is transverse to X_λ.

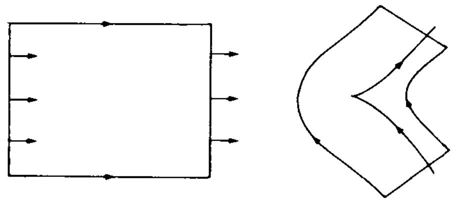


The vector field X_λ for a value $\mu < 0$ close to $\mu = 0$.

FIGURE 6

Possibly after modification of X_λ in the complement of some neighbourhood of $0 \in \mathbb{R}^2$ (which is irrelevant since we are interested in the germ of the family in $(0, 0)$), we may suppose that $X_\lambda \equiv X_0$ when $\|m\|$ is sufficiently big and that there exists a fixed neighbourhood A of $0 \in \mathbb{R}^2$ and a fixed neighbourhood B of $0 \in \mathbb{R}^3$ (parameter space) which we may choose arbitrarily small, so that:

(1) A is diffeomorphic to a rectangle, $X_\lambda \equiv X_0$ in a neighbourhood of ∂A, X_0 is tangent to ∂A on two opposite sides of ∂A and transverse with respect to ∂A on the two other sides, pointing resp. inward and outward.

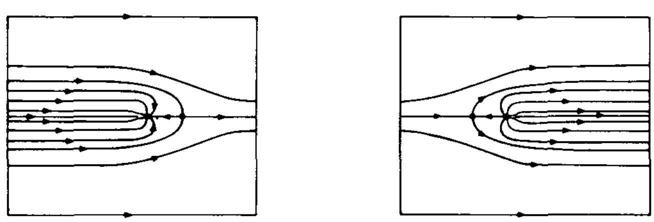


A as a rectangle choice of A

FIGURE 7

(2) The critical points of X_λ, if there exist critical points near $0 \in \mathbb{R}^2$, belong to A.

From now on we restrict the study of our family X_λ to $A \times B$. When $\lambda \in B \cap \{\mu > 0\} = B_+$ the topological type of X_λ inside A is trivial ($X_\lambda | A$ is equivalent to a 'flow box'). When $\lambda \in B \cap \{\mu < 0\} = B_-$ there may exist a certain number of cycles inside A. Each of these cycles will then transversally cut the segment $]e_\lambda, s_\lambda[$ and will border a disc containing e_λ in its interior and s_λ in its exterior. Let us also remark that the topological type of X_λ, up to C^0 equivalence, only depends on the number and the nature of the critical elements (critical points and cycles): as a matter of fact, pinching down the biggest cycle - the one that borders a disc containing the other cycles - to the point e_λ, one obtains a vector field on A having as unique critical elements the points e_λ and s_λ and satisfying condition (1) on ∂A. Such a vector field can only have one of the two topological types shown in figure 8:



e_λ is a sink e_λ is a source

FIGURE 8

In case the vector field has k cycles, which we suppose to be hyperbolic to fix our ideas, then the topological type of X_λ will only depend on the number k and on the nature of e_λ. For example, if e_λ is a source and $k = 2$, the smallest cycle will be stable, the biggest one unstable and the C^0 phase portrait of X_λ will look like:

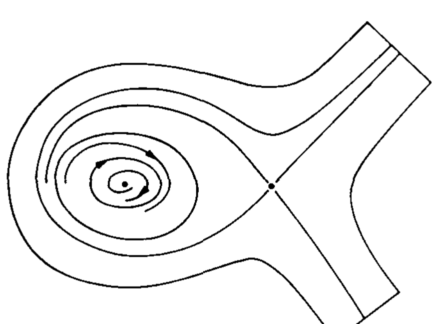


FIGURE 9

The nature of the critical points e_λ and s_λ is very easy to determine in the expression

$$X_\lambda = y\frac{\partial}{\partial x} + [x^2 + \mu + yK(y, \lambda)G(x, \lambda) + y^2 Q(x, y, \lambda)]\frac{\partial}{\partial y}, \tag{2}$$

with $G(x, \lambda) = \nu_0 + \nu_1 x + \alpha(\lambda)x^2 + x^3 + x^4 h(x, \lambda)$.

Let $x_\mu = \pm\sqrt{-\mu}$ be the abscissa of the resp. critical points. We look for the 1-jet of X_λ in the point $m_\mu = (x_\mu, 0)$. Putting $x = x_\mu + X$ and $y = Y$ we get

$$j^1 X_\lambda(m_\mu) = Y\frac{\partial}{\partial X} + (2x_\mu X + YG(x_\mu, \lambda))\frac{\partial}{\partial Y}. \tag{3}$$

The representative matrix of $j^1 X_\lambda(m_\mu)$ is equal to:

$$\begin{pmatrix} 0 & 1 \\ 2x_\mu & G(x_\mu, \lambda) \end{pmatrix} \tag{4}$$

and the equation of the eigenvalues reads:

$$\zeta^2 - G(x_\mu, \lambda)\zeta - 2x_\mu = 0. \tag{5}$$

For the point $s_\lambda = (\sqrt{-\mu}, 0)$, the product of the eigenvalues is negative: s_λ is a saddle point, as we mentioned before. For the point $e_\lambda = (-\sqrt{-\mu}, 0)$, this product is positive: e_λ is a node or a focus. The nature of e_λ (sink or source) depends on the sign of the trace $T(\lambda) = G(x_\mu, \lambda)$. We write $\gamma = -\sqrt{-\mu}$. The trace $T(\lambda)$ admits a development of the form

$$T(\lambda) = \nu_0 + \nu_1\gamma + \gamma^3 + \gamma^2 O(\|\lambda\|). \tag{6}$$

The equation $T(\lambda) = 0$ defines in B (if B is sufficiently small) a surface Σ_H contained in the half ball B_-. Observe that this surface, at least outside 0, is tangent to the plane $\{\mu = 0\}$, along the axis $0\nu_1$ having a quadratic contact; on the other hand Σ_H is transverse to the spheres $\|\lambda\|^2 = \varepsilon$ for $\varepsilon > 0$ sufficiently small. It is hence a cone on the intersection with one of those spheres. We will later on come back to this surface. Let us for the moment merely observe that if one crosses this surface transversally, that the trace at the point e_λ will annihilate in a regular way (the eigenvalues in e_λ are complex conjugate and cross the imaginary axis regularly). In other words we may expect that the point e_λ undergoes a Hopf bifurcation. We will make this precise later on, but it is possible to prove it directly by calculating a sufficient jet of X_λ at e_λ.

The only difficult problem is the determination of the number and the nature of the cycles, including their limit position as a saddle connection. In order to solve this equation, we'll use a *perturbation method* after applying a *blowing up* in the space $\mathbb{R}^2$ of variables (x, y) as well as in the space $\mathbb{R}^3$ of parameters (μ, ν_0, ν_1). More precisely, we use the following change of coordinates and of parameters

$$\Phi_m: \begin{cases} x = t^2\bar{x} \\ y = t^3\bar{y} \end{cases} \qquad \Phi_p: \begin{cases} \mu = t^4\bar{\mu} \\ \nu_0 = t^6\bar{\nu}_0 \\ \nu_1 = t^4\bar{\nu}_1 \end{cases} \tag{7}$$

We put $\bar{\lambda} = (\bar{\mu}, \bar{\nu}_0, \bar{\nu}_1)$; if $t \in]0, T]$ for some T and $\bar{\lambda} \in S^2$, then the parameter λ describes a neighbourhood B of 0 in $\mathbb{R}^3$.

Consider the dual form ω_λ of X_λ:

$$\omega_\lambda = y\,dy - [\mu + x^2 + yK(y,\lambda)[\nu_0 + \nu_1 x + \alpha x^2 + x^3 + x^4 h] + y^2 Q]\,dx. \tag{8}$$

This 1-form is the sum of a Hamiltonian part, dH_μ, with

$$H_\mu = \tfrac{1}{2}y^2 - \left(\mu x + \frac{x^3}{3}\right) \tag{9}$$

and a 'dissipative' part.

The change given in (7) transforms ω_λ into:

$$\frac{1}{t^6}\omega_\lambda = \bar\omega_{\bar\lambda,t} = d\bar H_{\bar\mu} - t^5\bar y(1 + t\theta(\bar y, t, \lambda))(\bar\nu_0 + \bar\nu_1\bar x + t^2\bar\alpha\bar x^2 + \bar x^3 + t^2\bar x^4\bar h)\,d\bar x + t^{2N+2}\bar y^2\bar Q\,d\bar x, \tag{10}$$

where $\theta(\bar y, t, \lambda)$ is a C^∞ function of order $O(\bar y)$, where $\bar h$ is a C^∞ function in $\bar x, t, \bar\lambda$, where $\bar\alpha$ is a C^∞ function in $\bar\lambda$, t and $\bar Q$ is a C^∞ function in $\bar x, \bar y, t, \bar\lambda$.

We will use (10) to study ω_λ in conic sectors around the axes 0μ and $0\nu_1$. In fact we'll work with $\bar\omega$ in fixed domains $\bar A$ in the space $(\bar x, \bar y)$; when $t \to 0$, the image $A_\lambda = \Phi_m(\bar A)$ has a diameter tending to 0 in the space of initial variables (x, y). This could give problems since we need to study X_λ in a fixed domain A; we'll come back to this in § 4.5.

4.1. *The behaviour of X_λ in a sector around the axis* 0μ. We take $\bar\mu = -1$ and $\bar\nu = (\bar\nu_0, \bar\nu_1) \in \mathbb{R}^2$. The 1-form $\bar\omega$, depending on $\bar\nu$ and t is

$$\bar\omega_{\bar\nu,t} = dH - t^5\bar y(\bar\nu_0 + \bar\nu_1\bar x + \bar x^3)\,d\bar x + O(t^6)\bar y\,d\bar x \tag{11}$$

with

$$H = \bar H_{-1} = \tfrac{1}{2}\bar y^2 + \bar x - \frac{\bar x^3}{3}.$$

The symbol $O(t^6)$ in (11) stands for a C^∞ function in all variables and of order 6 in t. In the expression (11), the form $\bar\omega_{\bar\nu,t}$ appears as a perturbation of order $O(t^5)$ of the exact form dH. The function H is a Morse function with two critical points: a centre at $e = (-1, 0)$ and a saddle point at $s = (1, 0)$:

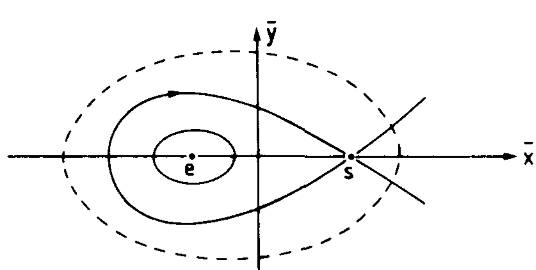


FIGURE 10

We fix a compact neighbourhood $\bar A$ of the singular disc defined by the saddle connection (in s). Let $\bar X_{\bar\nu,t}$ be the vector field dual to $\bar\omega_{\bar\nu,t}$. For $t = 0$, this vector field is the hamiltonian vector field of H: $\bar X = \bar X_{\bar\nu,0}$. For any $(\bar\nu, t)$, the vector field $\bar X_{\bar\nu,t}$ is transverse to $]e, s[$. Therefore, if one fixes any compact subset K of $\mathbb{R}^2(\bar\nu)$,

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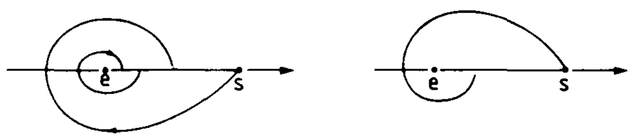
there exists some value $T(K)>0$ such that if $(t, \bar{\nu})\in[0, T(K)]\times K$, the vector field $\bar{X}_{\bar{\nu},t}$ admits a first return mapping (or Poincaré mapping) with respect to the segment $[e, s]$: $P_{\bar{\nu},t}$.

As a matter of fact either $P_{\bar{\nu},t}$ is defined on the whole segment $[e, s]$ or $P^{-1}_{\bar{\nu},t}$ is, as we can see here:



$P_{\bar{\nu},t}$ defined on $[e, s]$ $\quad$ $P^{-1}_{\bar{\nu},t}$ defined on $[e, s]$

FIGURE 11

The study of the cycles for small values of t can be done with the help of the following well known lemma.

LEMMA 1 (Perturbation lemma). *Let $\bar{\omega}_{\bar{\nu},t}$ be the 1-form given in* (11) *and let K be a compact subset of $\mathbb{R}^2$, the space of parameters $\bar{\nu}$; then there exists a value $T(K)>0$ such that for all $(t, \bar{\nu})\in[0, T(K)]\times K$ we have the following properties:*

(i) *$\bar{\omega}_{\bar{\nu},t}$ admits e, s as the only critical points inside $\bar{A}$: e is a focus and s a saddle point; the first return mapping $P_{\bar{\nu},t}$ on $[e, s]$ or its inverse $P^{-1}_{\bar{\nu},t}$ is defined on the entire segment $[e, s]$.*

(ii) *If we parametrize $[e, s]$ using the value of the function H we obtain a parametrization by $b\in[-\frac{2}{3}, \frac{2}{3}]$ $(H(e)=-\frac{2}{3}, H(s)=\frac{2}{3})$ which is regular on $]-\frac{2}{3}, \frac{2}{3}[$.*
For this parametrization, the mapping $P_{\bar{\nu},t}$ has the following development on its domain of definition:

$$P_{\bar{\nu},t}(b)=b+t^5\int_{\gamma_b}\omega_D+o(t^5), \tag{12}$$

where γ_b is the compact component of $\{H=b\}$, clockwise oriented for the integration, $\omega_D=\bar{y}(\bar{\nu}_0+\bar{\nu}_1 x+\bar{x}^3)\,d\bar{x}$, and $o(t^5)$ stands for a C^∞ function in b, $\bar{\nu}$, t, of order $o(t^5)$.

In order to calculate $\int_{\gamma_b}\omega_D$ we introduce the 1-forms

$$\omega_i=\bar{y}\bar{x}^i\,d\bar{x} \qquad \text{with } i>0. \tag{13}$$

By this: $\omega_D=\bar{\nu}_0\omega_0+\bar{\nu}_1\omega_1+\omega_3$. For the corresponding integrals we use the notation:

$$I_i(b)=\int_{\gamma_b}\omega_i, \tag{14}$$

and we write

$$G(b, \bar{\nu})=\bar{\nu}_0 I_0(b)+\bar{\nu}_1 I_1(b)+I_3(b). \tag{15}$$

The mapping $P_{\bar{\nu},t}(b)$ becomes:

$$P_{\bar{\nu},t}(b)=b+t^5 G(b, \bar{\nu})+o(t^5) \tag{16}$$

The cycles contained in $\bar{A}$ cut $[e, s]$ at the values of b which are fixed points of $P_{\bar{\nu},t}$. The equation of these fixed points is

$$0=P_{\bar{\nu},t}(b)-b=t^5 G(b, \bar{\nu})+o(t^5), \tag{17}$$

or otherwise said

$$G(b, \bar{\nu}) + \varepsilon(t) = 0, \tag{18}$$

where $\varepsilon(t)$ is a C^∞ function in all variables, tending to zero when $t \to 0$. The limiting position of the cycles, for $t \to 0$, is hence given by the solution of the equation:

$$G(b, \bar{\nu}) = 0. \tag{19}$$

We now first pay attention to this equation.

4.1(a). *Study of the surface* $\{G(b, \bar{\nu}) = 0\}$. Equation (19) defines a surface in the space (b, ν_0, ν_1). For $b = -\frac{2}{3}$, the three integrals I_0, I_1, I_3 become zero; however as $I_0(b) > 0$ for $b \neq -\frac{2}{3}$ and as I_1/I_0 and $I_3/I_0 \to -1$ for $b \to -\frac{2}{3}$ we can remove the degeneracy of equation (19) at $b = -\frac{2}{3}$ by changing (19) into the equivalent equation:

$$\tilde{G}(b, \bar{\nu}) = \frac{1}{I_0} G = 0. \tag{20}$$

The function $\tilde{G}(b, \bar{\nu})$ is written as

$$\tilde{G}(b, \bar{\nu}) = \bar{\nu}_0 - P(b)\bar{\nu}_1 - Q(b) \tag{21}$$

with

$$\begin{cases} P(b) = -\dfrac{I_1}{I_0} \quad \text{and} \quad Q(b) = -\dfrac{I_3}{I_0} \quad \text{for} \quad b \neq -\frac{2}{3} \\ P(-\frac{2}{3}) = Q(-\frac{2}{3}) = 1 \end{cases} \tag{22}$$

We denote by Σ the ruled surface defined by $\tilde{G}(b, \bar{\nu}) = 0$ for $b \in [-\frac{2}{3}, \frac{2}{3}]$. For each value b, the equation $\tilde{G} = 0$ defines a line Δ_b in the space of parameters $\bar{\nu}$ and the surface Σ is diffeomorphic to a strip $[-\frac{2}{3}, \frac{2}{3}] \times \mathbb{R}$ embedded in $\mathbb{R}^3$. The limiting lines, defined by $b = -\frac{2}{3}, \frac{2}{3}$ are projections of the boundary of Σ. Their respective equations are:

$$\text{for } h = \Delta_{-\frac{2}{3}}: \quad \bar{\nu}_0 = \bar{\nu}_1 + 1, \tag{23}$$

$$\text{for } c = \Delta_{\frac{2}{3}}: \quad \bar{\nu}_0 = \tfrac{5}{7}\bar{\nu}_1 + \tfrac{103}{77}. \tag{24}$$

(The coefficients in the equation of the line c can easily be calculated by integrating $\omega_0, \omega_1, \omega_3$ on the saddle connection, but one can also obtain their value without calculation as we'll see a bit later.)

We will now show that the surface Σ admits a critical locus of fold type, Σ_c, with as projection on the $\bar{\nu}$-space a simple arc l joining a point $h_2 \in h$ to a point $c_2 \in c$. Recall that the critical locus Σ_c of Σ for the projecton on the $\bar{\nu}$-plane has as equation:

$$\tilde{G}(b, \bar{\nu}) = \frac{\partial \tilde{G}}{\partial b}(b, \bar{\nu}) = 0;$$

Σ_c is of fold type, if at all points of Σ_c:

$$\frac{\partial^2 \tilde{G}}{\partial b^2}(b, \bar{\nu}) \neq 0.$$

We are first going to express $\tilde{G}$ as a function merely of P, using the fact that the 2 forms ω_0 and ω_1 generate the cohomology of $\mathbb{R}^2 \backslash \{s, e\}$ relative to dH. More explicitly, we have the following expression of ω_3 (this expression was suggested to us by H. Zoładek).

LEMMA 2.
$$\tfrac{11}{2}\,\omega_3 = -3H\omega_0 + \tfrac{15}{2}\omega_1 + \tfrac{3}{2}\,d(\bar{x}\bar{y}^3) - \tfrac{9}{2}\bar{x}\bar{y}\,dH. \tag{25}$$

Proof. Recalling that $H(\bar{x},\bar{y}) = \frac{1}{2}\bar{y}^2 + \bar{x} - \frac{1}{3}\bar{x}^3$, we get
$$\omega_3 = \bar{y}\bar{x}^3\,d\bar{x} = -3H\omega_0 + 3\omega_1 + \tfrac{3}{2}\bar{y}^3\,d\bar{x}. \tag{26}$$
Now
$$\bar{y}^3\,d\bar{x} = d(\bar{x}\bar{y}^3) - 3\bar{x}\bar{y}^2\,d\bar{y}. \tag{27}$$
From $dH = \bar{y}\,d\bar{y} + (1-\bar{x}^2)\,d\bar{x}$ we obtain:
$$-3\bar{x}\bar{y}^2\,d\bar{y} = -3\bar{x}\bar{y}(\bar{x}^2-1)\,d\bar{x} - 3\bar{x}\bar{y}\,dH, \tag{28}$$
hence
$$-3\bar{x}\bar{y}^2\,d\bar{y} = -3\omega_3 + 3\omega_1 - 3\bar{x}\bar{y}\,dH \tag{29}$$
Combining the equalities (26), (27) and (29) we find:
$$\omega_3 = -3H\omega_0 + 3\omega_1 + \tfrac{9}{2}\omega_1 - \tfrac{9}{2}\omega_3 - \tfrac{9}{2}\bar{x}\bar{y}\,dH + \tfrac{3}{2}\,d(\bar{x}\bar{y}^3) \tag{30}$$
eventually giving the required formula. □

The lemma implies that
$$\tfrac{11}{2}I_3 = -3bI_0 + \tfrac{15}{2}I_1, \tag{31}$$
therefore
$$Q(b) = \tfrac{6}{11}b + \tfrac{15}{11}P(b), \tag{32}$$
and
$$\tilde{G}(b,\bar{\nu}) = \bar{\nu}_0 - P(b)\bar{\nu}_1 - \tfrac{6}{11}b - \tfrac{15}{11}P(b). \tag{33}$$
From this formula for $\tilde{G}(b,\bar{\nu})$ it follows that
$$\frac{\partial\tilde{G}}{\partial b}(b,\bar{\nu}) = -P'(b)\bar{\nu}_1 - \tfrac{6}{11} - \tfrac{15}{11}P'(b), \tag{34}$$
$$\frac{\partial^2\tilde{G}}{\partial b^2}(b,\bar{\nu}) = -P''(b)\cdot(\bar{\nu}_1 + \tfrac{15}{11}), \tag{35}$$
with $P'(b) = \partial P(b)/\partial b$ and $P''(b) = \partial^2 P(b)/\partial b^2$.

In his work on 2-parameter families, R. I. Bogdanov shows that for all $b \in [-\frac{2}{3}, \frac{2}{3}[$: $P'(b) < 0$, (we'll also prove this a little later). Equation (34) permits us to calculate $\bar{\nu}_1$ along the critical locus:
$$\bar{\nu}_1(b) = -\frac{6}{11}\frac{1}{P'(b)} - \frac{15}{11}. \tag{36}$$
At each point $b_0 \in]-\frac{2}{3}, \frac{2}{3}[$ where
$$\tilde{G}(b_0,\bar{\nu}) = \frac{\partial\tilde{G}}{\partial b}(b_0,\bar{\nu}) = 0, \tag{37}$$
we have (bringing (36) into the expression (35)) that
$$\frac{\partial^2\tilde{G}}{\partial b^2}(b_0,\bar{\nu}) = \frac{6}{11}\frac{P''(b_0)}{P'(b_0)} \tag{38}$$

Consequently, the critical locus Σ_c consists of fold-points if we can prove that $P''(b) \neq 0$ for all $b \in]-\frac{2}{3}, \frac{2}{3}[$.

We remark also that if we differentiate the expression of $\bar{\nu}_1(b)$ given in (36), that we get

$$\frac{d\bar{\nu}_1}{db}(b) = \frac{6}{11} \frac{P''(b)}{(P'(b))^2} \tag{39}$$

Hence, the same condition $P''(b) \neq 0$ for all $b \in]-\frac{2}{3}, \frac{2}{3}[$ ensures that the component $\bar{\nu}_1(b)$ of Σ_c and henceforth the component $\bar{\nu}_1(b\ )$ of l is regular: by this l is a regular curve. (Later on, we'll show that l is the graph of a convex function $\bar{\nu}_0(\bar{\nu}_1)$.) Let us however first show that for all $b \in [-\frac{2}{3}, \frac{2}{3}[$: $P''(b) < 0$. We also recall that the analogous result on the first derivative has been proven by Bogdanov [**B2**], and after that by Il' Yashenko [**I**] (see also [**K, H**]).

We prove the two properties of P simultaneously.

THEOREM 3. *The function* $P(b) = -I_1(b)/I_0(b)$, *where* I_1, I_0 *are the elliptic integrals defined in* (14) *satisfies*: $P'(b) < 0$ *and* $P''(b) < 0$ *for all* $b \in [-\frac{2}{3}, \frac{2}{3}[$.

Remark. Using the proof given below, one can easily demonstrate that P' and P'' tend to $-\infty$ for $b \to +\frac{2}{3}$.

Proof. The function $P(b)$ is a solution of the differential equation

$$(9b^2 - 4)P' = 7P^2 + 3bP - 5, \tag{40}$$

as we show in the appendix 1. Therefore the graph of $P(b)$ belongs to an orbit of the following vector field Z defined on the space $\mathbb{R}^2$ of coordinates (b, P):

$$Z = (4 - 9b^2)\frac{\partial}{\partial b} - (7P^2 + 3bP - 5)\frac{\partial}{\partial P} \tag{41}$$

This vector field has 4 critical points:

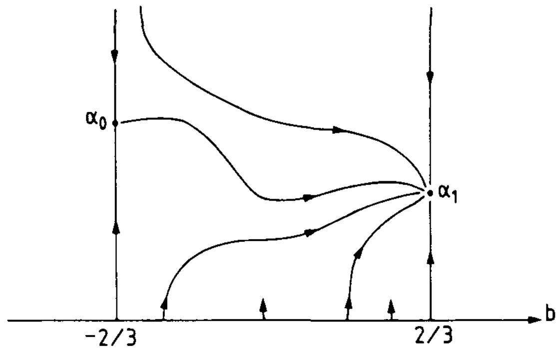
$$\alpha_0 = (-\tfrac{2}{3}, 1), \quad \alpha_1 = (\tfrac{2}{3}, \tfrac{5}{7}), \quad \alpha_0' = (-\tfrac{2}{3}, -\tfrac{5}{7}), \quad \alpha_1' = (-\tfrac{2}{3}, -1);$$

and admits the lines $\Delta_0 = \{b = -\frac{2}{3}\}$ and $\Delta_1 = \{b = \frac{2}{3}\}$ as invariant lines. Along these lines Z is normally hyperbolic and in restriction to Δ_0 and Δ_1 the critical points are also hyperbolic.

The 4 critical points are hence hyperbolic and one easily checks that α_0 and α_1' are saddle points, while α_0' and α_1 are nodes, respectively unstable and stable. The phase portrait of Z in the vertical strip $B = \{P \geq 0, -\frac{2}{3} \leq b \leq \frac{2}{3}\}$ can now easily be obtained taking into account the value of the vertical component of Z when $P = 0$ and when P is big (figure 12).

In particular we notice the existence of a unique Z-orbit lying in the interior of B and having the saddle point $\alpha_0 = (-\frac{2}{3}, 1)$ as an α-limit point; it is the unstable separatrix Γ of α_0, which tends to α_1 for $t \to \infty$.

As we noticed already that $P(b) \to 1$ for $b \to -\frac{2}{3}$ it follows that the graph of $P(b)$ is equal to Γ (of course, this implies that $P(b) \to \frac{5}{7}$ for $b \to \frac{2}{3}$, giving the required coefficients of the line c).



Tentative drawing of the phase portrait of Z in B.

FIGURE 12

Let us show that $P'(b)<0$ for all $b\in[-\frac{2}{3},\frac{2}{3}[$. For $b=-\frac{2}{3}$ we obtain $P'(-\frac{2}{3})=-\frac{1}{8}$ merely by development of the formula (41). For the other values of b, we make the following qualitative reasoning. We consider the equation: $7P^2+3bP-5=0$, describing the points where Z is horizontal. The equation defines a hyperbola, whose two connected components are the graphs of functions. The part of this hyperbola contained in the strip B is an arc S, graph of $P=\frac{1}{14}(-3b+(9b^2+140)^{1/2})$, $b\in[-\frac{2}{3},\frac{2}{3}]$. Along S we can solve b in terms of P: $b=(5-7P^2)/3P$ (since $P\neq 0$ on S). Hence Z is transverse to S, along S, and directed to the right. Finally, the extremities of S are the points α_0, α_1. We now study the position of S with respect to Γ. In α_0, the tangent to S has a slope equal to $-\frac{1}{4}$, which value is inferior to $P'(-\frac{2}{3})=-\frac{1}{8}$. Also, in the neighbourhood of α_0, the separatrix Γ is above S. But as, along S, Z is transverse to S and directed to the right, the orbit Γ is not permitted to cut S again for $t\to\infty$: the orbit is hence entirely situated in B, above S. But in this region, the vertical component of Z is negative. It follows that $P'(b)<0$ for all $b\in[-\frac{2}{3},\frac{2}{3}[$. (For $b\to\frac{2}{3}$, a local study in $(\frac{2}{3},1)$ reveals that $P'(b)\to-\infty$.)

Let us now show that $P''(b)<0$ for all $b[-\frac{2}{3},\frac{2}{3}[$. First of all, using a development up to order 2 of the equation (41) in $b=-\frac{2}{3}$, one obtains that $P''(-\frac{2}{3})=-\frac{55}{2304}<0$. Let us for a moment suppose that P'' would have a zero on $[-\frac{2}{3},\frac{2}{3}[$ and let $b_0>-\frac{2}{3}$ be the minimum of such points: $P''(b_0)=0$ and $P''(b)<0$ for all $b\in[-\frac{2}{3},b_0[$. We show that this is impossible since *Z is a quadratic vector field.* Therefore consider D, the tangent line of Γ in the point $m_0=(b_0,P(b_0))$. As $P''(b_0)=0$, the order of contact between D and T is at least 2. Let v be a vector orthogonal to D and $D(u)$ a linear parametrization of D. The function $\psi(u)=\langle Z(D(u)),v\rangle$ ($\langle\cdot,\cdot\rangle$ denoting the euclidean inner product on $\mathbb{R}^2$) has a zero of order at least 1 in u_0, with $D(u_0)=m_0$ (see appendix 2). As $P''(b)<0$ for all $b\in[-\frac{2}{3},b_0[$, the corresponding arc of Γ is situated below D. The line D hence cuts $\Delta_0=\{b=-\frac{2}{3}\}$ at a point n_0 above α_0. At this point, Z is directed downward. On the other hand, in the points of D with abscissa $<b_0$ but near b_0, Z is directed towards the half plane above D. From this it follows that the function $\psi(u)$ needs to possess a zero at some $u_1\neq u_0$ with $D(u_1)\in]n_0,m_0[$.

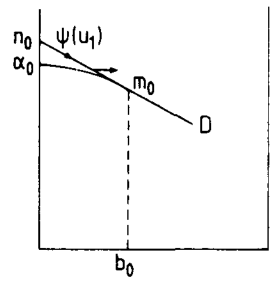


FIGURE 13

However, the vector field Z being quadratic, the function $\psi(u)$ is a polynomial of second degree in u; the existence of a double zero at u_0 and another zero at u_1 implies then that $\psi \equiv 0$ and hence that Γ is a line segment. This is of course not compatible with $P''(-\frac{2}{3}) < 0$, ending the proof of the theorem. □

Let us come back to the study of the surface Σ, by making precise a few supplementary properties. First, along the line h with equation $\bar{\nu}_0 = \bar{\nu}_1 + 1$, we have $P'(-\frac{2}{3}) = -\frac{1}{8}$ so that

$$\left.\frac{\partial \tilde{G}}{\partial b}\right|_h = \tfrac{1}{8}\bar{\nu}_1 - \tfrac{3}{8} \neq 0 \qquad \text{except at } h_2 = (4, 3). \tag{42}$$

At the point h_2:

$$\frac{\partial \tilde{G}}{\partial b}(h_2) = 0 \quad \text{and} \quad \frac{\partial^2 \tilde{G}}{\partial b^2}(h_2) = -P''(-\tfrac{2}{3})(3 + \tfrac{15}{11}) \neq 0. \tag{43}$$

Second let us state - and this will be required later - the condition

$$d\,\omega_D(\bar{\nu})|_s = 0, \tag{44}$$

that is $d\,\omega_D(\bar{\nu})$ restricted to $\{\bar{x} = 1\}$, defines a line t. Explicitly

$$d\,\omega_D = (\bar{\nu}_0 + \bar{\nu}_1 \bar{x} + \bar{x}^3)\; d\bar{y} \wedge d\bar{x} \tag{45}$$

and

$$d\,\omega_D(\bar{\nu})|_s = (\bar{\nu}_0 + \bar{\nu}_1 + 1)\; d\bar{y} \wedge d\bar{x}. \tag{46}$$

The equation of the line t is

$$\bar{\nu}_0 + \bar{\nu}_1 + 1 = 0. \tag{47}$$

It cuts the line c transversally at the point

$$c_2 = (\tfrac{4}{11}, -\tfrac{15}{11}) \tag{48}$$

since the function $\bar{\nu}_0 + \bar{\nu}_1 + 1$ restricted to c has a regular zero at c_2.

Finally let us come back to the curve l. On l, the function $\bar{\nu}_1(b) = (-\frac{6}{11})1/P - \frac{15}{11}$ (see (36)) goes from $\bar{\nu}_1(-\frac{2}{3}) = 3$ to $\bar{\nu}_1(\frac{2}{3}) = -\frac{15}{11}$ so that l joins the points h_2 and c_2. The curve l is the graph of a convex function $\bar{\nu}_0(\bar{\nu}_1)$ and therefore situated above the lines h and c. To prove this last point, we recall that l is the envelope of the lines Δ_b defined by

$$\tilde{G}(b, \bar{\nu}_0, \bar{\nu}_1) = \bar{\nu}_0 - P(b)\bar{\nu}_1 - \tfrac{6}{11}b - \tfrac{15}{11}P(b) = 0.$$

As $P(b)$ is invertible, we can choose its values as a parameter so that the lines can be parametrized by their slopes $P \in [\frac{5}{7}, 1]$:

$$\Delta_b \sim \Delta_P: \bar{\nu}_0 - P\bar{\nu}_1 - H(P) = 0 \tag{49}$$

where

$$H(P) = \tfrac{6}{11}b(P) - \tfrac{15}{11}P,$$

($b(P)$ stands for the inverse function of $P(b)$). We calculate the second derivative of $H(P)$: $H''(P) = \frac{6}{11}b''(P)$ with $b''(P) = -P''(b)/(P'(b))^3$ so that $H''(P) < 0$ for all $P \in]\frac{5}{7}, 1]$.

It is well known that the envelope of a family of lines like in (49), parametrized by their slopes and defined by $\bar{\nu}_0 = P\bar{\nu}_1 - (-H)$ for a convex function $-H$, is always the graph of a convex function: $\bar{\nu}_0(\bar{\nu}_1)$ is the Legendre transform of the function $-H$ ([**A3**]).

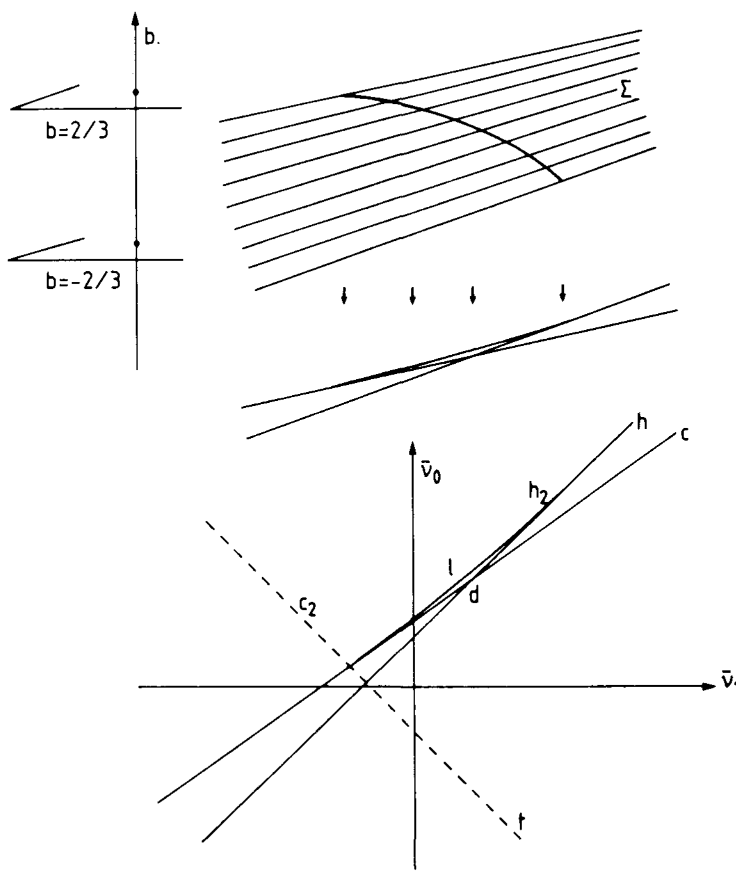


FIGURE 14

4.1(b). *From the surface Σ to the bifurcation set inside a conic neighbourhood of 0μ, $\mu < 0$.* The perturbation lemma permits us to deduce the nature of the bifurcations for the family $\bar{X}_{\bar{\nu},t}$ from the properties of the function $\tilde{G}$, established in § 4.1(a).

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THEOREM 4. *Let K be a compact neighbourhood of the curved triangle (d, h_2, c_2) in the $\bar{\nu}$-plane; let $\bar{A}$ be a compact neighbourhood of the singular disc $\{H \leq \frac{2}{3}\} \cap \{\bar{x} \leq 1\}$ in the $(\bar{x}, \bar{y})$-plane. There exists a value $T(K) > 0$ such that in $\bar{C}(K) =]0, T(K)] \times K \subset \mathbb{R}^3$ (with coordinates $(t, \bar{\nu})$) the bifurcation set of $\bar{X}_{\bar{\nu},t} | \bar{A}$ can be described as follows, up to a diffeomorphism of $\bar{C}(K)$ equal to the identity on $t = 0$:*

(i) *Bifurcations of codimension 1:*

$S_H =]0, T(K)[\times (h \backslash \{h_2\})$ *is a surface of Hopf bifurcation;*

$S_C =]0, T(K)[\times (c \backslash \{c_2\})$ *is a surface of generic saddle connection (when one crosses S_C, two separatrices of s cross generically);*

$S_L =]0, T(K)[\times l$ *is a surface of generic coalescence of 2 limit cycles.*

(ii) *Bifurcations of codimension 2:*

$]0, T(K)] \times \{h_2\}$ *is a curve of Hopf bifurcation of codimension 2;*

$]0, T(K)] \times \{c_2\}$ *is a curve of saddle connection of codimension 2;*

$S_H \cap S_L$ *is a curve of simultaneous Hopf bifurcation and saddle connection.*

Outside these bifurcation sets, the topological type of $\bar{X}_{\bar{\nu},t}$ is constant in $\bar{A}$.

Proof. We make use of the development of the Poincaré mapping on $[e, s]$, as given in lemma 1:

$$P_{\bar{\nu},t}(b) - b = \Delta_{\bar{\nu},t}(b) \qquad \text{where } \Delta_{\bar{\nu},t}(b) = t^5(G(b, \bar{\nu}) + \varepsilon(t))$$

with $\varepsilon(t)$ a C^∞ function in all variables, tending to 0 for $t \to 0$. For $b = -\frac{2}{3}$ (corresponding to the point e) we have $\Delta_{\bar{\nu},t}(-\frac{2}{3}) \equiv 0$. So this function is divisible by $(b + \frac{2}{3})$ and hence also by $I_0(b)$ since $I_0'(-\frac{2}{3}) \neq 0$ and $I_0(b) \neq 0$ for $b \in]-\frac{2}{3}, \frac{2}{3}]$. Using a new function tending to 0 for $t \to 0$ and which we denote by $\xi(b, t, \bar{\nu})$ we have

$$\Delta_{\bar{\nu},t}(b) = t^5 I_0(b)(\tilde{G}(b, \bar{\nu}) + \xi). \tag{50}$$

In the 3-parameter family $X_{\bar{\nu},t}$, the set of Hopf bifurcation is given by:

$$\frac{\partial \Delta_{\bar{\nu},t}}{\partial b}(-\tfrac{2}{3}) = 0. \tag{51}$$

On this set, the condition:

$$\frac{\partial^2 \Delta_{\bar{\nu},t}}{\partial b^2}(-\tfrac{2}{3}) = 0 \tag{52}$$

defines the Hopf bifurcation of codimension 2, while the condition

$$\frac{\partial^2 \Delta_{\bar{\nu},t}}{\partial b^2}(-\tfrac{2}{3}) \neq 0 \tag{53}$$

defines the set of Hopf bifurcation of codimension 1. Now, the Hopf bifurcation of codimension 1, defined by (51) and (53), is generic if equation (51) has rank 1. Also, the Hopf bifurcation of codimension 2 defined by (51) and (52) is generic if the defining equations form a system of rank 2 and if moreover, in the points of this bifurcation set:

$$\frac{\partial^3 \Delta_{\bar{\nu},t}}{\partial b^3}(-\tfrac{2}{3}) \neq 0. \tag{54}$$

Because of (50) we have:

$$\frac{\partial \Delta_{\bar{\nu},t}}{\partial b}(b) = t^5 \frac{\partial I_0}{\partial b}(\tilde{G}+\xi) + t^5 I_0\left(\frac{\partial \tilde{G}}{\partial b}+\frac{\partial \xi}{\partial b}\right). \tag{55}$$

In particular, for $b=-\frac{2}{3}$, as $I_0(-\frac{2}{3})=0$:

$$\frac{\partial \Delta_{\bar{\nu},t}}{\partial b}(-\tfrac{2}{3}) = t^5 \frac{\partial I_0}{\partial b}(-\tfrac{2}{3})(\tilde{G}(-\tfrac{2}{3}, \bar{\nu})+\xi(-\tfrac{2}{3}, t, \bar{\nu})). \tag{56}$$

Thus, for $t \neq 0$, condition (51) is equivalent to:

$$(\tilde{G}+\xi)|_{b=-\frac{2}{3}} = 0. \tag{57}$$

Equally, conditions (51) and (52) are equivalent to (57) and

$$\left(\frac{\partial \tilde{G}}{\partial b}+\frac{\partial \xi}{\partial b}\right)\Bigg|_{b=-\frac{2}{3}} = 0. \tag{58}$$

For $t=0$, equation (57) defines the line h, the equations (57), (58) define the point h_2. Finally the condition (54) on the bifurcation set is equivalent to

$$\frac{\partial^2 \tilde{G}}{\partial b^2} \neq 0. \tag{59}$$

If one considers expression (33) given by $\tilde{G}$ it is clear that equation (57) has rank 1 and that the equations (57), (58) have rank 2 in $(\bar{\nu}_0, \bar{\nu}_1)$ for $t=0$. By continuity, this remains true for t sufficiently small. The conclusions of theorem 4 with respect to the Hopf bifurcations hence follow if one uses the implicit function theorem.

In the same way we observe that the equations of the bifurcation set for the coalescence of two cycles

$$\Delta = \frac{\partial \Delta}{\partial b} = 0 \quad \text{and} \quad \frac{\partial^2 \Delta}{\partial b^2} \neq 0$$

are equivalent to the equations:

$$\tilde{G}+\xi = \frac{\partial \tilde{G}}{\partial b}+\frac{\partial \xi}{\partial b} = 0 \quad \text{and} \quad \frac{\partial^2 \tilde{G}}{\partial b^2}+\frac{\partial^2 \xi}{\partial b^2} \neq 0. \tag{60}$$

The equations have been studied for $\tilde{G}$ in § 4.1(a), for $t=0$. The study for $t \neq 0$ follows because of the implicit function theorem and the fact that for every order of derivation k

$$\frac{\partial^k \xi}{\partial b^k} \to 0 \qquad \text{for } t \to 0.$$

In an analogous way one deduces the bifurcation set of saddle connection of codimension 1 from the properties of $\tilde{G}$ along the line c. However the treatment in the neighbourhood of the curve of saddle connection of codimension 2 is more delicate and we postpone it to appendix 3.

Outside the described bifurcation sets, the vector field $\bar{X}_{\bar{\nu},t}$ only possesses hyperbolic critical elements (points and cycles) in $\bar{A}$: it is hence a Morse-Smale field of locally constant topological type. This ends the proof of the theorem. □

The image of $\bar{C}(K)=]0, T(K)]\times K$ by the mapping Φ_p restricted to $\bar{\mu}=-1$ ($\mu=-t^4$, $\nu_0=t^6\bar{\nu}_0$, $\nu_1=t^4\bar{\nu}_1$) is a cone $C_\mu(K)$, in a neighbourhood of the axis 0μ for $\mu<0$,

$$C_\mu(K)=\{(-t^4, t^6\bar{\nu}_0, t^4\bar{\nu}_1)\,|\,t\in\,]0, T(K)], \bar{\nu}\in K\}. \tag{61}$$

The bifurcation sets in $C_\mu(K)$ are the images by Φ_p of those described in theorem 4 and hence homeomorphic to cones on $h, l, c, c_2, h_2, d\subset\mathbb{R}^2$ (cones with as generating curves $t\to(-t^4, t^6\bar{\nu}_0, t^4\bar{\nu}_1)$ or, differently said, $\mu\to(\mu, (-\mu)^{3/2}\bar{\nu}_0, -\mu\bar{\nu}_1)$ with $\mu<0$).

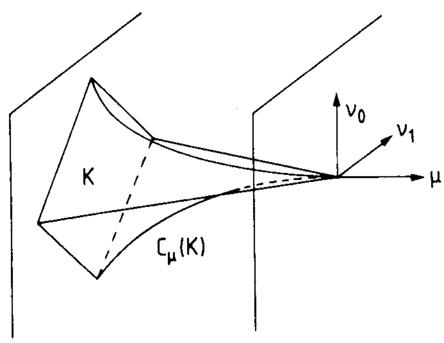

FIGURE 15

4.2. *The behaviour of X_λ in a sector around the axis $0\nu_1$.* We return to the expression (10) for the 1-form $\bar{\omega}_{\bar{\lambda},t}$. In order to make a study inside a cone in the neighbourhood of $0\nu_1$, we fix $\bar{\nu}_1=\pm1$ in (10), retaining $(t, \bar{\mu}, \bar{\nu}_0)$ as parameter. For $\bar{\nu}_1=1$ we reach a cone in the neighbourhood of $0\nu_1$, $\nu_1>0$, and for $\bar{\nu}_1=-1$, a cone in the neighbourhood of $0\nu_1$, $\nu_1<0$. As both cases can be treated similarly, we only consider $\bar{\nu}_1=1$. Expression (10) becomes:

$$\bar{\omega}_{t,\bar{\mu},\bar{\nu}_0}=d\bar{H}_{\bar{\mu}}-t^5(\bar{y}(\bar{\nu}_0+\bar{x})+\bar{x}^2\bar{y}\phi+\bar{y}^2\psi)\,d\bar{x}, \tag{62}$$

where ϕ and ψ are C^∞ functions in $\bar{x}$, $\bar{y}$, t, $\bar{\mu}$, $\bar{\nu}_0$ and

$$d\bar{H}_{\bar{\mu}}=d\left(\frac{1}{2}\bar{y}^2-\left(\bar{\mu}\bar{x}+\frac{\bar{x}^3}{3}\right)\right).$$

Observe that for fixed $t\neq0$, the 2-parameter family $\bar{\omega}$ is a family of the kind studied by R. Bogdanov. (With a minor difference in the choice of parameters, however not affecting the genericity of the family.)

We will see that for a fixed value $T>0$, one can apply the theory of Bogdanov *uniformly with respect to* $t\in\,]0, T]$, in the region of the parameters $\bar{\mu}$, $\bar{\nu}_0$ where traditionally the study can be made by 'perturbation of a Hamiltonian'. More precisely, in the halfplane $\bar{\mu}\le0$ there exists a fixed compact subset $\bar{B}$, diffeomorphic to a disc, having a contact of order 1 in $(0,0)$ with $0\bar{\nu}_0$ and such that the results of Bogdanov (concerning the curves of Hopf bifurcation, saddle connection and unicity of cycle in between the two curves of bifurcation for $(\bar{\mu}, \bar{\nu})\in\bar{B}$) are valid for any $(\bar{\mu}, \bar{\nu}_0)\in\bar{B}$.

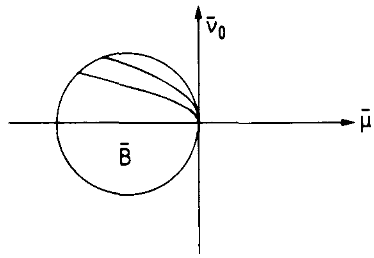

FIGURE 16

To prove this result, we use a secondary blowing up for each fixed $t \in]0, T]$. The blowing up is defined by:

$$\Phi'_m: \begin{cases} \bar{x} = \tau^2 \bar{\bar{x}} \\ \bar{y} = \tau^3 \bar{\bar{y}} \end{cases} \qquad \Phi'_p: \begin{cases} \bar{\mu} = -\tau^4 \\ \bar{\nu}_0 = \tau^2 \bar{\bar{\nu}}_0 \end{cases} \tag{63}$$

For each $\tau \neq 0$, the expression of $\bar{\omega}$ in the new coordinates $(\bar{\bar{x}}, \bar{\bar{y}})$ can be divided by τ^6:

$$\bar{\bar{\omega}}_{\tau,\bar{\bar{\nu}}_0} = \frac{1}{\tau^6} \bar{\omega} = dH - t^5 \tau \bar{\bar{y}}((\bar{\bar{\nu}}_0 + \bar{\bar{x}}) + \varepsilon(\tau))\, d\bar{\bar{x}} \tag{64}$$

where

$$H(\bar{\bar{x}}, \bar{\bar{y}}) = \tfrac{1}{2}\bar{\bar{y}}^2 + \bar{\bar{x}} - \frac{\bar{\bar{x}}^3}{3}$$

and $\varepsilon(\tau)$ is a C^∞ function in all variables which tends to 0 for $\tau \to 0$.

As in the preceding paragraph, the dual vector field $\bar{\bar{X}}_{\tau,\bar{\bar{\nu}}_0}$ possesses a first return mapping $P_{\tau,\bar{\bar{\nu}}_0}(b)$ with a development:

$$P(b) - b = t^5 \tau(\bar{\bar{\bar{G}}}(b, \bar{\bar{\nu}}_0) + \varepsilon(\tau)), \tag{65}$$

where

$$\bar{\bar{\bar{G}}} = \bar{\bar{\nu}}_0 I_0 + I_1. \tag{66}$$

Again as in § 4.1, the study of the bifurcation is equivalent to the detection of the zeros of the functions $\bar{\bar{\bar{G}}} + \varepsilon(\tau)$.

The results on the existence of bifurcation curves and on the unicity of the cycle for $(\bar{\bar{x}}, \bar{\bar{y}})$ in a fixed neighbourhood $\bar{\bar{A}}$ of the singular disc $\{H \leq \frac{2}{3}\} \cap \{\bar{\bar{x}} \leq 1\}$ and for $\bar{\bar{\nu}}_0$ in a fixed compact K' are valid for $\tau > 0$ sufficiently small ($0 < \tau \leq \tau_0$ for a certain τ_0).

Indeed, as $\varepsilon(\tau)$ is C^∞ in all variables, it tends to zero as well as all its derivatives $\partial^k \varepsilon / \partial b^k$ in a uniform way when $(b, \bar{\bar{\nu}}_0, \bar{\bar{x}}, \bar{\bar{y}})$ belongs to a fixed compact set $[-\frac{2}{3}, \frac{2}{3}] \times K' \times A$ and $t \in [0, T]$. The properties of $\bar{\bar{\bar{G}}}/I_0 = \bar{\bar{\nu}}_0 - P$ (e.g. $\bar{\bar{\bar{G}}}/I_0 = 0 \Rightarrow \partial(\bar{\bar{\bar{G}}}/I_0)/\partial b \neq 0$) are still satisfied by $(1/I_0)(\bar{\bar{\bar{G}}} + \varepsilon(\tau))$ for small τ, let us say $\tau \in]0, \tau_0]$. The image of the rectangle $K' \times]0, \tau_0]$ in the $(\bar{\mu}, \bar{\nu}_0)$-space contains a region $\bar{B}$ diffeomorphic to a disc.

Finally, the result remains valid in the cone on $\bar{B}$ defined by

$$C^+_{\nu_1} = \{(t^4 \bar{\mu}, t^6 \bar{\nu}_0, t^4) \mid t \in]0, T], (\bar{\mu}, \bar{\nu}_0) \in \bar{B}\}$$

for the vector field $\bar{X}_{t,\bar{\lambda}}$ in the region $A_\lambda = \Phi_p \cdot \Phi'_p(\bar{\bar{A}})$ (depending on λ). Similarly we can obtain a cone $C^-_{\nu_1}$, around $0\nu_1$ for $\nu_1 < 0$.

4.3. *Conclusions of* 4.1 *and* 4.2. Let $C^+_{\nu_1}$, $C^-_{\nu_1}$ be the two cones obtained in § 4.2. We can now choose the compact K in the $\bar{\nu}$-plane of § 4.1 in a way that the union $C_\mu(K) \cup C^-_{\nu_1}$ contains a cone $C(D)$ on a disc D belonging to a sphere $S_{\varepsilon_0} = \{\mu^2 + \nu_0^2 + \nu_1^2 = \varepsilon_0^2\}$ with ε_0 sufficiently small. (The cone structure refers as before to the mapping Φ_{p^*}.) The disc is chosen inside the half sphere $S = S_{\varepsilon_0} \cap \{\mu \le 0\}$, in such a way that it contains the half circle $\{\nu_0 = 0\}$ and that the boundary ∂D is tangent with a contact of order 1, to ∂S in the points $\alpha = \{\nu_0 = \mu = 0, \nu_1 = 1\}$ and $\beta = \{\nu_0 = \mu = 0, \nu_1 = 1\}$.

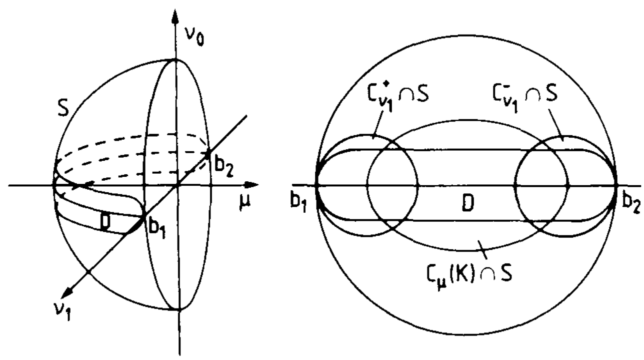

FIGURE 17

We moreover impose on the choice of D that in the interior of D, the curve of Hopf bifurcation and the curve of saddle connection described in § 4.2 are connected with the curves $H = S_H \cap D$ and $C = S_C \cap D$ described in § 1 (inside $C_\mu(K) \cap S$). We also ask D to contain the curve $L = S_L \cap S$ (see § 4.1).

The topological type of X_λ is known in a region A_λ for each $\lambda \in C(D)$. On the boundary $\partial C(D) = C(\partial D)$ the vector field X_λ has on $[e_\lambda, s_\lambda]$ a first return mapping without fixed points. As on the other hand X_λ has no critical points when $\mu > 0$, it only remains to handle the parameter values outside the cone $C(D)$ with $\mu < 0$ and in a neighbourhood of the origin.

4.4. *The region in* $\{\mu < 0\}$ *without cycles.* Take $B_\varepsilon = \{(\mu, \nu_0, \nu_1) \mid \mu^2 + \nu_0^2 + \nu_1^2 \le \varepsilon\}$ for $\varepsilon > 0$, and $S_\varepsilon = \partial B_\varepsilon$ as above. The number ε_0 has been defined in § 3. We also go on working with the disc $D \subset S = S_{\varepsilon_0} \cap \{\mu \le 0\}$ introduced in § 4.3.

THEOREM 5. *If $\varepsilon > 0$ is sufficiently small and $\lambda = (\mu, \nu_0, \nu_1)$ is such that $\lambda \in B_\varepsilon$, $\lambda \notin B_\varepsilon \cap C(\mathring{D})$ and $\mu < 0$, then X_λ does not admit a cycle in the neighbourhood A of 0 in $\mathbb{R}^2$, (A defined at the beginning of § 4, $\mathring{D}$ the interior of D).*

Proof. We resume the proof of Bogdanov [**B1**]. (As a matter of fact, one may observe that the normal form which we use for X_λ makes the proof even easier.) As we have remarked in § 4.3, the disc D has been chosen in such a way that the first return mapping of X_λ on $[e_\lambda, s_\lambda]$ has no fixed points other than e_λ for $\lambda \in C(\partial D)$. More precisely, this mapping has the following property: either $P^{-1}(x) < x$ ($x \in]e_\lambda, s_\lambda]$) for $\nu_0 > 0$ and the orbits coming from e_λ are expanding spirals, or $P(x) < x$ ($x \in]e_\lambda, s_\lambda]$) for $\nu_0 < 0$ and the orbits tending to e_λ are contracting spirals.

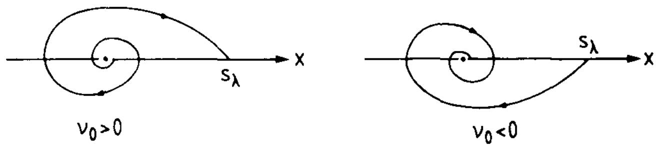

FIGURE 18

Let $\varepsilon>0$ be a number which we will make precise later on. We consider the set of $\lambda=(\mu, \nu_0, \nu_1)$ with $\mu<0$ and $\lambda\in B_\varepsilon\cap C(\mathring{D})$. Let us focus our attention to the region $\{\nu_0>0\}$ as the $\{\nu_0<0\}$-case can be studied in the same way. First of all, if $\lambda'\in B_\varepsilon\cap C(\partial D)\cap\{\nu_0>0\}$, there exists a positive semi-orbit starting in a point $x_{\lambda'}\in]e_{\lambda'}, s_{\lambda'}[$ and tending to $s_{\lambda'}$ without cutting $]e_{\lambda'}, s_{\lambda'}[$ again (see the left-hand figure of figure 18). We denote by $D_{\lambda'}$ the singular disc bounded by the piece of this orbit between $x_{\lambda'}$ and $s_{\lambda'}$ and the segment $[x_{\lambda'}, s_{\lambda'}]$. For continuity reasons:

$$\operatorname{Sup}\{\operatorname{diam} D_{\lambda'} \mid \lambda'\in B_\varepsilon\cap C(\partial D)\cap\{\nu_0>0\}\}=O(\varepsilon). \tag{67}$$

Take now a value $\lambda\in B_\varepsilon$, $\lambda\notin B_\varepsilon\cap C(\mathring{D})$ with $\mu<0$ and $\nu_0>0$. We are going to compare X_λ to $X_{\lambda'}$ where $\lambda'=(\mu, \nu_0', \nu_1)$ (same values for μ, ν_1) is such that $\lambda'\in C(\partial D)\cap\{\nu_0>0\}$; λ' is necessarily in B_ε.

$$X_\lambda - X_{\lambda'} = yK(y,\lambda)((\nu_0-\nu_0')R(\lambda,\lambda',x,y))\frac{\partial}{\partial y} \tag{68}$$

with

$$R(\lambda,\lambda',x,y)$$
$$=(\alpha(\lambda)-\alpha(\lambda'))x^2+(h(x,\lambda)-h'(x,\lambda'))x^4+\frac{y}{K(y,\lambda)}(Q(x,y,\lambda)-Q(x,y,\lambda')). \tag{69}$$

Using the Taylor formula of order 1 with integral remainder term for $\alpha(\lambda)-\alpha(\lambda')$, etc...., one obtains that for $(x,y)\in A$:

$$R(\lambda,\lambda',x,y)=(\nu_0-\nu_0')\tilde{R}(\lambda,\lambda',x,y) \tag{70}$$

with $\tilde{R}=O(\|m\|)$ for $m=(x,y)$. We therefore can write $X_\lambda-X_{\lambda'}$ in the form:

$$X_\lambda-X_{\lambda'}=yK(y,\lambda)(\nu_0-\nu_0')(1+\tilde{R}(\lambda,\lambda',x,y))\frac{\partial}{\partial y}, \tag{71}$$

with $\tilde{R}=O(\|m\|)$. In particular if $(x,y)\in D_{\lambda'}$, the estimates (67) and (71) imply that $\tilde{R}(\lambda,\lambda',x,y)=O(\varepsilon)$. Hence if $\varepsilon>0$ is chosen sufficiently small and if $(x,y)\in D_{\lambda'}$, with λ, λ' related as before, one has:

$$K(y,\lambda)(\nu_0-\nu_0')(1+\tilde{R})>0, \tag{72}$$

Now fix such a value ε; in the disc $D_{\lambda'}$ we pass from $X_{\lambda'}$ to X_λ by adding a $\partial/\partial y$-component of the same sign as y. That this implies the impossibility for X_λ to have a periodic orbit in A can be shown as follows: Let Γ be such an orbit; it cuts $]e_\lambda, s_\lambda[=]e_{\lambda'}, s_{\lambda'}[$ at a point u_λ. Let Γ' be the $X_{\lambda'}$-orbit through u_λ followed for negative time until it reaches $]e_{\lambda'}, s_{\lambda'}[$ again at the point v_λ. Necessarily $v_\lambda<u_\lambda$ and the singular discs bounded by $[v_\lambda, u_\lambda]$ and the piece of Γ' in between u_λ and v_λ is contained in $D_{\lambda'}$.

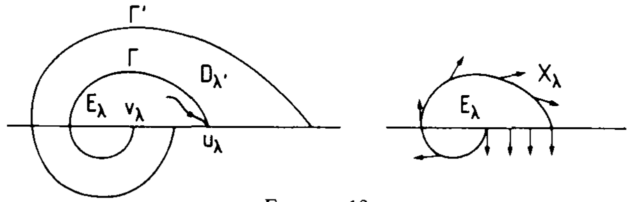

FIGURE 19

We can see from formula (72), valid in D', that the orbit Γ, emerging from u_λ, for $t<0$ enters the disc E_λ. As along Γ', between u_λ and v_λ, the vector field X_λ is transverse to Γ' and points outward from E_λ except in the points of the x-axis and as it is also transverse to $[u_\lambda, v_\lambda]$ pointing outward from E_λ, the orbit Γ cannot leave E_λ for negative times and hence cannot return to the point u_λ. □

4.5. *Final remarks.* X_λ being a generic family which we suppose to have the normal form (21) of § 3, we have found in the preceding paragraphs:

(i) a fixed neighbourhood A of $0\in\mathbb{R}^2$ (phase space);

(ii) a fixed neighbourhood B_ε in the parameter space;

(iii) a bifurcation set $\Sigma\subset B_\varepsilon$, having a conic structure with base in ∂B_ε.

The following properties have been verified:

(iv) For each $\lambda\in B_\varepsilon$, the vector field X_λ has resp. 0, 1 or 2 critical points in A, depending on whether $\mu>0$, $\mu=0$ or $\mu<0$. The vector field is fixed (independently of λ) in a neighbourhood of ∂A and has been described in § 4.1.

(v) The disc obtained by intersecting B_ε with the $\{\mu=0\}$-plane belongs to Σ. In this disc, outside 0, there occurs a saddle-node bifurcation. In the half disc $B_\varepsilon^+=B_\varepsilon\cap\{\mu\geq 0\}$ the vector field X_λ is trivial in A. On the other hand, we have proven in § 4 that in B_ε^-, outside a cone $C(D)$, the field has no other critical elements than the critical points: the topological type of X_λ in A is hence well determined there.

(vi) The other parts of the bifurcation set Σ are contained in $C(D)$. If $\lambda\in C(D)\backslash\Sigma$, we have proven that the topological type of $X_\lambda\,|\,A_\lambda$, with $A_\lambda\subset A$ a certain neighbourhood of $0\in\mathbb{R}^2$ (see § 3) is well defined. A problem could be that the diameter of A_λ tends to 0 for $\lambda\to 0$. However, as we know, the vector field X_λ has no critical elements in $A\backslash A_\lambda$ so that $X_\lambda\,|\,A$ has to be topologically equivalent to $X_\lambda\,|\,A_\lambda$.

By all this, $B_\varepsilon\backslash\Sigma$ is divided into a finite number (exactly 6) of open connected components, in which the topological type of $X_\lambda\,|\,A$ is constant, and does not depend on the family. Also on the different parts of Σ as described in § 4.1 and § 4.2 (9 surfaces and 5 curves) $X_\lambda\,|\,A$ has a constant topological type, independent on the family.

In order to prove that the family X_λ is (fibre) C^0 equivalent to the polynomial family $\tilde{X}_\lambda$ defined in the introduction, one starts by choosing a homeomorphism of B_ε into itself sending the bifurcation set Σ of X_λ onto the bifurcation set $\tilde{\Sigma}$ of $\tilde{X}_\lambda$. The existence of such a homeomorphism has been shown in § 4.1 and § 4.2. From now on we may hence suppose that X_λ and $\bar{X}_\lambda$ have the same bifurcation set Σ. As we have observed that the topological type of X_λ and of $\bar{X}_\lambda$ in some fixed neighbourhood of 0 is the same on each of the connected components of $B_\varepsilon\backslash\Sigma$ and on each of the different parts of Σ we obviously obtain a (fibre)-C^0-equivalence between X_λ and $\bar{X}_\lambda$.

Appendix 1

Differential equation for $P(b)$. Let $H(x, y)=\frac{1}{2}y^2+x-x^3/3$. This function has two critical points $e=(-1,0)$ and $s=(1,0)$; $H(e)=-\frac{2}{3}$, $H(s)=\frac{2}{3}$, for all $b\in[-\frac{2}{3},\frac{2}{3}]$, $\{H=b\}$ contains a compact connected component γ_b. We define $\omega_i=yx^i\,dx$, for $i\geq 0$, and $I_i(b)=\int_{\gamma_b}\omega_i$, using clockwise orientation. The function $P(b)$ is defined as $P(b)=-I_1(b)/I_0(b)$, extended to be 1 for $b=-\frac{2}{3}$. We intend to show that $P(b)$ verifies the equation

$$(9b^2-4)P'(b)=7P^2+3bP-5 \tag{1}$$

For this, we will first establish some recurrence relations between the integrals I_i. Let us consider $\gamma_b^+=\{(x, y)\in\gamma_b\,|\,y\geq 0\}$. The cycle γ_b cuts the x-axis at points of abscissa $c(b)\leq c_1(b)$, roots ≤ 1 of the equation

$$b=x-\frac{x^3}{3} \tag{2}$$

The half-cycle γ_b^+ is the graph of $y=(2b-2x+(2x^3/3))^{1/2}$ with $x\in[c(b), c_1(b)]$. Introducing the radical

$$R(w, b)=\left(2b-2w+\frac{2w^3}{3}\right)^{\frac{1}{2}} \qquad \text{for } w\in[c(b), c_1(b)] \tag{3}$$

and $J_i(b)=\frac{1}{2}I_i(b)$ we get the expression

$$J_i(b)=\int_{c(b)}^{c_1(b)} w^iR(w, b)\,dw. \tag{4}$$

(i) Derivation with respect to b gives us:

$$J_i'(b)=\int_c^{c_1}\frac{w^i}{R}\,dw, \qquad \text{where } c=c(b),\ c_1=c_1(b),\ R=R(w, b).$$

On the other hand: $J_i=\int_c^{c_1}(w^iR^2/R)\,dw$ and as $R^2=2b-2w+(2w^3/3)$ this implies:

$$J_i=2bJ_i'-2J_{i+1}'+\tfrac{2}{3}J_{i+3}' \qquad \text{for } i\geq 0 \tag{5}$$

(ii) Partial integration leads to

$$J_i=\left[\frac{1}{i+1}w^{i+1}R\right]_c^{c_1}+\frac{1}{i+1}\int_c^{c_1}\frac{w^{i+1}(1-w^2)}{R}\,dw$$

hence

$$J_i=\frac{1}{i+1}(J_{i+1}'-J_{i+3}') \qquad \text{for all } i\geq 0 \tag{6}$$

(iii) We can eliminate J_{i+3}' between (5) and (6):

$$(2i+5)J_i=-4J_{i+1}'+6bJ_i'. \tag{7}$$

In particular:

$$\begin{cases}5J_0=-4J_1'+6bJ_0'\\ 7J_1=4J_2'+6bJ_1'\end{cases} \tag{8}$$

(iv) However, $J_2(b)\equiv J_0(b)$. Indeed: from $dH=y\,dy+(1-x^2)\,dx$ we obtain

$$\omega_0-\omega_2=y\,dH-y^2\,dy \tag{9}$$

and integrating along γ_b gives:

$$J_0(b) = J_2(b) \tag{10}$$

This relation permits us to eliminate J_2 from (8), and to obtain the following equations for J_0, J_1:

$$\begin{cases} 5J_0 = 6bJ_0' - 4J_1' \\ 7J_1 = -4J_0' + 6bJ_1' \end{cases} \tag{11}$$

On the interval $]-\frac{2}{3}, \frac{2}{3}[$ one can calculate J_0', J_1' in terms of J_0, J_1:

$$\begin{cases} (9b^2-4)J_0' = \frac{15}{2}bJ_0 + 7J_1 \\ (9b^2-4)J_1' = 5J_0 + \frac{21}{2}bJ_1 \end{cases} \tag{12}$$

implying (1) since $P'(b) = (J_0'J_1 - J_1'J_0)/J_0^2$.

Appendix 2

In the proof of theorem 3 in § 4 we have used the following result:

LEMMA. *Let $m \in \mathbb{R}^2$ be a regular point of a C^∞ vector field Z on $\mathbb{R}^2$ $(Z(m) \neq 0)$. Let $C :]-\varepsilon, \varepsilon[\to \mathbb{R}^2$ (for $\varepsilon > 0$) be a regular C^∞ path with $C(0) = m$. The Z-orbit through $m(Z(t, m))$ has then a contact of order $k \geq 1$ with C at m if and only if the function*

$$\psi(u) = \left\langle Z(C(u)), \left(\frac{dC}{du}(u)\right)^{\perp}\right\rangle$$

has a zero of order $k-1$ at $u = 0$. ($\langle\ ,\ \rangle$ denotes the euclidean inner product of $\mathbb{R}^2$ and $(dC/du)^\perp$ is the vector obtained by rotating dC/du through an angle $\pi/2$.)

Proof. The order of contact between two curves does not depend on the choice of local coordinates around m nor on the regular parametrization of C. We hence choose a coordinate system $W(x, y)$ in which $m = (0, 0)$ and $Z(x, y) = \partial/\partial x$. As the curve defined by C is tangent to the x-axis at $(0, 0)$, we may suppose that it is given as the graph of a function c depending on the variable x. Finally, we may as well replace the euclidean inner product on $\mathbb{R}^2$ by any riemannian matrix g around m, in order to define the function ψ. Indeed, if

$$\psi_g(u) = g\left(Z(C(u)), \left(\frac{dC}{du}(u)\right)^{\perp}\right)$$

($\perp$ being calculated with respect to g) one has $\psi_g(u) = \xi(u) \cdot \psi(u)$, where $\xi(u)$ is non-zero around 0, so that the 2 functions ψ_g and ψ have at $u = 0$ a zero of the same order. If for g we choose the euclidean inner product in the chart W and we use the x-variable as parameter, then

$$Z = \frac{\partial}{\partial x}, \quad \frac{dC}{dx} = \left(1, \frac{dc}{dx}\right), \quad \left(\frac{dC}{dx}\right)^{\perp} = \left(-\frac{dc}{dx}, 1\right),$$

hence

$$\psi_g(x) = \left\langle Z(C(x)), \left(\frac{dC}{dx}\right)^{\perp}\right\rangle = -\frac{dc}{dx},$$

implying the result.

Appendix 3. *Study of the bifurcation set in the neighbourhood of the line of saddle connection of codimension* 2.

A3.1. *Preliminary calculations.* We want to study (see § 4.1):

$$X_{\bar{\nu},t}=\bar{y}\frac{\partial}{\partial\bar{x}}-(1-\bar{x}^2)\frac{\partial}{\partial\bar{y}}+t^5\bar{y}(\bar{\nu}_0+\bar{\nu}_1\bar{x}+\bar{x}^3)\frac{\partial}{\partial\bar{y}}+O(t^6)\bar{y}\frac{\partial}{\partial\bar{y}}$$

in some neighbourhood of the saddle connection through the saddle point $s=(1,0)$ for $t\in\,]0,T[$ and $(\bar{\nu}_0,\bar{\nu}_1)\in V$ where $T>0$ and V, a neighbourhood of $c_2=(\frac{4}{11},-\frac{15}{11})$, are yet to be chosen.

We introduce the coordinates $x=\bar{x}-1$, $y=\bar{y}$ in order to put the saddle point s in the origin, and we write $t^5=\varepsilon$:

$$X_{\bar{\nu},\varepsilon}=y\frac{\partial}{\partial x}+(x+2)\times\frac{\partial}{\partial y}+\varepsilon y(\bar{\nu}_0+\bar{\nu}_1(1+x)+(1+x)^3)\frac{\partial}{\partial y}+o(\varepsilon).$$

The linear part $j_1(X_{\bar{\nu},\varepsilon})(0)=y\,\partial/\partial x+[2x+\varepsilon(\bar{\nu}_0+\bar{\nu}_1+1)y]\,\partial/\partial y$ and its eigenvalues are

$$\tfrac{1}{2}[\varepsilon(\bar{\nu}_0+\bar{\nu}_1+1)\pm(8+\varepsilon^2(\bar{\nu}_0+\bar{\nu}_1+1)^2)^{1/2}].$$

The ratio of the eigenvalues is $-(1+(\bar{D}/2)\varepsilon+o(\varepsilon))$ where $\bar{D}=(\bar{\nu}_0+\bar{\nu}_1+1)$.

The *hyperbolicity factor* of the saddle point is

$$\alpha(\bar{\nu},\varepsilon)=\frac{\bar{D}}{2}\varepsilon+o(\varepsilon).$$

The hyperbolicity factor is defined to be the number $\alpha(\bar{\nu},\varepsilon)$ such that

$$j_1(X_{\bar{\nu},\varepsilon})(0)\sim*\,u\frac{\partial}{\partial u}-(1+\alpha(\bar{\nu},\varepsilon))v\frac{\partial}{\partial v},$$

where $\sim*$ stands for linear equivalence. (This means a linear conjugacy and a multiplication with a strictly positive constant.)

Let us call $\bar{\alpha}(\bar{\nu},\varepsilon)=(\bar{D}/2)\varepsilon$ the '*reduced hyperbolicity factor*'. We can also evaluate the integral $G(b,\bar{\nu})$ (see § 4.1)

$$\begin{aligned}G(b,\bar{\nu})&=\bar{\nu}_0I_0(b)+\bar{\nu}_1I_1(b)+I_3(b)\\&=(\bar{\nu}_0-\tfrac{6}{11}b)I_0(b)+(\bar{\nu}_1+\tfrac{15}{11})I_1(b).\end{aligned}$$

In terms of $\bar{b}=\frac{2}{3}-b$, $\bar{\bar{\nu}}_0=\bar{\nu}_0-\frac{4}{11}$, $\bar{\bar{\nu}}_1=\bar{\nu}_1+\frac{15}{11}$:

$$G(\bar{b},\bar{\nu})=(\bar{\bar{\nu}}_0+\tfrac{6}{11}\bar{b})I_0+\bar{\bar{\nu}}_0I_1$$

while $\bar{D}=\bar{\bar{\nu}}_0+\bar{\bar{\nu}}_1$. We develop $I_0(\bar{b})$ and $I_1(\bar{b})$ as (see [**R**]):

$$\begin{cases}I_0=\alpha_0+\alpha_1\bar{b}\ln\bar{b}+\alpha_2\bar{b}+\cdots\\I_1=\beta_0+\beta_1\bar{b}\ln\bar{b}+\beta_2\bar{b}+\cdots\end{cases}$$

For G this gives:

$$G=\bar{\bar{\nu}}_0\alpha_0+\bar{\bar{\nu}}_1\beta_0+[\bar{\bar{\nu}}_0\alpha_1+\bar{\bar{\nu}}_1\beta_1]\bar{b}\ln\bar{b}+[\tfrac{6}{11}\alpha_0+\bar{\bar{\nu}}_0\alpha_2+\bar{\bar{\nu}}_1\beta_2]\bar{b}+\cdots$$

A direction calculation (based on the equations for I_0, I_1 in appendix 1) gives:

$$\beta_0/\alpha_0=-\tfrac{5}{7},\qquad \alpha_1/\beta_1=1.$$

Hence:

$$G(\bar{b},\bar{\nu})=\bar{\eta}-\bar{\alpha}\bar{b}\ln\bar{b}+\bar{\beta}\bar{b}+\cdots$$

with

$$\bar{\eta}(\bar{\bar{\nu}}_0, \bar{\bar{\nu}}_1) = \alpha_0(\bar{\bar{\nu}}_0 - \tfrac{5}{7}\bar{\bar{\nu}}_1)$$
$$\bar{\alpha}(\bar{\bar{\nu}}_0, \bar{\bar{\nu}}_1) = -\alpha_1(\bar{\bar{\nu}}_0 + \bar{\bar{\nu}}_1) = -\alpha_1\bar{D}$$
$$\bar{\beta}(\bar{\bar{\nu}}_0, \bar{\bar{\nu}}_1) = \tfrac{6}{11}\alpha_0 + \bar{\bar{\nu}}_0\alpha_2 + \bar{\bar{\nu}}_1\beta_2.$$

We remark that $\bar{\beta}(0, 0) = \frac{6}{11}\alpha_0 > 0$.

This turns out to be the condition needed to prove the statements concerning the bifurcation set in the neighbourhood of the line of saddle connections of codimension 2 (see theorem 4 of § 4.1(b)). The proof of this fact is part of a more general theorem which will appear in [**R**]. Moreover the problem has been studied by Cherkas in [**C**] for analytic vector fields. We hence will limit ourselves to a survey of the complete elaboration, with emphasis on those calculations which concern our specific problem.

A3.2. *Final elaboration.* Let Γ denote the separatrix loop (saddle connection) of the saddle point $s = (0, 0)$ for the system

$$X_{\bar{\nu},\varepsilon} = y\frac{\partial}{\partial x} + (x+2)x\frac{\partial}{\partial y} + \varepsilon y((\bar{\bar{\nu}}_0 + \tfrac{4}{11}) + (\bar{\bar{\nu}}_1 - \tfrac{15}{11})(1+x) + (1+x)^3)\frac{\partial}{\partial y} + o(\varepsilon),$$

at $\varepsilon = 0$. We write $\lambda = (\bar{\bar{\nu}}, \varepsilon)$. We want to study the limit cycles of $X_\lambda = X_{\bar{\nu},\varepsilon}$ in the neighbourhood of Γ, for $(\bar{\bar{\nu}}_0, \bar{\bar{\nu}}_1)$ near $(0, 0)$ and $\varepsilon > 0$ small. As we already calculated:

$$j_1(X_\lambda)(0) \sim * \; u\frac{\partial}{\partial u} - (1 + \alpha(\lambda))v\frac{\partial}{\partial v},$$

where $\alpha(\lambda) = \alpha(\bar{\bar{\nu}}, \varepsilon) = (\bar{D}/2)\varepsilon + o(\varepsilon)$ with $\bar{D} = \bar{\bar{\nu}}_0 + \bar{\bar{\nu}}_1$.

Let (u, v) denote C^∞ coordinates around $s = (0, 0)$ in which $\{u = 0\}$ represents the stable separatrix, $\{v = 0\}$ represents the unstable separatrix and the first quadrant $\{u > 0, v > 0\}$ represents the interior of the loop. Let σ be a segment inside $\{v = v_0, u \geq 0\}$ for some v_0 cutting the v-axis in $(0, v_0)$, and τ a segment inside $\{u = u_0, v \geq 0\}$ for some u_0, cutting the u-axis in $(u_0, 0)$. We parametrize σ by means of $u(\sigma \cap \{u = 0\} = \{(0, v_0)\})$ and τ by means of $v(\tau \cap \{v = 0\} = \{(u_0, 0)\})$. We also denote by σ, resp. τ, the same segment in (x, y)-coordinates parametrized by u, resp. v.

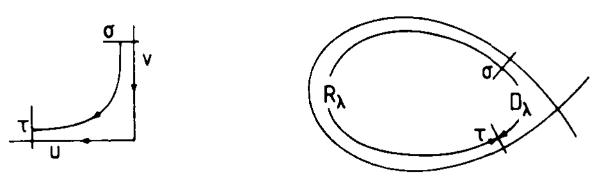

FIGURE 20

Let us denote by D_λ the Dulac mapping of the saddle point from σ to τ, while R_λ denotes the Poincaré mapping from σ to τ along the regular part of the loop. We express both mappings in terms of u and v.

$$v = D_\lambda(u), \qquad v = R_\lambda(u).$$

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($D_\lambda(u)$ - resp. $R_\lambda(u)$ - is the v-value of the point on τ where the positive - resp. negative - semi-orbit of a point of σ with parameter value u cuts τ for the first time.)

Let us define $\Delta_\lambda(u) = D_\lambda(u) - R_\lambda(u)$. Knowing that $P_\lambda = D_\lambda \circ R_\lambda^{-1}$ is the Poincaré mapping of Γ with respect to τ (in terms of v), we see that $\Delta_\lambda(u)$ represents the distance $(P_\lambda - I)(v)$ in terms of $u(u = R_\lambda^{-1}(v))$. Let us define $\omega(\lambda) = (u^{-\alpha(\lambda)} - 1)/\alpha(\lambda)$ for $u \geq 0$. To simplify the notation we - for a moment - change $\alpha(\lambda)$ into a and $\omega(\lambda)$ into ω. We now use a result which is proven in [**R**] and state it in terms of the notation introduced above.

PROPOSITION. (i) $\Delta_\lambda(u) = \varepsilon\tilde{\Delta}_\lambda(u)$.

(ii) *If λ is sufficiently small, then*

$$-\tilde{\Delta}_\lambda(u) = \tilde{\eta}(\lambda) + \tilde{\alpha}(\lambda)u\omega + \tilde{\beta}(\lambda)u + \tilde{\gamma}(\lambda)u^2\omega^2 + \tilde{\delta}(\lambda)u^2\omega + \tilde{\psi}(u, \lambda),$$

where $\varepsilon\tilde{\alpha}(\lambda) = \alpha(\lambda)$ is the hyperbolicity factor; $\tilde{\eta}$, $\tilde{\beta}$, $\tilde{\gamma}$ and $\tilde{\delta}$ are C^∞ functions in λ, $\tilde{\psi}$ is C^2 and is 2-flat with respect to u at $u = 0$. ($\tilde{\psi}(0, \lambda) = \partial\tilde{\psi}(0, \lambda)/\partial u = 0$.)

(iii)

$$-\tilde{\Delta}_\lambda(u) = \eta(\bar{\bar{\nu}}) - \frac{\bar{D}}{2} u \ln u + \beta(\bar{\bar{\nu}})u + \gamma(\bar{\bar{\nu}})u^2 \ln^2 u + \delta(\bar{\bar{\nu}})u^2 \ln u + \psi(u, \bar{\bar{\nu}}) + O(\varepsilon),$$

where $\varepsilon\bar{D}/2$ is the reduced hyperbolicity factor, η, β, γ, δ are C^∞ functions in $\bar{\bar{\nu}}$, ψ is C^2 and is 2 flat with respect to u at $u = 0$.

From the 'perturbation lemma' in § 4.1 and in terms of § A3.1 we see that

$$-\tilde{\Delta}_\lambda(u(\bar{b})) = G(\bar{b}, \bar{\bar{\nu}}) + O(\varepsilon).$$

(Here $u(\bar{b})$ denotes the reparametrization on the segment σ from $\bar{b}$ to u; the minus sign before $\tilde{\Delta}_\lambda$ comes from changing b to $\bar{b} = \frac{2}{3} - b$.) So $-\tilde{\Delta}_{(\bar{\bar{\nu}},0)}(u(\bar{b})) = G(\bar{b}, \bar{\bar{\nu}})$, implying that

$$G(\bar{b}, \bar{\bar{\nu}}) = \eta(\bar{\bar{\nu}}) - \frac{\bar{D}}{2}u(\bar{b}) \ln u(\bar{b}) + \beta(\bar{\bar{\nu}})u(\bar{b}) + \gamma(\bar{\bar{\nu}})[u(\bar{b}) \ln u(\bar{b})]^2$$
$$+ \delta(\bar{\bar{\nu}})(u(\bar{b}))^2 \ln u(\bar{b}) + \psi(u(\bar{b}), \bar{\bar{\nu}})$$

and hence

$$\eta(\bar{\bar{\nu}}) = \alpha_0(\bar{\bar{\nu}}_0 - \tfrac{5}{7}\bar{\bar{\nu}}_1)$$
$$\alpha_1\bar{D} = -\frac{\bar{D}}{2}\left[\frac{\partial u}{\partial \bar{b}}(0)\right]$$
$$\tfrac{6}{11}\alpha_0 + \bar{\bar{\nu}}_0\alpha_2 + \bar{\bar{\nu}}_1\beta_2 = \beta(\bar{\bar{\nu}}) + \frac{\bar{D}}{2}\left[\frac{\partial u}{\partial \bar{b}}(0) \cdot \ln\left(\frac{\partial u}{\partial \bar{b}}(0)\right)\right],$$

so $\beta((0,0)) = \frac{6}{11}\alpha_0 > 0$, (while $\eta(0,0)) = 0$ and $\bar{D}((0,0)) = 0$). Let us now go on with the study of the zeros of

$$-\tilde{\Delta}_\lambda(u) = \tilde{\eta}(\lambda) + \tilde{\alpha}(\lambda)u\omega + \tilde{\beta}(\lambda)u + \tilde{\gamma}(\lambda)u^2\omega^2 + \tilde{\delta}(\lambda)u^2\omega + \tilde{\psi}(u, \lambda),$$

where $\tilde{\eta}$, $\tilde{\alpha}$, $\tilde{\beta}$, $\tilde{\gamma}$, $\tilde{\delta}$ are C^∞, $\tilde{\psi}$ is C^2 and 2-flat with respect to u in $u = 0$, $\tilde{\eta}(0) = 0$, $\tilde{\alpha}(0) = 0$, $\tilde{\beta}(0) > 0$. As

$$\tilde{\eta}(\lambda) = \alpha_0(\bar{\bar{\nu}}_0 - \tfrac{5}{7}\bar{\bar{\nu}}_1) + O(\varepsilon),$$
$$\tilde{\alpha}(\lambda) = \tfrac{1}{2}(\bar{\bar{\nu}}_0 + \bar{\bar{\nu}}_1) + O(\varepsilon),$$

for sufficiently small $\varepsilon > 0$, we may as well describe the number of zeros of $\tilde{\Delta}_\lambda(u)$ in terms of $(\tilde{\alpha}, \tilde{\eta})$ near $(0,0)$ instead of $(\bar{\nu}_0, \bar{\nu}_1)$ near $(0,0)$. Put

$$\xi_\lambda = \tilde{\alpha}(\lambda)u\omega + \tilde{\beta}(\lambda)u + \tilde{\gamma}(\lambda)u^2\omega^2 + \tilde{\delta}(\lambda)u^2\omega + \tilde{\psi}(u, \lambda),$$

then $-\tilde{\Delta}_\lambda(u) = 0 \Leftrightarrow \xi_\lambda = -\tilde{\eta}(\lambda)$.

Let $\dot{f}$ denote the derivative of a function $f(u)$ with respect to u.

$$\dot{\omega} = \overline{\left(\frac{u^{-\alpha} - 1}{\alpha}\right)}^{\,\cdot} = -u^{-1-\alpha},$$

$$\dot{\overline{u\omega}} = \omega - u^{-\alpha} = (1-\alpha)\omega - 1 \qquad (\text{since } u^{-\alpha} = 1 + \alpha\omega),$$

$$\dot{\overline{(u^i\omega^j)}} = o(1) \qquad \text{whenever } i \geq 2 \text{ and } j \geq 0.$$

$\alpha = \varepsilon\tilde{\alpha}$ and we take $\varepsilon > 0$. So

$$\dot{\xi}_\lambda = \tilde{\alpha}[(1-\alpha)\omega - 1] + \tilde{\beta} + o(1) = \left(\frac{1}{\varepsilon} - \tilde{\alpha}\right)(u^{-\alpha} - 1) + \tilde{\beta} - \tilde{\alpha} + o(1).$$

(i) *Case* $\tilde{\alpha} \geq 0$. $x^{-\alpha} - 1 \geq 0$. Because of our hypotheses $\tilde{\beta}(0) > 0$ and $\tilde{\alpha}(0) = 0$ we see that

$$\exists A > 0, \exists U > 0, \exists E > 0 \text{ such that}$$
$$\forall \varepsilon \in]0, E[\text{ and } \forall(\tilde{\alpha}, u) \in [0, A] \times [0, U]: \dot{\xi}_\lambda > 0.$$

(Indeed for $\tilde{\alpha} > 0$: $\dot{\xi}_\lambda \to \infty$ when $u \to 0$; for $\tilde{\alpha} = 0$: $\dot{\xi}_\lambda \to \tilde{\beta}$ when $u \to 0$.)

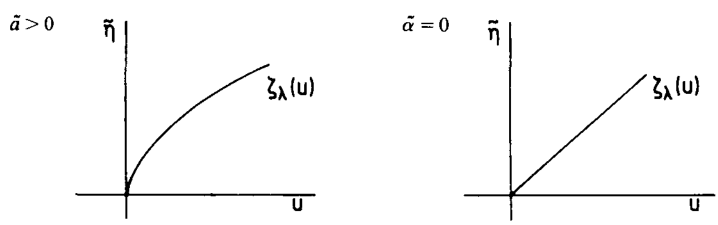

FIGURE 21

For $\tilde{\eta} \geq 0 \Rightarrow$ no $u > 0$ is solution of $\xi_\lambda(u) = -\tilde{\eta}$

$\tilde{\eta} < 0 \Rightarrow \exists! u > 0$ solution of $\xi_\lambda(u) = -\tilde{\eta}$.

The bifurcation diagram for $\tilde{\alpha} \geq 0$ is:

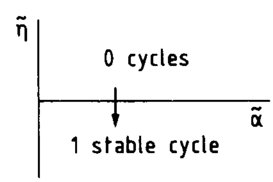

FIGURE 22

For each $\tilde{\alpha} \geq 0$ fixed we have the creation of 1 stable limit cycle when $\tilde{\eta}$ goes from positive to negative values; $\{\tilde{\eta} = 0, \tilde{\alpha} \geq 0\}$ is a half line of saddle connections with non-zero divergence at the saddle point. (Remark: the cycle is a stable one since $\partial(\tilde{\Delta}_\lambda(u))/\partial u < 0$ at the fixed point.)

(ii) *Case* $\tilde{\alpha}<0$.

$$\dot{\overline{(u^2\omega^2)}}=(2-2\alpha)u\omega^2-2u\omega,$$
$$\dot{\overline{(u^2\omega)}}=(2-\alpha)u\omega-u.$$

Hence

$$\dot{\xi}_\lambda=\frac{1}{\varepsilon}(1-\alpha)(u^{-\alpha}-1)+\tilde{\beta}-\tilde{\alpha}+\tilde{\gamma}[(2-2\alpha)u\omega^2-2u\omega]+\tilde{\delta}[(2-\alpha)u\omega-u]+\dot{\tilde{\psi}}.$$

As $\dot{\overline{u\omega^2}}=(1-2\alpha)\omega^2-2\omega$:

$$\ddot{\xi}_\lambda=-\frac{\alpha}{\varepsilon}(1-\alpha)u^{-\alpha-1}+\tilde{\gamma}[(2-2\alpha)[(1-2\alpha)\omega^2-2\omega]-2(1-\alpha)\omega+2]$$
$$+\tilde{\delta}[(2-\alpha)(1-\alpha)\omega-(2-\alpha)-1]+\ddot{\tilde{\psi}},$$

$$\ddot{\xi}_\lambda=-\frac{\alpha}{\varepsilon}(1-\alpha)u^{-1-\alpha}+2\tilde{\gamma}(1-\alpha)(1-2\alpha)\omega^2+(1-\alpha)[-6\tilde{\gamma}+(2-\alpha)\tilde{\delta}]\omega$$
$$+[2\tilde{\gamma}+(\alpha-3)\tilde{\delta}]+\ddot{\tilde{\psi}}.$$

We can find a certain bounded function $O(1)$ so that:

$$\ddot{\xi}_\lambda=-\frac{\alpha}{\varepsilon}(1-\alpha)u^{-1-\alpha}+\omega^2O(1)$$

From this we will prove that $\exists A'>0$, $\exists U'>0$, $\exists E'>0$ such that for all $\varepsilon\in]0,E[$ and for all $\tilde{\alpha}\in]-A',0[$ the graph of ξ_λ on $[0,U']$ looks like:

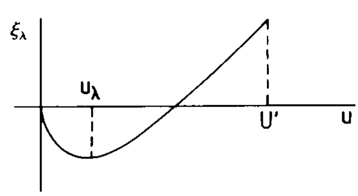

FIGURE 23

i.e. u_λ is a unique strict minimum: $\dot{\xi}_\lambda(u_\lambda)=0$, $\ddot{\xi}(u_\lambda)>0$, $\dot{\xi}_\lambda>0$ on $]u_\lambda, U']$ and $\dot{\xi}_\lambda<0$ on $[0,u_\lambda[$ with $\dot{\xi}_\lambda\to-\infty$ for $u\to 0$.

Therefore, consider:

$$\frac{1}{\omega^2}\ddot{\xi}_\lambda=-\frac{\alpha(1-\alpha)}{\varepsilon\omega^2u^{1+\alpha}}+O(1)=\frac{|\tilde{\alpha}|(1+\varepsilon|\tilde{\alpha}|)}{\omega^2u^{1-|\tilde{\alpha}|}}+O(1);$$

$$\omega=\frac{u^{-\alpha}-1}{\alpha}=\frac{1-u^{|\alpha|}}{|\alpha|}\le|\ln u|$$

for $U'<1$ and E' sufficiently small. Choosing some $\delta>0$, taking $-\delta/2\le\tilde{\alpha}<0$, and U' sufficiently small:

$$\frac{1}{\omega^2}\ddot{\xi}_\lambda\ge\frac{|\tilde{\alpha}|}{(\ln u)^2u^{1-\delta/2}}+O(1)\ge\frac{|\tilde{\alpha}|}{u^{1-\delta}}+O(1).$$

Take $M>0$ so that $O(1)\geq -M$ on $[0, U']\times[-\delta, 0]$, then:

$$\frac{1}{\omega^2}\ddot{\xi}_\lambda \geq M \quad \text{if} \frac{|\tilde{\alpha}|}{u^{1-\delta}}\geq 2M \quad \Leftrightarrow \quad u\leq\left(\frac{|\tilde{\alpha}|}{2M}\right)^{\frac{1}{1-\delta}}$$

Let us write $u\leq C\cdot|\tilde{\alpha}|^\mu$ with $C=(1/2M)^{1/(1-\delta)}$ and $\mu=1/(1-\delta)$. For these values of U', δ, M let us consider $\dot{\xi}_\lambda$ on $[C|\tilde{\alpha}|^\mu, U']$:

$$\dot{\xi}_\lambda=\frac{1}{\varepsilon}\alpha(1-\alpha)\omega+\tilde{\beta}-\tilde{\alpha}+o(1).$$

As $\alpha<0$, the function $\alpha\omega=u^{-\alpha}-1=e^{-\alpha\ln u}-1$ is strictly increasing and negative, so that for all $u\in[C|\tilde{\alpha}|^\mu, U']$:

$$\alpha\omega(C|\tilde{\alpha}|^\mu)\leq\alpha\omega(u)\leq 0.$$

And

$$\begin{aligned}\alpha\omega(C|\tilde{\alpha}|^\mu)&=e^{-\alpha\ln(C|\tilde{\alpha}|^\mu)}-1=e^{|\alpha|\ln(C|\tilde{\alpha}|^\mu)}-1\\&=O(|\alpha|\ln(C|\tilde{\alpha}|^\mu))\\&\Rightarrow\alpha\omega(u)=O(|\alpha|\ln(C|\tilde{\alpha}||\ln(C|\tilde{\alpha}|^\mu))\quad\text{for } u\in[C|\tilde{\alpha}|^\mu, U'].\end{aligned}$$

Hence

$$\dot{\xi}_\lambda=(1-\alpha)O|\tilde{\alpha}|\ln(C(|\tilde{\alpha}|^\mu))+\tilde{\beta}-\tilde{\alpha}+o(1).$$

For U' sufficiently small: $|o(1)|\leq\tilde{\beta}/3$; and for A' sufficiently small:

$$|-\tilde{\alpha}+(1-\alpha)O(|\tilde{\alpha}|\ln(C|\tilde{\alpha}|^\mu))|\leq\tilde{\beta}/3,$$

implying that

$$\dot{\xi}_\lambda\geq\tilde{\beta}/3>0$$

(all this independent of the values of ε).

As a conclusion we see that for $-A'\leq\tilde{\alpha}<0$ fixed, there is a bifurcation value (corresponding to a generic coalescence of limit cycles) for

$$\xi_\lambda(u_\lambda)=-\tilde{\eta}(\lambda).$$

This bifurcation occurs at $\tilde{\eta}=\Gamma(\tilde{\alpha}, \varepsilon)$; Γ is a C^∞ function for $\tilde{\alpha}<0$ because of the implicit function theorem applied to the Poincaré mapping of the C^∞ vector field X_λ in the neighbourhood of the semi-stable limit cycle.

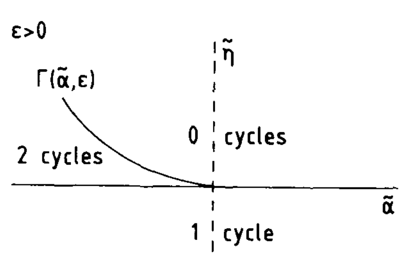

FIGURE 24

Taking $\Gamma(0, \varepsilon)=0$ we will now say something about the behaviour of $\Gamma(\tilde{\alpha}, \varepsilon)$ in the neighbourhood of $\tilde{\alpha}=0$:

$$u_\lambda \text{ is given by } \dot{\xi}_\lambda(u_\lambda)=0,$$

i.e.

$$\tilde{\alpha}(1-\alpha)\omega(u_\lambda)+\tilde{\beta}-\tilde{\alpha}+o(1)=0$$

$$\Leftrightarrow |\tilde{\alpha}|\omega(u_\lambda)=\frac{1}{1-\alpha}(\tilde{\beta}-\tilde{\alpha}+o(1)).$$

Now

$$\tilde{\eta}(\lambda)=u_\lambda[|\tilde{\alpha}|\omega(u_\lambda)-(\tilde{\beta}+o(1)]$$
$$=u_\lambda\left[\frac{\alpha\tilde{\beta}}{1-\alpha}-\frac{\tilde{\alpha}}{1-\alpha}+o(1)\right]\leq u_\lambda.$$

And as $|\tilde{\alpha}||\ln u_\lambda|\geq|\tilde{\alpha}|\omega(u_\lambda)\geq\tilde{\beta}/2$ for A' and U' sufficiently small we see that

$$u_\lambda\leq e^{-\tilde{\beta}/2|\tilde{\alpha}|}.$$

Hence

$$\Gamma(\tilde{\alpha}, \varepsilon)\leq e^{-\tilde{\beta}/2|\tilde{\alpha}|},$$

meaning that Γ is ∞-flat for $\tilde{\alpha}=0$.

Final remark. In some neighbourhood $V\times[0, E']$ of $(\tilde{\alpha}, \tilde{\eta}, \varepsilon)=(0,0,0)$, the surface $\{\tilde{\eta}=0\}$ represents a C^∞ surface of bifurcation. Also $\tilde{\eta}=\Gamma(\tilde{\alpha}, \varepsilon)$ (with $\Gamma(\tilde{\alpha}, 0)$ defined as the graph of the line l in § 4.1(a)) represents a surface of bifurcation. We proved that Γ is C^∞ outside $(0,0)\times[0, E']$, while along $(0,0)\times[0, E']$ the surface is ∞-tangent to $\{\tilde{\eta}=0\}$ in the sense that there exists some C^∞ flat function $\tilde{\Gamma}(\tilde{\alpha})$ with $0\leq\Gamma(\tilde{\alpha}, \varepsilon)\leq\tilde{\Gamma}(\tilde{\alpha})$. We do not however know if Γ is C^∞.

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