DESARGUES' THEOREM IN n-SPACE

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Introduction

Two sets of $r + 2$ points, P_i, P'_i, each spanning a projective space of $r + 1$ dimensions, $[r + 1]$, which has no solid ([3]) common with that spanned by the other, are said to be *projective* from an $[r - 1]$, if here is an $[r - 1]$ which meets the $r + 2$ joins $P_iP'_i$. It is to be proved that the two sets are projective, if and only if the $r + 2$ intersections A_i of their corresponding $[r]$s lie in a line a. A_i are said to be the *arguesian points* and a the *arguesian line* of the sets. When $r = 1$, the proposition becomes the well-known Desargues' two-triangle theorem (**3**) in a plane. Therefore in analogy with the same we name it as the *Desargues' theorem in* $[2r]$. Following Baker (**1**, pp. 8—39), we may prove this theorem in the same synthetic style by making use of the axioms and the corresponding proposition of incidence in $[2r + 1]$ or with the aid of the Desargues' theorem in a plane and the axioms of $[2r]$ only. But the use of symbols makes its proof more concise; the algebraic approach adopted here is due to the referee (Arts. 2, 3, 5, 6, 7). Pairs of sets of $r + p$ points each projective from an $[r - 1]$ are also introduced to serve as a basis for a much more thorough investigation.

1. Synthetic Outline

Following Coxeter (**4**, p. 7), first we observe that the theorem is obvious almost when the 2 $[r + 1]$s of the 2 sets meet in a line a and therefore both lie in a $[2r + 1]$, because in this case the projections from the transversal $[r - 1]$ of the $r + 2$ joins of their corresponding points are the $r + 2$ points A_i, which all lie in a. The theorem for the sets in a $[2r]$ arises as a limiting case.

2. The Two Theorems

To avoid the considerations of continuity, we may formulate the theorems for $[2r + 1]$ and $[2r]$ separately as follows:

I. *Given 2 sets of $r + 2$ points, P_i, P'_i, each spanning $[r + 1]$, and between them spanning $[2r + 1]$, the necessary and sufficient condition that there should*

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be an $[r-1]$ *which meets the* $r+2$ *lines* $P_i P'_i$ *is that each of the* $r+2$ *pairs of* $[r]$*ss uch as* $[P_0, \cdots, P_{i-1}, P_{i+1}, \cdots, P_{r+1}]$ *and* $[P'_0, \cdots, P'_{i-1}, P'_{i+1}, \cdots, P'_{r+1}]$ *should have a common point* A_i.

II. *Given* 2 *sets of* $r+2$ *points,* P_i, P'_i, *each spanning* $[r+1]$, *and between them spanning* $[2r]$, *the necessary and sufficient condition that there should be an* $[r-1]$ *which meets the* $r+2$ *lines* $P_i P'_i$ *is that the* $r+2$ *points of concurrence* A_i *of the pairs of* $[r]$*s such as* $[P_0, \cdots, P_{i-1}, P_{i+1}, \cdots, P_{r+1}]$ *and* $[P'_0, \cdots, P'_{i-1}, P'_{i+1}, \cdots, P'_{r+1}]$ *should be collinear.*

3. The Proofs of these Theorems

We may use Baker's method (**6**) of "point symbols" and "algebraic symbols", or treat "points" as being represented by vectors with $2r+2$ components, the vectors P_i and kP_i corresponding to the same point. There are either two (in $[2r+1]$) or three (in $[2r]$) identities (or "syzygies") connecting the $2r+4$ point symbols (or vectors) P_i, P'_i.

Theorem I. Assume the $r+2$ lines $P_iP'_i$ are met by an $[r-1]$, and that by an adjustment of multipliers the points in which the $[r-1]$ meets the lines are $P_i + P'_i$. Since these $r+2$ points lie in $[r-1]$, there are 2 syzygies, say $\sum_0^{r+1} (P_j + P'_j) = 0$, $\sum_0^{r+1} k_j(P_j + P'_j) = 0$. From these we deduce $r+2$ relations such as

$$\sum_{j=0}^{r+1} (k_i - k_j)(P_j + P'_j) = 0.$$

This is the condition that the 2 $[r]$*s* quoted in the theorem have common the point $A_i \equiv \sum (k_i - k_j)P_j \equiv -\sum (k_i - k_j)P'_j$. The converse can be proved equally simply.

Theorem II. The algebraic part of the argument from the existence of the $[r-1]$ to the collinearity of the $r+2$ points is identical with that above.

For the converse we assume that among the $2r+4$ points there are 3 syzygies, say $\sum (P_i + P'_i) = 0$, $\sum h_iP_i + \sum h'_iP'_i = 0$, $\sum k_iP_i + \sum k'_iP'_i = 0$. It has to be shown that if these are such that the $r+2$ points A_i are collinear, then the $r+2$ lines have a transversal $[r-1]$. The plane in which the 2 $[r+1]$*s* meet is $U^0H^0K^0$ where

$$U^0 = \sum P_j = -\sum P'_j, \qquad H^0 = \sum h_j P_j, \qquad K^0 = \sum k_j P_j,$$

and the points A_i are

$$A_i = (h_ik'_i - h'_ik_i)U^0 + (k_i - k'_i)H^0 + (h'_i - h_i)K^0.$$

The $r+2$ points of this form are collinear, if and only if multipliers p, q, r

can be found such that

$$p(h_i k_i' - h_i' k_i) + q(k_i - k_i') + r(h_i' - h_i) = 0.$$

From this the existence of the transversal $[r-1]$ can be deduced.

4. The Associated Arguesian Lines

The pair of sets of $r+2$ points P_i, P_i' projective from an $[r-1]$ give rise to $2^{r+1}-1$ more such pairs obviously projective from the same $[r-1]$. For there are 2 choices for every point, P_i or P_i', to belong to a set independent of each other. For example, $r+2$ pairs are of the type $[P_0', P_1, \cdots, P_{r+1}]$, $[P_0, P_1', \cdots, P_{r+1}']$; $\binom{r+2}{2}$ pairs of the type $[P_0', P_1', P_2, \cdots, P_{r+1}]$, $(P_0, P_1, P_2', \cdots, P_{r+1}']$, and so on. Evidently every subset of $r+1$ points belongs to 2 sets, e.g., $[P_1, \cdots, P_{r+1}]$ belongs to $[P_0, \cdots, P_{r+1}]$ and $[P_0', P_1, \cdots, P_{r+1}]$. Thus: *there are in all 2^{r+1} arguesian lines, one for each pair of such sets, and $2^r(r+2)$ arguesian points, $r+2$ on each line and each common to 2 lines, such that every line meets $r+2$ other lines, skew to each other.*

It is assumed here that no r lines P_iP_i' lie in a $[2r-2]$. For otherwise a number of arguesian points coincide and the picture is no longer general. For example, if the lines for $i = 1, \cdots, r$ lie in a $[2r-2]$, the 2 $[r-1]$s $[P_1, \cdots, P_r]$, $[P_1', \cdots, P_r']$ meet in a point which coincides with the 4 arguesian points $[P_0, \cdots, P_r] \cdot [P_0', \cdots, P_r']$, $[P_1, \cdots, P_{r+1}] \cdot [P_1', \cdots, P_{r+1}']$, $[P_0', P_1, \cdots, P_r] \cdot [P_0, P_1', \cdots, P_r']$, $[P_1, \cdots, P_r, P_{r+1}'] \cdot [P_1', \cdots, P_r', P_{r+1}]$.

5. Redundant Coordinates

Let (x_i, x_i') in the symbol $(x_iP_i + x_i'P_i')$ be taken as coordinates initially in $[2r+3]$. The 2 syzygies $U = \sum (P_i + P_i') = 0, K = \sum k_i(P_i + P_i') = 0$ correspond to projections from $[2r+3]$ on to $[2r+1]$ from the points whose symbols are U and K. Thus $\sum (u_i x_i + u_i' x_i') = 0$ represents a prime in $[2r+1]$ only if $\sum (u_i + u_i') = 0$ and $\sum k_i(u_i + u_i') = 0$, and a quadratic form in x_i, x_i' represents a quadric in $[2r+1]$ only if in $[2r+3]$ it represents a quadric cone with the line UK as vertex. It represents a quadric in a subspace of $[2r+1]$ only if the subspace is the projection of a space in $[2r+3]$ that is tangent to the quadric in $[2r+3]$ at every point of UK.

6. Case $r = 2$

a) Take the two tetrads of points A, B, C, D; A', B', C', D' connected by the two syzygies $U = A + A' + B + B' + C + C' + D + D' = 0$,

$$K = a(A + A') + b(B + B') + c(C + C') + d(D + D') = 0.$$

The first arguesian line contains the 4 points

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$$
\begin{array}{ll}
(b-a)B+(c-a)C+(d-a)D, \text{ say} & .000, \\
(a-b)A+(c-b)C+(d-b)D, & 0.00, \\
(a-c)A+(b-c)B+(d-c)D, & 00.0, \\
(a-d)A+(b-d)B+(c-d)C, & 000..
\end{array}
$$

Interchanging D, D', we find another arguesian line through $(b-a)B+(c-a)C+(d-a)D'$, say .001, etc. The 8 lines and 16 points can be exhibited as the rows and columns of the scheme

.000	0.00	00.0	000.
1.00	.100	001.	00.1
10.0	01.0	.010	0.01
100.	010.	0.10	.001

i.e., they lie in a solid s, and are two tetrads of generators (**6**) of a quadric surface q and their 16 common points.

b) The equations of s are $k(x+x')+l(y+y')+m(z+z')+n(t+t')=0$ where $k+l+m+n=0$, $ak+bl+cm+dn=0$, i.e.

(i) $$\left\|\begin{matrix} x+x' & y+y' & z+z' & t+t' \\ 1 & 1 & 1 & 1 \\ a & b & c & d \end{matrix}\right\| = 0$$

The transversal of the two solids $ABCD$, $A'B'C'D'$ is the line $[A+A', B+B', C+C', D+D']$. s is the "harmonic conjugate" of this w.r. to A, A'; B, B'; C, C'; D, D', viz., the solid

$$[A''=A-A',\ B''=B-B',\quad C''=C-C',\quad D''=D-D'].$$

c) The equation of q is

(ii) $$\left\|\begin{matrix} xx' & yy' & zz' & tt' \\ 1 & 1 & 1 & 1 \\ a & b & c & d \\ a^2 & b^2 & c^2 & d^2 \end{matrix}\right\| = 0$$

since it is satisfied by the 16 points $(b-a)B+(c-a)C+(d-a)D$ etc., i.e., $(0, b-a, c-a, d-a, 0, 0, 0, 0)$ etc., and since further, in the [7] in which the points $A, \cdots, D'$ are independent, the [5] with equations (i) passes through U, K and lies in the tangent primes at those points to the quadric sixfold of which the equation is (ii).

d) Further it can be seen immediately that the 4 points of each of the 4 sets such as .000, .100, .010, .001 are coplanar. These points in fact lie in the plane of which the equations are $x-x'=0$, together with equations (i); the 4 such planes are the faces [1] of the tetrahedron $T''=A''B''C''D''$.

[1] This observation is due to the referee.

e) So far it has been assumed that the basic figure lies in [5]. If it lies in [4], there will be an additional syzygy connecting $A, \cdots, D'$ and the figure in [4] may therefore be considered as the projection of that in [5] from some point, say H. The figure of the 8 arguesian lines will therefore not be affected, unless H lies in s.

7. Case $r = 3$

a) Following the same line of argument, we find in [7] a configuration of 40 points which are collinear by sets of five on 16 lines.

b) Using in [5] redundant coordinates $(x, y, z, t, u, x', y', z', t', u')$ and two points of projection U, K from [7], we find: the figure lies in the [4] s whose equations are

$$\left\| \begin{matrix} x+x' & y+y' & z+z' & t+t' & u+u' \\ 1 & 1 & 1 & 1 & 1 \\ a & b & c & d & e \end{matrix} \right\| = 0,$$

which is the "harmonic conjugate" of the plane $A + A', \cdots, E + E'$.

c) It lies [2] on the pencil of quadrics determined by

$$\left\| \begin{matrix} xx' & yy' & zz' & tt' & uu' \\ 1 & 1 & 1 & 1 & 1 \\ a & b & c & d & e \\ a^2 & b^2 & c^2 & d^2 & e^2 \end{matrix} \right\| = 0.$$

d) The 8 points .0000, .1000, .0100, .0010, .0001, .1001, .0101, .0011 lie in a solid, viz., $x - x' = 0$, and are *associated*, for they all lie on the quadrics (quadric surfaces in the solid) in the system

$$\left\| \begin{matrix} xx' & yy' & zz' & tt' & uu' \\ 1 & 1 & 1 & 1 & 1 \\ a & b & c & d & e \end{matrix} \right\| = 0.$$

The 5 such solids form the common self-polar simplex [2] of the pencil of quadrics.

e) The system of quadrics (in [4]) in this case is "general", and the 16 lines form the general 16_5 figure lying on the Segre [2] quartic surface (**2**, pp. 166—72). In case of general r we shall obtain a system of $r - 1$ linearly independent quadrics in $[r + 1]$, but they are not "general".

8. The Dual of S-configurations

a) Let us recall the tetrads of coplanar arguesian points (Art. 6d) .000, .100, .010, .001 and study their symbols as follows:

[2] These observations are due to the referee.

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$.000 = (b-a)B + (c-a)C + (d-a)D$ is equivalent to $(b-a)B'' + (c-a)C'' + (d-a)D''$ (Art. 6b) because of their connecting syzygies (Art. 6a). Similarly

$.100 = (a-b)B'' + (c-a)C'' + (d-a)D''$,
$.010 = (b-a)B'' - (c-a)C'' + (d-a)D''$,
$.001 = (b-a)B'' + (c-a)C'' - (d-a)D''$.

Thus they form a quadrangle whose diagonal triangle is $B''C''D''$. Similarly behave the other such 3 tetrads of coplanar arguesian points in the other 3 respective faces of the tetrahedron T'' which is then self-polar for the quadric q (Art. 6c).

b) In the same manner we may observe that the octad of arguesian points (Art. 7d) form the pair of tetrahedra, $T_A = (.0000, .1001, .0101, .0011)$, $T'_A = (.1000, .0100, .0010, .0001)$ desmic with the tetrahedral face $T''_A = B''C''D''E''$ of the 4-simplex $S'' = A''B''C''D''E''$, where $A'' = A - A'$, etc. Thus they form a *closed set* (**5**) w. r. to their diagonal tetrahedron T''_A such that all quadrics, for which T''_A is self-polar, passing through one of the 8 points pass through all of them. Similarly behave the other 4 such octads of arguesian points in the other 4 respective solid faces of S''.

c) Now we are in a position to state the general proposition as follows (Arts. 3, 4, 7b): *The $2^r(r+2)$ arguesian points arising from a pair of sets of $r+2$ points P_i, P'_i projective from an $[r-1]$ distribute into $r+2$ sets of 2^r each such that the points of a set form the vertices of the dual of an r-dimensional S-configuration* [3] *whose diagonal r-simplex forms a prime face of the $(r+1)$-simplex with vertices at the $r+2$ points $P_i - P'_i$ which determine the "harmonic conjugate" $[r+1]$ of the transversal $[r-1]$ of the $r+2$ lines $P_iP'_i$* (**5**).

d) The preceding proposition indicates the construction of the system of $r-1$ linearly independent quadrics in $[r+1]$ referred to above (Art. 7e) as follows:

Construct the system of quadrics for which the simplex $S'' = P''_0 \cdots P''_{r+1}$ is self-polar, where $P''_i = P_i - P'_i$. Let them further pass through 3 arguesian points, one in each of 3 prime faces of S''. This system then contains all the $3 \cdot 2^r$ arguesian points in the 3 faces of S'' considered. For, the vertices of the dual of an r-dimensional S-configuration form a *closed set* (**5**) of 2^r points w. r. to their common diagonal r-simplex such that all the $(r-1)$-quadrics in the $[r]$ of the r-simplex, for which it is self-polar, and which pass through one of them, pass through all of them. Again each arguesian line has just one arguesian point common with each prime face of S'' and therefore has 3 points common with the system of quadrics which then contain all the arguesian lines as required.

[3] "Dual of an r-dimensional S-configuration": the system of points $(\pm 1, \pm 1, \ldots, \pm 1)$, cf. (**5**).

9. Projective Sets of $r + 3$ $(r > 2)$ Points

a) Consider a pair of sets of $r + 3$ points, P_i, P'_i, each spanning an $[r + 2]$ which has no [5] common with that spanned by the other, projective from an $[r - 1]$ such that it meets the $r + 3$ joins $P_iP'_i$. They give rise to $r + 3$ arguesian lines, one for each pair of their corresponding subsets of $r + 2$ points each, which then evidently lie in a plane, referred as the *arguesian plane* of the sets.

b) Further one pair of such projective sets gives rise to $2^{r+2} - 1$ more such pairs (cf. Art. 4). Thus: *Given a pair of sets, of $r + 3$ points each, projective from an $[r - 1]$, there arise 2^{r+2} arguesian planes and $2^{r+1}(r + 3)$ arguesian lines, $r + 3$ lines in each plane, each line common to 2 planes which then lie in a solid such that there are $2^{r+1}(r + 3)$ such solids, each containing $2r + 5$ lines. There are $2^r\binom{r+3}{2}$ arguesian points. $(r + 2)^2$ in each solid, each lying on 4 lines and 4 planes which lie in a [4] such that there are $2^r\binom{r+3}{2}$ such [4]s in all, each containing 4 solids, 4 planes, $4(r + 2)$ lines and $2r^2 + 6r + 5$ points.*

c) We may introduce here an *arguesian triangle* too as one formed by a triad of arguesian lines, that being possible in arguesian planes only with vertices at 3 arguesian points therein. Evidently there are $2^{r+2}\binom{r+3}{3}$ such triangles, $\binom{r+3}{3}$ in each plane. Thus: *Through every vertex of an arguesian triangle PQR there passes just one other arguesian plane determined by the other 2 arguesian lines through it. The 3 such planes meet in pairs at the vertices of another arguesian triangle $P'Q'R'$ such that P, Q, R, P', Q', R' constitute a "5-dimensional octahedron" with the 2 skew planes PQR, $P'Q'R'$ as a pair of its opposite planes, the other 3 pairs being $PQ'R$, $P'QR'$, PQR', $P'Q'R$; $P'QR$, $PQ'R'$.*

d) This 5-*dimensional octahedron*, or, say, 5-*octahedron*, occurs in many contexts, e.g., as the Grassmann representative in [5] of the lines through the vertices and in the faces of a tetrahedron in a solid. But for immediate reference we may note down here its make-up expressed symbolically following Baker (**6**; **2**, p. 104) as follows: 6(., 4, 4, 8, 5)12(2, ., 2, 5, 4)8(3, 3, ., 3, 3) 12(4, 5, 2, ., 2)6(5, 8, 4, 4, .). That is, it has 6 vertices, 12 edges, 8 planes, 12 solids, 6 [4]s as its elements such that each vertex lies on 4 edges, 4 planes, 8 solids and 5 [4]s; each edge contains 2 vertices and lies in 2 planes, 5 solids and 4 [4]s; and so on. From the above considerations we find that there are in all $2^{r-1}\binom{r+3}{3}$ 5-octahedra whose relations with the arguesian points and lines w.r. to the arguesian triangles may be represented by the scheme:

$$2^r\binom{r+3}{2}(.,4,4r+4,r+1) \quad 2^{r+1}(r+3)\left(r+2,.,2\binom{r+2}{2},\binom{r+2}{2}\right)$$
$$2^{r+2}\binom{r+3}{3}(3,3,.,1) \quad 2^{r-1}\binom{r+3}{3}(6,12,8,.).$$

e) Now each 5-octahedron represents a [5] which contains, besides its 6 vertices, $4(r + 2)$ more arguesian points, and besides its 12 edges, $8r$

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more arguesian lines. To sum up, the configuration of all the arguesian points, lines, planes, and their solids, [4]s and [5]s may be put down in the following scheme:

$$2^r\binom{r+3}{2}(., 4, 4, 4r+8, 2r^2+6r+5, (2r^3+6r^2+7r+3)/3)$$
$$2^{r+1}(r+3)(r+2, ., 2, 2r+5, (r+2)^2, \binom{r+2}{2}(2r+3)/3)$$
$$2^{r+2}(\binom{r+3}{2}, r+3, ., r+3, \binom{r+3}{2}, \binom{r+3}{3})$$
$$2^{r+1}(r+3)((r+2)^2, 2r+5, 2, ., r+2, \tfrac{1}{2}(r+2)(r+1))$$
$$2^r\binom{r+3}{2}(2r^2+6r+5, 4r+8, 4, 4, ., r+1)$$
$$2^{r-1}\binom{r+3}{3}(4r^2+8r+6, 8r+12, 8, 12, 6, .).$$

f) The above proposition in regard to pairs of sets of $r+3$ points projective from an $[r-1]$ holds good rather obviously in $[2r+1]$ as well as in $[2r+2]$.

10. Projective Sets of $r+p$ Points $(r > 2, 1 < p < 2r+1)$

Now it follows as an immediate consequence of what precedes that: *If 2 sets of $r+p$ points, every subset of $r+1$ points of either set spanning an $[r]$ which has no line common with the corresponding $[r]$ of the other, be projective from an $[r-1]$ such that it meets all the $r+p$ joins of their corresponding points, the $\binom{r+p}{r+1}$ points of intersection of their corresponding $[r]$s all lie in a $[p-1]$, by $(r+2)$s on $\binom{r+p}{r+2}$ lines therein, each common to $p-1$ of them.*

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