GENERALISED HESSIAN, MAX FUNCTION AND WEAK CONVEXITY

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In this paper, a second-order characterisation of \( \eta \)-convex \( C^{1,1} \) functions is derived in a Hilbert space using a generalised second-order directional derivative. Using this result it is then shown that every \( C^{1,1} \) function is locally weakly convex, that is, every \( C^{1,1} \) real-valued function \( f \) can be represented as \( f(x) = h(x) - \eta \|x\|^2 \) on a neighbourhood of \( x \) where \( h \) is a convex function and \( \eta > 0 \). Moreover, a characterisation of the generalised second-order directional derivative for max functions is given.

1. INTRODUCTION

In this paper, characterisations of the generalised Hessian and the generalised second-order directional derivative introduced in [11] for certain max functions are obtained. It is shown how the twice weakly Gâteaux differentiability of max functions can be characterised. A necessary and sufficient condition for a real valued \( C^{1,1} \) function to be \( \eta \)-convex is presented in a Hilbert space using the generalised second-order directional derivative. It is then shown that every \( C^{1,1} \) function is locally weakly convex in a Hilbert space. This extends the corresponding results given in Hiriart-Urruty [3] and Vial [10].

Let \( X \) be a Banach space. The class of \( C^{1,1} \) functions is defined to be the set of all real valued continuously Gâteaux differentiable functions with locally Lipschitz gradients on \( X \), denoted by \( C^{1,1}(X) \). Consider the max function of the form \( f(x) = [\max(g(x),0)]^2 \), where \( x \in X \) and \( g : X \rightarrow \mathbb{R} \). If \( g \) is twice continuously differentiable, then it is known that \( f \) is a \( C^{1,1} \) function and various generalised Hessians for the function \( f \) were given, for example, in Hiriart-Urruty, Strodiot and Nguyen [4] and Yang and Jeyakumar [11]. In this paper, we study the generalised Hessian introduced in [11] for \( f \) when \( g \) is a \( C^{1,1} \) function. It is worth noting that squares of max functions appear in augmented Lagrangian function methods and smoothing

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approximation methods (see [9, 13, 14]). Thus generalised differentiabilities of max functions may be useful in studying optimisation methods.

In Hiriart-Urruty [3] and Vial [10], it was shown that in a finite dimensional space every $C^{1,1}$ function is locally weakly convex. This result is useful in establishing relations between $C^{1,1}$ functions and so-called lower-$C^2$ functions. We reprove this result in a Hilbert space by first obtaining a necessary and sufficient condition for $\eta$-convex functions. This generalises a corresponding characterisation for convex $C^{1,1}$ functions in [11] to generalised convex functions and extends a result of finite dimensional spaces in [10] to infinite dimensional spaces.

2. A Generalised Second-Order Directional Derivative

Let $X^*$ be the dual space of $X$ and $\langle \cdot , \cdot \rangle$ be the canonical pair between $X^*$ and $X$. Let $g : X \rightarrow \mathbb{R}$ be a locally Lipschitz function and $x \in X$. The Michel-Penot generalised directional derivative of $g$ at $x$ in the direction $u \in X$ is defined by

$$g^\circ(x; u) = \sup_{z \in X} \lim_{s \downarrow 0} g(x + sz + su) - g(x + sz),$$

and $g$ is said to be semi-regular at $x$ if the one-sided direction derivative

$$g'(x; u) = \lim_{s \downarrow 0} \frac{g(x + su) - g(x)}{s},$$

exists and is equal to $g^\circ(x; u)$ for every $u \in X$. (See Michel and Penot [7].)

It is known that the max function of semi-regular functions is semi-regular and that the semi-regularity condition can be used to establish strong calculus rules. We now give the following notion of a second-order directional derivative of a $C^{1,1}$ function $f$ in terms of the gradient function $\nabla f$. (See Yang and Jeyakumar [11] and Yang [12].)

**Definition 1:** Let $f : X \rightarrow \mathbb{R}$ be a $C^{1,1}$ function and let $x \in X$. Then the generalised second-order directional derivative of $f$ at $x$ in the directions $(u, v) \in X \times X$, denoted by $f^{\infty}(x; u, v)$, is defined by

$$f^{\infty}(x; u, v) = \sup_{z \in X} \lim_{s \downarrow 0} \frac{(\nabla f(x + sz + su), v) - (\nabla f(x + sz), v)}{s}. \tag{1}$$

The generalised Hessian of $f$ at $x \in X$ for each $u \in X$, denoted by $\partial^{\infty} f(x)(u)$, is defined by

$$\partial^{\infty} f(x)(u) = \{z^* \in X^* : f^{\infty}(x; u, v) \geq (z^*, v), \forall v \in X\}. \tag{2}$$

The following proposition summarises some basic properties of the generalised second-order directional derivative and the generalised Hessian which are used in the sequel (see [11]).
PROPOSITION 1. Let \( f : X \to \mathbb{R} \) be \( C^{1,1} \) and \( x, u, v \in X \). Then the following properties hold

(i) \( f^\infty(x;u,v) \) is finite and bi-sublinear as a function of \( u \) and \( v \);

(ii) \( \partial f(x)(u) \) is a nonempty, convex and weak*-compact subset of \( X^* \);

(iii) \(-f)^\infty(x;u,v) = f^\infty(x;-u,v) = f^\infty(x;u,-v)\);

(iv) \( f^\infty(x;u,\alpha v) = f^\infty(x;\alpha u,v), \quad \forall \alpha \in \mathbb{R} \setminus \{0\} \).

The function \( f \) is said to be twice weakly Gâteaux differentiable at \( x \) if \( f \) is continuously Gâteaux differentiable near \( x \) and its gradient function \( \nabla f \) is weakly Gâteaux differentiable at \( x \), that is, there exists a linear function \( D^2 f(x) : X \to X^* \) such that for each \( v \in X^{**}, u \in X \), the following holds:

\[
\lim_{s \to 0} \frac{\langle \nabla f(x + su), v \rangle - \langle \nabla f(x), v \rangle}{s} = \langle D^2 f(x)(u), v \rangle.
\]

Examples of \( C^{1,1} \) functions appear, for example, in penalty function methods, augmented Lagrangian methods, proximal point methods and smooth approximation methods. We now give some examples of \( C^{1,1} \) functions.

EXAMPLE 1. Let \( X = \mathbb{R} \) and let \( g : \mathbb{R} \to \mathbb{R} \) be a locally Lipschitz function. Then the function \( f : \mathbb{R} \to \mathbb{R}, \) defined by

\[
f(x) = \int_0^x g(t)dt, \quad x \in \mathbb{R},
\]

is a \( C^{1,1} \) function. If in addition \( g \) is increasing, then \( f \) is a convex \( C^{1,1} \) function.

EXAMPLE 2. Let \( X \) be a Hilbert space and let

\[
h(x) = \frac{1}{2} \|x\|^2, \quad x \in X.
\]

Then \( h \) is \( C^{1,1} \). Furthermore, it is twice weakly Gâteaux differentiable. We have

(3) \[ h^\infty(x;u,v) = \langle u,v \rangle, \quad \forall u,v \in X. \]

EXAMPLE 3. Let \( C \) be a subset of \( X \). Define the following functions, for each \( z \in X \),

\[
d_C(z) = \inf\{\|x - y\| : y \in C\},
\]

\[
\phi(x) = \frac{1}{2} d_C^2(x),
\]

\[
P_C(z) = \{y \in C : \|x - y\| = \inf_{z \in C} \|x - z\|\}.\]
Two special cases:

(i) $C = \{0\}$, we have $\phi(x) = 1/2 \|x\|^2$ which was considered in Example 2;

(ii) $C = E_1$, a closed interval in $\mathbb{R}$ (bounded or unbounded), then $d^2_{\mathcal{E}_1}(x)$
can be used in formulating exterior point methods and augmented Lagrangian methods, see [9]. In particular, if $C = (-\infty,0]$, then $\phi(x) = 1/2[\max\{x,0\}]^2$.

If $C$ is a closed convex subset of a Hilbert space, then $P_C(\cdot)$ is single-valued, Lipschitz with Lipschitz constant $L(P_C(\cdot)) = 1$ and

$$\nabla \phi(\cdot) = (I - P_C)(\cdot),$$

see Holmes [5]. Hence $\phi(x)$ is a $C^{1,1}$ function. The generalised second-order directional derivative of $\phi(x)$ was calculated in [12] under certain regularity conditions. We now obtain an estimate of the generalised second-order directional derivative for this function without regularity conditions.

**Proposition 2.** Let $X$ be a Hilbert space. If $C$ is a closed convex subset of $X$, then

$$\phi^\infty(x;u,u) \leq 0, \quad \forall u \in X.$$  

**Proof:** Since $P_C$ is Lipschitz with Lipschitz constant $L(P_C(\cdot)) = 1$ (see Example 3), we have from (4)

$$\begin{align*}
(d^2_{P_C})^\infty(x;u,u) &= \sup_{x \in X} \lim_{z \to 0} \left( (2(P_C - I)(x + su + sz),u) - (2(P_C - I)(x + sz),u) \right) \\
&= \sup_{x \in X} \lim_{z \to 0} \frac{2(P_C(x + su + sz) - P_C(x + sz), -u) - 2s(u,u)}{z} \\
&= 2 \sup_{x \in X} \lim_{z \to 0} \frac{(P_C(x + su + sz) - P_C(x + sz), -u) - 2(u,u)}{z} \\
&\leq 0, \quad \forall x, u \in X.
\end{align*}$$

Then (5) holds. 

3. MAX FUNCTION AND GENERALISED HESSIAN

In this section, we study generalised differentiability properties of the max functions of the form

$$m_p(x) = \max\{g(x),0\}, \quad x \in X,$$
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where $X$ is a Banach space, $g : X \rightarrow \mathbb{R}$ and $p \geq 2$. It is known that the max function $m_p(x)$ is (Gâteaux) differentiable if $g$ is (Gâteaux) differentiable. Indeed, we have

$$\nabla m_p(x) = p[\max\{g(x), 0\}]^{p-1}\nabla g(x), \quad \forall x \in X.$$  

When $g$ has twice differentiability properties and $p = 2$, various generalised Hessians of the function $m_2$ have been obtained, for example, in [2, 4, 11, 14]. We are now able to obtain a characterisation of the generalised Hessian of $m_p$ in terms of the generalised Hessians of $g$ when $g$ is $C^{1,1}$ function. Moreover, we obtain necessary and sufficient conditions for $m_p$ to be twice weakly Gâteaux differentiable.

**Theorem 1.** Let $g : X \rightarrow \mathbb{R}$ be $C^{1,1}$ and $p \geq 2$. Then $m_p(x) = [\max\{g(x), 0\}]^p$ is $C^{1,1}$ and for each $u \in X$, the generalised second-order directional derivative of $m_p$ at $x$ is given by

$$m_p^{\infty}(x;u,v) = \begin{cases} pg(x)g^{\infty}(x;u,v) + p(\nabla g(x), u)(\nabla g(x), v), & \text{if } g(x) > 0; \\ 0, & \text{if } g(x) < 0; \\ p\max\{(\nabla g(x), u)(\nabla g(x), v), 0\}, & \text{if } g(x) = 0. \end{cases}$$

**Proof:** Since $g$ is $C^{1,1}$, it is clear from (7) that $m_p$ is $C^{1,1}$. For simplicity, we prove the results for the case $p = 2$. We shall consider the following three cases:

**Case I.** Let $g(x) > 0$. Then we have from (7) that the equality, $\nabla g(x) = 2g(x)\nabla g(x)$, holds in a neighbourhood of $x$. Since $g$ is $C^{1,1}$, it is semi-regular and so, we get

$$m_2^{\infty}(x;u,v) = \sup_{\delta \in X} \lim_{s \to 0} -\frac{1}{\delta} \left\{2g(x + su + sz)(\nabla g(x + su + sz), v) - 2g(x + sz)(\nabla g(x + sz), v)\right\}$$

$$= \sup_{\delta \in X} \lim_{s \to 0} -\frac{1}{\delta} \left\{2g(x + sz)((\nabla g(x + su + sz) - \nabla g(x + sz), v)) + 2(g(x + su + sz) - g(x + sz))(\nabla g(x + sz), v)\right\}$$

$$= \sup_{\delta \in X} \lim_{s \to 0} -\frac{1}{\delta} \{2g(x)((\nabla g(x + su + sz) - \nabla g(x + sz), v)) + 2(g(x + su + sz) - g(x + sz))(\nabla g(x), v)\}$$

$$= \sup_{\delta \in X} \lim_{s \to 0} -\frac{1}{\delta} \{2g(x)((\nabla g(x + su + sz) - \nabla g(x + sz), v)) + \lim_{s \to 0} \frac{1}{\delta} 2(g(x + su) - g(x))(\nabla g(x), v)\}$$

$$= 2g(x)g^{\infty}(x;u,v) + 2(\nabla g(x), u)(\nabla g(x), v),$$

thus the result holds.
CASE II. Let \( g(x) < 0 \). Then we obtain \( m_2(x) = 0 \) in a neighbourhood of \( x \). Hence the result is true.

CASE III. Let \( g(x) = 0 \). In fact, when \( p = 2 \), (7) becomes

\[
\nabla m_2(x) = 2 \max \{ g(x), 0 \} \nabla g(x), \quad \forall x \in X.
\]

For each \( x \in X \), we get

\[
\lim_{s \downarrow 0} \frac{\max \{ g(x + sz), 0 \} \left( (\nabla g(x + su + sz), v) - (\nabla g(x + sz), v) \right)}{s} = 0.
\]

Thus we have

\[
m_2^\infty(x; u, v) = \sup_{x \in X} \lim_{s \downarrow 0} \sup \left\{ 2 \max \{ g(x + sz), 0 \} (\nabla g(x + su + sz), v) - 2 \max \{ g(x + sz), 0 \} (\nabla g(x + sz), v) \right\}
\]

\[
= \sup_{x \in X} \lim_{s \downarrow 0} \left[ 2 \max \{ g(x + sz), 0 \} \left( (\nabla g(x + su + sz), v) - (\nabla g(x + sz), v) \right) + 2 \left( \max \{ g(x + su + sz), 0 \} - \max \{ g(x + sz), 0 \} \right) (\nabla g(x + sz), v) \right]
\]

\[
= \sup_{x \in X} \lim_{s \downarrow 0} \left[ 2 \max \{ g(x + su + sz), 0 \} - \max \{ g(x + sz), 0 \} \right] (\nabla g(x), v).
\]

Since \( g \) is \( C^{1,1} \), \( \max \{ g, 0 \} \) is semi-regular, thus we obtain

\[
m_2^\infty(x; u, v) = \frac{\max \{ g(x + su)(\nabla g(x)), 0 \} - \max \{ g(x)(\nabla g(x)), 0 \}}{s}
\]

\[
= 2 \max \{ (\nabla g(x), u)(\nabla g(x), v), 0 \}.
\]

Then the proof is complete. \( \square \)

REMARK 1. From the Hahn-Banach Theorem [5], we get the following inclusions of the generalised Hessian,

\[
\delta^\infty m_p(x)(u)
\]

\[
= \begin{cases} \{ pg(x)^{p-1} x + (p-1)g(x)^{p-2}(\nabla g(x), u) \nabla g(x) : x^* \in \delta^\infty g(x)(u) \}, & \text{if } g(x) > 0; \\ \{ 0 \}, & \text{if } g(x) < 0; \\ \{ \beta p(p-1)g(x)^{p-2}(\nabla g(x), u) \nabla g(x) : \beta \in [0,1] \}, & \text{if } g(x) = 0. \end{cases}
\]
REMARK 2. It follows from a second-order chain rule (see [12, Theorem 2]) that

$$\partial^\infty m_p(x)^{(u)}$$

$$\subseteq \begin{cases} 
pg(x)^{p-1}x + p(p-1)g(x)^{p-2}(\nabla g(x), u)\nabla g(x) : \\
x^* \in \partial^\infty g(x)^{(u)}), & \text{if } g(x) > 0; \\
\{0\}, & \text{if } g(x) < 0; \\
\{p(p-1)g(x)^{p-2}(\nabla g(x), u)\nabla g(x) : \beta \in [0, 1]\}, & \text{if } g(x) = 0.
\end{cases}$$

and that (9) holds with equality if $\nabla g(x)$ is onto. Comparing (8) with (9), we see that the onto condition used in [11] is only sufficient.

Using Theorem 1, we obtain characterisations of twice weakly Gâteaux differentiability of the max function $m_p$ when the function $g$ is $C^{1,1}$.

PROPOSITION 3. Let $X$ be a reflexive Banach space and let $g$ be $C^{1,1}$ and $x \in X$ be a point satisfying $g(x) = 0$. Then the function $m_p$ is twice weakly Gâteaux differentiable at $x$ if and only if $\nabla g(x) = 0$ and $g$ is twice weakly Gâteaux differentiable at $x$.

PROOF: From Theorem 1, $\partial^\infty m_p(x)^{(u)}$ is single-valued for all $u \in X$ if and only if $\nabla g(x) = 0$ and $\partial^\infty g(x)^{(u)}$ is single-valued for all $u \in X$. Then the conclusion holds.

We finish this section with a couple of numerical examples to show the structure of the generalised Hessian of max functions.

EXAMPLE 4. Let $m_p(x) = [\max\{z, 0\}]^p, x \in \mathbb{R}$ and $p \geq 2$. Then we have

$$\partial^\infty m_p(x)^{(u)} = \begin{cases} 
\{p(p-1)x^{p-2}u\}, & \text{if } x > 0; \\
\{0\}, & \text{if } x < 0; \\
\{\beta p(p-1)x^{p-2}u : \beta \in [0, 1]\}, & \text{if } x = 0.
\end{cases}$$

EXAMPLE 5. Let $m_2(x) = [\max\{\int_0^2 t^2 \sin(1/t) dt + 1, 0\}]^2, x \in \mathbb{R}$. Then our generalised Hessian $\partial^\infty m_2(x)^{(u)}$ at $x = 0$ is

$$\partial^\infty m_2(0)(u) = \{0\}.$$
DEFINITION 2. Let $C$ be a convex subset of $X$ and let $f : C \to \mathbb{R}$. The function $f$ is said to be $\eta$-convex on $C$ if there exist a real number $\eta$ and a convex function $h : C \to \mathbb{R}$ such that $f(x) = h(x) + \eta \|x\|^2$, $\forall x \in C$.

Note that if $\eta > 0$, then $f$ is said to be strongly convex on $C$; if $\eta = 0$, then $f$ is convex on $C$; if $\eta < 0$, then $f$ is said to be weakly convex on $C$, see Vial [10] and Jeyakumar [6].

DEFINITION 3. (i) $f : X \to \mathbb{R}$ is said to be locally weakly convex on $X$ if for each $x \in X$, there exists $r > 0$ such that $f$ is weakly convex on an open ball centred at $x$ with radius $r$, denoted by $U^o(x, r)$;

(ii) $f$ is said to be globally weakly convex if $f$ is weakly convex on $X$.

The following characterisation for a $C^{1,1}$ function to be convex is given in [11].

LEMMA 1. Let $X$ be a Banach space and let $f : X \to \mathbb{R}$. Then $f$ is convex on $X$ if and only if

$$f^{oo}(x; u, -u) \geq 0, \forall x, u \in X.$$ 

We first obtain a characterisation of $\eta$-convexity in terms of the generalised second-order directional derivative. It is worth noting that this result paves the way to establishing and generalising connections between a $C^{1,1}$ function and weak convexity in a Hilbert space.

THEOREM 2. Let $X$ be a Hilbert space and let $f : X \to \mathbb{R}$ be $C^{1,1}$. Then $f$ is $\eta$-convex on $X$ if and only if

$$f^{oo}(x; u, -u) \geq -2\eta \|u\|^2, \forall x, u \in X.$$ 

PROOF: Let $f$ be a $C^{1,1}$ function. If $f$ is $\eta$-convex on $X$, then there exist a real number $\eta$ and a convex function $h : X \to \mathbb{R}$ such that $f(x) = h(x) + \eta \|x\|^2$, $\forall x \in X$. Since $f$ and $\eta \|\cdot\|^2$ are $C^{1,1}$, the function $h$ is also $C^{1,1}$. Note from (3) that

$$\left(\|\cdot\|^2\right)^{oo}(x; u, -u) = -2\|u\|, \forall u \in X.$$ 

Hence from the triangle inequality, we obtain

$$f^{oo}(x; u, -u) \leq h^{oo}(x; u, -u) + \left(\eta \|\cdot\|^2\right)^{oo}(x; u, -u)$$

$$\leq h^{oo}(x; u, -u) - 2\eta \|u\|^2, \forall x, u \in X.$$ 

From Lemma 1, $h^{oo}(x; u, -u) \leq 0, \forall x, u \in X$, so we have

$$f^{oo}(x; u, -u) \leq -2\|u\|^2, \forall x, u \in X.$$ 

Conversely, if (10) holds, then

$$f(x) = \left(f(x) - \eta \|x\|^2\right) + \eta \|x\|^2, \forall x, u \in X,$$
and the function \( f(x) - \eta \|x\|^2 \) is convex on \( X \) since

\[
(f - \eta \|\cdot\|^2)\bigg|_{(x; u, -u)} \leq f^\infty(x; u, -u) + 2\eta \|u\|^2 \leq 0, \quad \forall x, u \in X.
\]

Thus \( f \) is \( \eta \)-convex on \( X \).

Clearly Theorem 2 is an extension of Lemma 1. Moreover, when \( \eta = 0 \), Theorem 2 reduces to Lemma 1. As an immediate application of Theorem 2, let \( g : \mathbb{R} \rightarrow \mathbb{R} \) be a locally Lipschitz function. Then the function \( f \) defined in Example 1 is \( \eta \)-convex if and only if \( g^\circ(x; -1) \leq 2\eta, \forall x \in \mathbb{R} \). The following corollary shows that Theorem 2 generalises a result in [10, Proposition 4.11] where twice differentiability is required.

**Corollary 1.** Let \( X \) be a Hilbert space and let \( f : X \rightarrow \mathbb{R} \) be twice weakly Gâteaux differentiable. Then \( f \) is \( \eta \)-convex on \( X \) if and only if

\[
\langle D^2 f(x)(u), u \rangle \geq 2\eta \|u\|^2, \quad \forall x, u \in X.
\]

**Proof:** This follows from the fact that \( f \) is twice weakly Gâteaux differentiable, thus

\[
f^\infty(x; u, -u) = -\langle D^2 f(x)(u), u \rangle, \quad \forall x, u \in X.
\]

Now we establish that in a Hilbert space every \( C^{1,1} \) function is locally weakly convex using our generalised second-order directional derivative.

**Theorem 3.** Let \( X \) be a Hilbert space. If \( f : X \rightarrow \mathbb{R} \) is a \( C^{1,1} \) function, then \( f \) is locally weakly convex on \( X \).

**Proof:** Let \( f : X \rightarrow \mathbb{R} \) be a \( C^{1,1} \) function. Then for any fixed \( \bar{x} \in X \), it follows from the locally Lipschitz condition of \( \nabla f \) that there exist \( L(\nabla f, \bar{x}) > 0 \) and \( r > 0 \) such that

\[
\|\nabla f(y) - \nabla f(x)\| \leq L(\nabla f, \bar{x}) \|y - x\|, \quad \forall y, x \in U^\circ(\bar{x}, r).
\]

Let \( \eta \geq (L(\nabla f, \bar{x}))/2 \). Then for any \( u \in X, x \in U^\circ(\bar{x}, r) \), we have

\[
f^\infty(x; u, -u) = \sup_{x \in X} \limsup_{s \to 0} \frac{\langle \nabla f(x + su + sz), u \rangle - \langle \nabla f(x + sz), -u \rangle}{s}
\leq L(\nabla f, \bar{x}) \|u\|^2 \leq 2\eta \|u\|^2.
\]

So,

\[
(f + \eta \|\cdot\|^2)\bigg|_{(x; u, -u)} \leq f^\infty(x; u, -u) + \left(\eta \|\cdot\|^2\right)^\infty(x; u, -u)
= f^\infty(x; u, -u) - 2\eta \|u\|^2
\leq 0, \quad \forall x \in U^\circ(\bar{x}, r), \ u \in X.
\]
From Lemma 1, \( f + \eta \|x\|^2 \) is convex on \( U^\circ(\bar{x}, r) \). Then \( f(x) = (f(x) + \eta \|x\|^2) - \eta \|x\|^2 \), in which \( f + \eta \|x\|^2 \) is convex on \( U^\circ(\bar{x}, r) \). Hence \( f \) is locally weakly convex on \( X \).

It is well known that the function \(-d^2_C(x)\) is globally weakly convex, where \( C \) is a closed convex subset of a Hilbert space. We present a proof of this result using our generalised second-order directional derivative. Recall that \(-d^2_C(x)\) is a \( C^{1,1} \) function, see Example 3.

**Proposition 4.** Let \( X \) be a Hilbert space. If \( C \) is a closed convex subset of \( X \), then \(-d^2_C(x)\) is globally weakly convex.

**Proof:** Observe that

\[
-d^2_C(x) = \left(2 \|x\|^2 - d^2_C(x)\right) - 2 \|x\|^2.
\]

Thus we need to prove that \( x \mapsto 2 \|x\|^2 - d^2_C(x) \) is convex on \( X \). From Proposition 2, we have

\[
(-d^2_C)^\circ (x; u, -u) = (d^2_C)^\circ (x; u, u) \leq 0, \quad \forall x, u \in X.
\]

Then from (3)

\[
\left(2 \|\cdot\|^2 - d^2_C\right)^\circ (x; u, -u) \leq \left(2 \|\cdot\|^2\right)^\circ (x; u, -u) + (-d^2_C)^\circ (x; u, -u)
\]

\[
\leq -4(u, u)
\]

\[
\leq 0, \quad \forall x, u \in X.
\]

From Lemma 1, the function \( x \mapsto 2 \|x\|^2 - d^2_C(x) \) is convex on \( X \). Therefore \(-d^2_C(x)\) is globally weakly convex.

**Corollary 2.** Let \( X \) be a Hilbert space and let \( g : X \to \mathbb{R} \) be a convex function. Then \( m_2(x) = -[\max\{g(x), 0\}]^2 \) is globally weakly convex.

**Proof:** Let \( C = \{x \in X : g(x) \leq 0\} \). Thus \( C \) is a closed convex subset and \( d^2_C(x) = [\max\{g(x), 0\}]^2 \). The conclusion follows from Proposition 4.

5. Discussion

Let \( X \) be a Hilbert space and let \( f : X \to \mathbb{R} \). Then the following classes of functions are introduced and studied in [3, 6, 8, 10]:

(i) the function \( f \) is said to be **locally difference convex** on \( X \) if for every \( \bar{x} \in X \), there exist a convex neighbourhood \( N(\bar{x}) \) of \( \bar{x} \), and convex functions \( p_N, q_N : X \to \mathbb{R} \) such that \( f(x) = p_N(x) - q_N(x), \quad \forall x \in N(\bar{x}) \). This class of functions is denoted by \( \text{LDC}(X) \). The function \( f \) is said to be **difference convex** on \( X \) if there exist two convex functions \( p, q : X \to \mathbb{R} \) such that \( f(x) = p(x) - q(x), \quad \forall x \in X \);
(ii) the function \( f \) is said to be lower-\( C^2 \) on \( X \) if for every \( \overline{x} \in X \), there exist a convex neighbourhood \( N(\overline{x}) \) of \( \overline{x} \), a convex function \( p_N \) and a quadratic convex function \( q_N \) such that \( f(x) = p_N(x) - q_N(x) \), \( \forall x \in N(\overline{x}) \). This class of functions is denoted by \( LC^2(X) \).

It follows from the previous definitions that every locally weakly convex function is locally difference convex. In general, a quadratic convex function in a Hilbert space has the form

\[
\langle A(u), u \rangle + \langle b, u \rangle + c,
\]

where \( b \in X, c \in \mathbb{R} \) and \( A : X \to X \) satisfies \( \langle A(x), z \rangle \geq 0, \langle A(x), y \rangle = \langle A(y), x \rangle \). In particular \( \|x\|^2 = \langle x, x \rangle \) is a quadratic convex function. Hence it follows from Theorem 3 that every \( C^{1,1} \) function is lower-\( C^2 \). It is clear that every lower-\( C^2 \) function is locally difference convex. Therefore we have established that

\[
C^{1,1}(X) \subset LC^2(X) \subset LDC(X),
\]

where \( X \) is a Hilbert space. This result was initially given in Hiriart-Urruty [3] and Vial [10] in a finite dimensional space.

REFERENCES


