



LOCALLY NORMAL SUBGROUPS OF TOTALLY DISCONNECTED GROUPS. PART I: GENERAL THEORY

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Abstract

Let G be a totally disconnected, locally compact group. A closed subgroup of G is **locally normal** if its normalizer is open in G . We begin an investigation of the structure of the family of closed locally normal subgroups of G . Modulo commensurability, this family forms a modular lattice $\mathcal{LN}(G)$, called the **structure lattice** of G . We show that G admits a canonical maximal quotient H for which the quasicentre and the abelian locally normal subgroups are trivial. In this situation $\mathcal{LN}(H)$ has a canonical subset called the **centralizer lattice**, forming a Boolean algebra whose elements correspond to centralizers of locally normal subgroups. If H is second-countable and acts faithfully on its centralizer lattice, we show that the topology of H is determined by its algebraic structure (and thus invariant by every abstract group automorphism), and also that the action on the Stone space of the centralizer lattice is universal for a class of actions on profinite spaces. Most of the material is developed in the more general framework of Hecke pairs.

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1. Introduction

The aim of the present article is to establish foundations for a study of general nondiscrete totally disconnected locally compact (t.d.l.c.) groups in terms of their local structure. We define a canonical lattice in terms of this local structure that,

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in the case of a p -adic Lie group, is equivalent to the lattice of ideals in the Lie algebra. Inspired by earlier work of Wilson [26] on just-infinite groups and by Barnea *et al.* [2] on abstract commensurators of profinite groups, we develop properties of this lattice. This article, which concerns general t.d.l.c. groups, is the first in a series. The second paper in the series [8] uses the tools developed here to study the more special class of compactly generated t.d.l.c. groups with few normal subgroups and, in particular, those that are topologically simple.

Most of the definitions and results in this first part apply more generally to a **Hecke pair** (G, U) , that is a group G with a commensurated subgroup U , in a way that depends only on the (abstract) commensurability class of U as a subgroup of G . Although our principal interest is in t.d.l.c. groups, we state and prove results in this more general framework in situations where it is unnecessary to use the topology of the group. For example, in the case $G = U$, most of the results we obtain will be valid for residually finite groups in general, with profinite groups as a special case.

1.1. Background. Since the solution of Hilbert's fifth problem by Gleason, Montgomery–Zippin and Yamabe, connected locally compact groups are known to have a far-reaching structure theory, ensuring notably that they are all projective limits of connected Lie groups. This deep result extends the powerful machinery of Lie theory to all connected locally compact groups, thereby yielding decomposition theorems and classification results.

In contrast, the solution of Hilbert's fifth problem yields no information about **totally disconnected** locally compact groups, that is, those in which the connected component of each element is a singleton. Understanding this case would go a long way towards an understanding of general locally compact groups because each general group is an extension of a totally disconnected group by a connected one. While a rather rich and deep theory was developed for some special classes of t.d.l.c. groups (profinite groups, reductive groups over local fields, p -adic analytic groups, ...), for a long time the only general result known was van Dantzig's theorem, which asserts that compact open subgroups exist and form a basis of identity neighbourhoods. Since the class of t.d.l.c. groups includes all discrete groups, it seemed during this time that no meaningful structure theory could be developed beyond the connected or almost connected case.

Three developments over the last 20 years have changed the situation. First, new general results were obtained with the aid of the **scale function**, introduced by Willis in [25]. The scale function on the t.d.l.c. group G measures the effect on compact open subgroups of conjugation by elements of G ; it takes positive-integer values and is nontrivial only when G is not discrete. The scale function and associated ideas elicit significant new features of general nondiscrete t.d.l.c.

groups such as tidy subgroups, contraction groups, parabolic subgroups and so on, and have resolved several open questions.

A complementary approach was initiated by Barnea *et al.* [2], based on the following observation: since any two compact open subgroups of a t.d.l.c. group G are commensurate, G admits a natural homomorphism to the group of abstract commensurators of its compact open subgroups. By definition, abstract commensurators can be recovered from any compact open subgroup and are thus purely local objects depending only on arbitrarily small identity neighbourhoods of G . This observation thus produces a **local approach** to t.d.l.c. groups.

The third development is that techniques for isolating the nondiscrete case from the discrete one have been found. At the global level, Caprace and Monod [7] identified simple pieces of a t.d.l.c. group G under the hypothesis that G has no discrete quotients while, at the local level, Burger and Mozes [6] introduced, in analogy with the kernel of the adjoint representation of a Lie group, the **quasicentre** of G , which contains in particular any discrete normal subgroup of G . The former of the two motivates the focus of our next paper [8] on simple groups, while the latter was important in the approach introduced in [2] and is developed further in Sections 3 and 6 below as part of the foundation for a general theory of the local structure of t.d.l.c. groups.

1.2. Locally normal subgroups and their centralizers. The central concept in the general local theory is that of a **locally normal** subgroup of a t.d.l.c. group, that is, a subgroup whose normalizer has finite index in an open subgroup. A major part of [2] focuses on t.d.l.c. groups in which every open compact subgroup is (hereditarily) just infinite; this corresponds exactly to the condition that every nontrivial compact locally normal subgroup is open. Away from this case, the arrangement of (closed) locally normal subgroups becomes more complex.

The intersection of two locally normal subgroups is locally normal. Moreover, if H and K are closed locally normal subgroups of G , then they admit respective compact open subgroups $H' < H$ and $K' < K$ which normalize each other, and such that the product $H'K'$, which is also a compact subgroup, is locally normal. It is thus natural to consider the collection of all compact locally normal subgroups modulo the equivalence relation given by **local equivalence**: two subgroups H and K are **locally equivalent** if their intersection $H \cap K$ is open in both H and K . In this way, the collection of all equivalence classes of compact locally normal subgroups of G , which is denoted by $\mathcal{LN}(G)$, endowed with the partial ordering induced by the inclusion of subgroups, is easily seen to form a modular lattice. We call it the **structure lattice** of G . There are two elements that we regard as trivial (assuming G is nondiscrete): the minimum element, which is the class of the trivial subgroup, and the maximum element, which is the class of compact

open subgroups. The symbols 0 and ∞ will be used to denote the minimum and maximum elements respectively of any bounded lattice, including $\mathcal{LN}(G)$. A nontrivial subset of a bounded lattice is one that contains elements other than 0 and ∞ .

Its structure lattice is a *local* invariant of a t.d.l.c. group G , since it is determined by any open compact subgroup, which admits a natural action of the topological automorphism group of G . In case G is a p -adic Lie group, the structure lattice $\mathcal{LN}(G)$ is canonically isomorphic to the lattice of ideals in the \mathbf{Q}_p -Lie algebra of G (see Proposition 2.5 below). Hence the structure lattice may be viewed as a group theoretic analogue for *arbitrary* t.d.l.c. groups of the lattice of ideals in the p -adic case. Although it is difficult to determine the structure lattice of a particular t.d.l.c. group G explicitly, the action of G on $\mathcal{LN}(G)$ provides an important tool for elucidating the overall structure of G . Moreover, under the right conditions we obtain two natural G -invariant subsets of $\mathcal{LN}(G)$ that are Boolean algebras when equipped with the induced order and, when nontrivial, constitute valuable additional tools. The first is the **local decomposition lattice** $\mathcal{LD}(G)$, which consists of all direct factors of open compact subgroups of G up to equivalence. The second is the **centralizer lattice** $\mathcal{LC}(G)$: an element of $\mathcal{LC}(G)$ is given by the equivalence class of the centralizer $C_U(K)$ for some open compact subgroup U of G and closed normal subgroup K of U . In general, the structure lattice depends on the topology of the open compact subgroups of G , but the centralizer and decomposition lattices depend only on the Hecke pair structure (G, U) , where U is any open compact subgroup of G .

Following Burger and Mozes [6], we define the **quasicentre** $\text{QZ}(G)$ of a locally compact group G to be the collection of all elements whose centralizer is open (see Lemma 3.1 below for its basic properties).

THEOREM I. *Let G be a t.d.l.c. group.*

- (i) (See Theorem 4.5) *Suppose that G has trivial quasicentre. Then $\mathcal{LD}(G)$ is a Boolean algebra.*
- (ii) (See Theorem 5.2) *Suppose in addition that G has no nontrivial abelian locally normal subgroups. Then $\mathcal{LC}(G)$ is a Boolean algebra and $\mathcal{LD}(G)$ is a subalgebra of $\mathcal{LC}(G)$.*

This opens up a new perspective: using the Stone representation theorem, a group action on a Boolean algebra \mathcal{A} can equivalently be regarded as a group action by homeomorphisms on a topological space $\mathfrak{S}(\mathcal{A})$, called the **Stone space** of \mathcal{A} , that is **profinite**, in other words, compact and totally disconnected. An important property of the G -actions on the Stone spaces of $\mathcal{LD}(G)$ and $\mathcal{LC}(G)$ is

encapsulated in the following definition. Let G be a locally compact group and \mathfrak{X} be a profinite space on which G acts faithfully by homeomorphisms. The G -action is called **weakly decomposable** if for every nonempty clopen subset $\alpha \subsetneq \mathfrak{X}$, the stabilizer of α in G is open, and the pointwise stabilizer of α is nontrivial. Say that G is **faithful weakly decomposable** if it admits a faithful weakly decomposable action on some profinite space.

THEOREM II (See Theorem 5.18). *Let G be a t.d.l.c. group such that $\text{QZ}(G) = 1$.*

- (i) *Suppose that the only abelian locally normal subgroup of G is trivial, and suppose that G acts faithfully on $\mathcal{LC}(G)$. Then the induced action of G on $\mathfrak{S}(\mathcal{LC}(G))$ is a faithful weakly decomposable action.*
- (ii) *Suppose \mathfrak{X} is a profinite space with a faithful weakly decomposable action of G . Then the only abelian locally normal subgroup of G is trivial and there exists a G -equivariant continuous surjective map from $\mathfrak{S}(\mathcal{LC}(G))$ to \mathfrak{X} .*

In particular, if G is faithful weakly decomposable, then $\mathfrak{S}(\mathcal{LC}(G))$ is the unique maximal compact G -space for which the action is faithful and weakly decomposable.

1.3. Radical theory. The hypotheses of Theorem I are satisfied by all compactly generated, topologically simple t.d.l.c. groups: for (i) this is [2, Theorem 4.8], and for (ii) this will be proved in [8, Theorem A]. We show, by developing a suitable ‘radical theory’, that an *arbitrary* t.d.l.c. group has a canonical maximal quotient which satisfies the hypotheses of Theorem I(i) or (ii).

THEOREM III. *Let G be a t.d.l.c. group.*

Then G has closed characteristic subgroups $\text{QZ}^\infty(G) \leq R_{[A]}(G) \leq G$ with the following properties:

- (i) *(See Theorem 6.2) $\text{QZ}^\infty(G)$ is the unique smallest closed normal subgroup N of G such that G/N has trivial quasicentre.*
- (ii) *(See Theorem 6.11) $R_{[A]}(G)$ is the unique smallest closed normal subgroup M of G such that G/M has trivial quasicentre and no nontrivial abelian locally normal subgroups.*

In case G is a p -adic Lie group, $R_{[A]}(G)$ is the unique largest closed normal subgroup whose Lie algebra is the soluble radical of $\text{Lie}(G)$; see Proposition 6.18 below.

Both $\text{QZ}^\infty(G)$ and $\text{R}_{[\text{A}]}(G)$ are well behaved with respect to open subgroups, in the sense that $\text{QZ}^\infty(H) = \text{QZ}^\infty(G) \cap H$ and $\text{R}_{[\text{A}]}(H) = \text{R}_{[\text{A}]}(G) \cap H$ for any open subgroup H of G (see Propositions 6.3 and 6.15). Thus if the canonical quotients provided by Theorem III turn out to be trivial, we have additional local information about the structure of G . For instance it is easily seen that if $\text{R}_{[\text{A}]}(G) = G$, then the open compact subgroups of G are Fitting-regular in the sense of [22].

1.4. Group topologies. We now specialize to the case of first-countable t.d.l.c. groups and give some results on the relationship between the commensurated compact subgroups and the compact locally normal subgroups. For instance (Lemma 7.4), a closed subgroup K of a first-countable profinite group U is commensurated by U if and only if U normalizes an open subgroup of K . Hence, any compact commensurated subgroup of a first-countable t.d.l.c. group G is commensurate with a locally normal subgroup, whose commensurability class is thus a G -fixed point in $\mathcal{LN}(G)$. As a consequence, the commensurated compact locally normal subgroups have an interpretation in terms of group topologies.

THEOREM IV (See Section 7.1). *Let G be a first-countable t.d.l.c. group. Then there is a natural bijection between the elements of $\mathcal{LN}(G)$ that are fixed by the action of G , and refinements of the topology of G that are locally compact and compatible with the group structure.*

When G is faithful weakly decomposable, we obtain a topological rigidity result of a similar kind to those for semisimple Lie groups over real and p -adic fields given in [17].

THEOREM V (See Theorem 7.6). *Let G be a second-countable t.d.l.c. group with trivial quasicycle, and suppose that G is faithful weakly decomposable. Then the topology of G is the coarsest group topology on G for which it is a Baire space. In particular, it is preserved by every automorphism of G as an abstract group. Consequently, the topology of G is the unique σ -compact locally compact group topology on G , and the set of all locally compact group topologies of G is in natural bijection with the set of fixed points of G acting on $\mathcal{LN}(G)$ by conjugation.*

2. Locally normal subgroups

2.1. Definition and first properties. Let G be a t.d.l.c. group, which is assumed to be Hausdorff by convention. Then van Dantzig's theorem

[13, Theorem II.7.7] asserts that the open compact subgroups of G form a base of identity neighbourhoods. Denote this set by $\mathcal{B}(G)$. Any two elements of $\mathcal{B}(G)$ are commensurate, as an elementary topological argument shows. Hence, letting U be any open compact subgroup of G , the pair (G, U) satisfies the following condition.

A **Hecke pair (of groups)** (G, U) is a group G together with a commensurated subgroup U of G , that is, $U \cap gUg^{-1}$ has finite index in U for all $g \in G$.

Most of our arguments will be made in terms of general Hecke pairs, as the results then apply directly to residually finite groups and not just to their profinite completions. When specializing to the case that U is an open compact subgroup of the t.d.l.c. group G , we say that (G, U) is a **t.d.l.c. Hecke pair**.

Given a Hecke pair (G, U) , a **locally normal** subgroup of (G, U) is a subgroup K of G such that the normalizer $N_U(K)$ has finite index in U . A subgroup K of G is **bounded** if $K \cap U$ is commensurate with K . When (G, U) is t.d.l.c., subgroups that are bounded and closed are precisely the compact ones.

The most elementary properties of locally normal subgroups collected in the following will frequently be used without comment.

LEMMA 2.1. *Let (G, U) be a Hecke pair and K, L be subgroups of G .*

- (i) *If K and L are locally normal, so is $K \cap L$.*
- (ii) *If (G, U) is t.d.l.c. and K is compact and locally normal, so is every open subgroup of K .*
- (iii) *If K is locally normal, so is the centralizer $C_V(K)$ for any $V \leq G$ commensurate with U .*
- (iv) *If K is bounded, then there is $V \leq G$ commensurate with U that contains K . If K is also locally normal, then V may be chosen to normalize K . When (G, U) is t.d.l.c. and K is compact and locally normal, then we can take $V \in \mathcal{B}(G)$.*

Proof. (i) is clear since $K \cap L$ is normalized by $N_U(K) \cap N_U(L)$.

(ii) An open subgroup of K may be written as $K \cap U$ for some $U \in \mathcal{B}(G)$. Since any element of $\mathcal{B}(G)$ is locally normal, (ii) follows from part (i).

(iii) Let $W = U \cap V$. If K is locally normal, then $C_V(K)$ is a subgroup of V , so that $N_W(K) \leq N_W(C_V(K))$ and we have

$$|U : N_U(C_V(K))| \leq |U : W| |W : N_W(C_V(K))| \leq |U : W| |W : N_W(K)| < \infty;$$

hence $C_V(K)$ is locally normal.

(iv) If K is bounded, then $W = \bigcap_{k \in K} kUk^{-1}$ is commensurate with U and normalized by K , so that $V = KW$ is a subgroup commensurate with U and containing K . If K is locally normal, then $W = N_U(K)$ is commensurate with U and $V = KW$ is a subgroup commensurate with U and containing K as a normal subgroup. When (G, U) is t.d.l.c. and K is compact and locally normal, then W open and compact with either definition, and so V is an open compact subgroup of G . \square

Notice that the converse to Lemma 2.1(ii) does not hold: namely, the fact that K is compact and contains a locally normal subgroup as an open subgroup does not generally imply that K itself is locally normal, since otherwise all finite subgroups of G would be locally normal and hence contained in the quasicycle $\text{QZ}(G)$.

2.2. Local equivalence and the structure lattice. Let (G, U) be a Hecke pair and let H be a subgroup of G . Write $[H]$ for the class of all subgroups K of G such that $K \cap U$ is commensurate with $H \cap U$ and say that H and K are **locally equivalent** if $[H] = [K]$. Although H might not be bounded, and such will be the case in later arguments, it is the bounded subgroups of $[H]$ that are of interest. There is a natural partial ordering on local equivalence classes, given by $[H] \geq [K]$ if $H \cap K \cap U$ has finite index in $K \cap U$. Note that, when (G, U) is t.d.l.c. and H is closed, the closed subgroups in $[H]$ are those that have an open subgroup in common with H .

A subgroup H in G is locally equivalent to the trivial subgroup if and only if $H \cap U$ is finite; in the t.d.l.c. case, this occurs if and only if H is discrete. We shall later be interested in subgroups H of t.d.l.c. groups G such that $H \cap U = 1$ for all $U \in \mathcal{B}(G)$. Since every compact (and hence every finite) subgroup of G is contained in some open compact subgroup, the latter property holds if and only if H is discrete and torsion-free.

For a given a Hecke pair (G, U) , let $\mathcal{LN}(G, U)$ denote the set of local equivalence classes of subgroups of G that have locally normal representatives. Unless otherwise stated, when choosing a representative of $\alpha \in \mathcal{LN}(G, U)$, we only take representatives that are bounded and locally normal. Then the partially ordered set $\mathcal{LN}(G, U)$ is a lattice. The greatest lower bound and least upper bound of $\alpha, \beta \in \mathcal{LN}(G, U)$ are respectively

$$\alpha \wedge \beta = [H \cap K] \quad \text{and} \quad \alpha \vee \beta = [HK],$$

where representatives $H \in \alpha$ and $K \in \beta$ are chosen such that $H, K \trianglelefteq V$ for some $V \trianglelefteq U$ of finite index. That $[H \cap K]$ and $[HK]$ are independent of how H, K and V are chosen and that they are respectively the greatest lower bound and least

upper bound of α and β is easily verified. We call $(\mathcal{LN}(G, U), \leq)$ the **abstract structure lattice** of (G, U) .

If (G, U) is a t.d.l.c. Hecke pair, write $\mathcal{LN}(G)$ for the set of elements of $\mathcal{LN}(G, U)$ that have compact locally normal representatives. Then $\mathcal{LN}(G)$ is a sublattice of $\mathcal{LN}(G, U)$ that does not depend on the choice of U so long as it is compact and open. We define $\mathcal{LN}(G)$ with the induced operations to be the **structure lattice** of G .

It is clear that $\mathcal{LN}(G, U)$ is determined entirely by the group structure of U , so effectively $\mathcal{LN}(G, U) = \mathcal{LN}(U, U)$. We retain the notation $\mathcal{LN}(G, U)$ in order to distinguish the abstract (Hecke pair) structure lattice from the topological structure lattice $\mathcal{LN}(G)$, which is more useful in the study of t.d.l.c. groups.

EXAMPLE 2.2.

- (a) Let T be a regular tree of finite degree d , with $d \geq 3$, and let $G = \text{Aut}(T)$. Then G is a t.d.l.c. group, with topology generated by the pointwise stabilizers of finite subtrees of T . Let F be a finite subtree of T , let U be the pointwise stabilizer of F , and let T_1, \dots, T_n be the connected components of $T - F$. Then U decomposes as a direct product

$$U = K_1 \times K_2 \times \cdots \times K_n$$

where K_i is the pointwise stabilizer in G of all vertices outside T_i . Evidently K_i is compact and normal in U , so it is locally normal in G , and since each tree T_i is infinite and G induces an infinite group of automorphisms on it, each group K_i is infinite. Thus the given decomposition of U provides 2^n distinct elements of the structure lattice of G . By varying the finite tree F , we see that $\mathcal{LN}(G)$ is infinite in this case.

- (b) Let $G = \mathbf{Q}_p^n$ for $n \geq 2$. Then the every subgroup of G is locally normal, since G is abelian. The compact subgroups of G are all isomorphic to \mathbf{Z}_p^m where $0 \leq m \leq n$. It is easily seen that two compact subgroups of G are commensurate if and only if they span the same subspace of G regarded as a \mathbf{Q}_p -vector space, and conversely every \mathbf{Q}_p -subspace of G is spanned by a compact subgroup. Consequently, the structure lattice of G is naturally isomorphic to the lattice of \mathbf{Q}_p -subspaces of G . In particular, $\mathcal{LN}(G)$ is uncountable.
- (c) At the other extreme, let G be any t.d.l.c. group with a hereditarily just-infinite open subgroup U , for instance $G = \mathbf{Q}_p$ or $G = \text{PSL}_n(\mathbf{Q}_p)$. Then given a closed locally normal subgroup K , the intersection $K \cap U$ is normal in an open subgroup of U , and hence K is either finite or open. Hence $\mathcal{LN}(G)$ is the two-element lattice.

When G is a t.d.l.c. group, the lattice structure of $\mathcal{LN}(G)$ is preserved by the topological automorphism group $\text{Aut}(G)$ of G . More generally, the structure of $\mathcal{LN}(G, U)$ is preserved by those automorphisms of G that preserve the commensurability class of U . In particular, G acts on $\mathcal{LN}(G)$ (when defined) and $\mathcal{LN}(G, U)$ by conjugation. If U is infinite, there are two ‘trivial’ elements of $\mathcal{LN}(G, U)$ and of $\mathcal{LN}(G)$, namely the class of the trivial group, denoted by 0 , and the class of the subgroups containing a subgroup of U of finite index, denoted by ∞ .

The use of the term *structure lattice* is borrowed from Wilson [26], whose ideas developed in the context of just-infinite groups were an important source of inspiration for the present work. The structure lattice is a purely ‘local’ invariant of a t.d.l.c. group, that is, it is completely determined by any open compact subgroup. It is also built out of lattices of normal subgroups of open compact subgroups in a natural sense. Similar observations apply more generally to the abstract structure lattice.

LEMMA 2.3. *Let (G, U) be a Hecke pair.*

- (i) *Let X be a finite subset of $\mathcal{LN}(G, U)$, and let \mathcal{L} be the sublattice generated by X . Then there is a subgroup V of U of finite index and a choice of representatives Y for the elements of X , so that every element of Y is a normal subgroup of V , and \mathcal{L} is isomorphic to the lattice of subgroups of V generated by Y modulo commensuration. If $\mathcal{L} \subseteq \mathcal{LN}(G)$, then we can take $V \in \mathcal{B}(G)$.*
- (ii) *Let I be a lattice identity that is satisfied by the normal subgroup lattice of V (modulo commensuration), for every finite index subgroup V of U . Then I is satisfied by $\mathcal{LN}(G, U)$. In particular, $\mathcal{LN}(G, U)$ is a modular lattice.*

Proof. (i) Let $X = \{[K_1], \dots, [K_n]\}$, let $V = \bigcap N_U(K_i)$, and let $L_i = V \cap K_i$ for $i \in \{1, \dots, n\}$. Then $Y = \{L_1, \dots, L_n\}$ consists of normal subgroups of V , and the meet and join operations in \mathcal{L} can evidently be given by $[K_i] \wedge [K_j] = [L_i \cap L_j]$ and $[K_i] \vee [K_j] = [L_i L_j] = [\langle L_i, L_j \rangle]$. Thus the map $L \mapsto [L]$ gives a surjective map from the lattice generated by Y to \mathcal{L} , and two subgroups have the same image if and only if they are commensurate. If (G, U) is t.d.l.c. and $\mathcal{L} \subseteq \mathcal{LN}(G)$, then we can choose K_1, \dots, K_n to be closed, so V is closed, and hence $V \in \mathcal{B}(G)$.

(ii) This is immediate from (i), recalling that the lattice of normal subgroups in any group is modular (see [14, Theorem 8.3]). \square

If G is a t.d.l.c. group and U is an open compact subgroup, we see that $\mathcal{LN}(G)$ is a sublattice of $\mathcal{LN}(G, U)$. (In particular, $\mathcal{LN}(G)$ inherits the modular

property from $\mathcal{LN}(G, U)$.) In general, elements of $\mathcal{LN}(G)$ may have nonclosed representatives, since a finite index subgroup of a profinite group is not necessarily closed. However, we can recognize the elements of $\mathcal{LN}(G)$ inside $\mathcal{LN}(G, U)$ by the existence of closed representatives (without reference to normalizers).

PROPOSITION 2.4. *Let (G, U) be a t.d.l.c. Hecke pair and let $\alpha \in \mathcal{LN}(G, U)$. Then $\alpha \in \mathcal{LN}(G)$, that is, α has a compact locally normal representative, if and only if there is closed subgroup H of G such that $[H] = \alpha$.*

Proof. If $\alpha \in \mathcal{LN}(G)$, then certainly α has a closed representative. It remains to show the converse. Suppose H is a closed subgroup of G such that $[H] = \alpha$. Since $\alpha \in \mathcal{LN}(G, U)$, there is some bounded locally normal subgroup K of G such that $[K] = \alpha$. We see that $H \cap U$ is closed and $[H \cap U] = \alpha$, so we may assume $H \leq U$. Now $H \cap K$ has finite index in K , so the image of K in U/H is a finite set of cosets, say k_1H, \dots, k_nH . Since H is closed, there is a sufficiently small open normal subgroup V of U such that the cosets k_1H, \dots, k_nH remain pairwise distinct modulo V . It then follows that

$$H \cap K \cap V = K \cap V.$$

Note that $K \cap V$ is locally normal and $[K \cap V] = \alpha$. So by replacing K by $K \cap V$, we may assume $K \leq H$. Since H is a closed subgroup of G , we in fact have $\overline{K} \leq H$. Now K has finite index in \overline{K} , since K is commensurate with H , so $[\overline{K}] = \alpha$; moreover, the normalizer of \overline{K} in G is a closed subgroup containing $N_G(K)$, so the normalizer of \overline{K} is open. Hence \overline{K} is the desired compact locally normal representative of α . \square

We have seen that the structure lattice $\mathcal{LN}(G)$ of a t.d.l.c. group G is a modular lattice that can be recovered from any identity neighbourhood in G . In the special case where G is a p -adic Lie group, the structure lattice can be described in terms of the \mathbf{Q}_p -Lie algebra of G , as follows.

PROPOSITION 2.5. *Let G be a p -adic Lie group. Then the structure lattice $\mathcal{LN}(G)$ is G -equivariantly isomorphic to the lattice of ideals in the \mathbf{Q}_p -Lie algebra $\text{Lie}(G)$.*

Indeed, that proposition readily follows from the following known properties of the Lie correspondence in p -adic Lie groups.

PROPOSITION 2.6. *Let G be a p -adic Lie group, and $\mathfrak{g} = \text{Lie}(G)$ denote its \mathbf{Q}_p -Lie algebra.*

- (i) For any two closed subgroups $H_1, H_2 \leq G$, we have $[H_1] = [H_2]$ if and only if $\text{Lie}(H_1) = \text{Lie}(H_2)$.
- (ii) If $H \leq G$ is closed with $N_G(H)$ open, then $\text{Lie}(H)$ is an ideal.
- (iii) If \mathfrak{h} is an ideal of \mathfrak{g} , then there exists a compact locally normal subgroup H with $\text{Lie}(H) = \mathfrak{h}$.

Proof. We first recall that every closed subgroup of G is a Lie group, whose Lie algebra is a subalgebra of \mathfrak{g} , see [5, Ch. III, Section 8, Theorem 2].

We next observe that every vector subspace of \mathfrak{g} is closed: this is easily seen by induction on dimension, using the fact that every proper open subgroup of \mathbf{Q}_p is compact. Therefore, every Lie subalgebra of \mathfrak{g} is the Lie algebra of a closed subgroup of G by [5, Ch. III, Section 7, Theorem 2].

This being recorded, assertion (i) now follows from [5, Ch. III, Section 7, Theorem 2], while assertions (ii) and (iii) follow from [5, Ch. III, Section 7, Proposition 2]. \square

3. Quascentralizers and their stability properties

3.1. Motivation. Let (G, U) be a Hecke pair. Our principal aim in the present section and Sections 4 and 5 is to obtain two subsets of the structure lattice of (G, U) , namely the **local decomposition lattice** $\mathcal{LD}(G, U)$ and the **centralizer lattice** $\mathcal{LC}(G, U)$ that are Boolean algebras, provided that (G, U) satisfies some unavoidable hypotheses. The local decomposition lattice will have as representatives the direct factors of finite index subgroups of U ; more generally, one can define the local decomposition lattice $\mathcal{LD}(G, U; K)$ relative to a locally normal subgroup K , with representatives given by those direct factors of finite index subgroups of K that are locally normal in G . The centralizer lattice will have as representatives the centralizers of all locally normal subgroups. Here are some observations that will lead us to the required hypotheses.

- (1) Given a direct factor K of a group U , observe that there may be many different subgroups L such that $K \times L = U$: consider for instance the direct factorizations of $U = \mathbf{Z}_p^2$ as $\mathbf{Z}_p \times \mathbf{Z}_p$. However, if the centre of U is trivial, then K has a unique complement, namely $L = C_U(K)$.
- (2) *A priori*, the map

$$\mathcal{LN}(G, U) \rightarrow \mathcal{LN}(G, U); [K] \mapsto [C_G(K)]$$

is not well defined: given a locally normal subgroup K and a finite index subgroup V of U , it could be the case that $C_G(K \cap V)$ lies in a

local equivalence class that is strictly above the local equivalence class of $C_G(K)$. So instead, we define a subgroup $QC_G([K])$ that is the union of all centralizers of representatives of $[K]$, and then define $[K]^\perp$ to be $[QC_G([K])]$. We see that under suitable conditions, every representative of $[K]$ actually has the same centralizer, so the map $[K] \mapsto [C_G(K)]$ is in fact well defined.

- (3) There is a natural approach to obtaining a Boolean algebra from a bounded lattice \mathcal{L} , provided that \mathcal{L} is *pseudocomplemented*. A **pseudocomplementation** is a function $\perp : \mathcal{L} \rightarrow \mathcal{L}$ such that:

(*) For all $\alpha, \beta \in \mathcal{L}$, we have $\alpha \wedge \beta = 0$ if and only if $\beta \leq \alpha^\perp$.

If a pseudocomplementation exists, it is necessarily unique. Moreover, the set $\mathcal{A} = \{\alpha^\perp \mid \alpha \in \mathcal{L}\}$ forms a Boolean algebra, with the restriction of \perp serving as the complementation map on \mathcal{A} (in particular, $\perp^3 = \perp$). We obtain hypotheses such that $\mathcal{LN}(G, U)$ is pseudocomplemented, and the pseudocomplementation will be given by $[K] \mapsto [C_G(K)]$.

These observations lead naturally to the definitions of the *quasicentre* (see 3.2), *quasicentralizers* (see 3.3) and *C-stability* (see 3.4). The purpose of the present section (Section 3) is to perform the technical groundwork needed to verify that the definitions have the properties we desire of them.

Since the lattices we are describing are ‘local’ invariants of the Hecke pair (G, U) , for the proofs, there will be no loss of generality in assuming $G = U$, and we may freely replace G or U by a finite index subgroup when it is convenient to do so.

3.2. The quasicentre. As observed in Section 3.1, to ensure unique complementation in the local decomposition lattice of the Hecke pair (G, U) , we need the finite index subgroups of U (or at least all sufficiently small finite index subgroups of U) to have trivial centre. When (G, U) is a t.d.l.c. Hecke pair, this becomes the question as to whether the **quasicentre** of G , defined as the set of all elements whose centralizer is open, is discrete. The notion of quasicentre is due to Burger and Mozes [6]; its basic properties are as follows.

LEMMA 3.1. *Let G be a locally compact group. Then:*

- (i) $QZ(G)$ is a (not necessarily closed) characteristic subgroup of G .
- (ii) $QZ(O) \leq QZ(G)$ for any open subgroup $O \leq G$.
- (iii) For any closed normal subgroup N of G , the image of $QZ(G)$ under the projection to G/N is contained in $QZ(G/N)$.

- (iv) $\text{QZ}(G)$ contains every discrete subgroup of G whose normalizer in G is open. In particular, all discrete normal subgroups and all finite locally normal subgroups are contained in $\text{QZ}(G)$.
- (v) For any discrete normal subgroup N of G , we have $\text{QZ}(G)/N = \text{QZ}(G/N)$. In particular, if $\text{QZ}(G)$ is discrete, then $\text{QZ}(G/\text{QZ}(G)) = 1$.

Proof. (i) For all $g, h \in G$, we have $C_G(gh) \supseteq C_G(g) \cap C_G(h)$. This implies that $gh \in \text{QZ}(G)$ as soon as $g, h \in \text{QZ}(G)$. Part (i) follows.

(ii) Let $O \leq G$ be open. Since $C_G(g) \supseteq C_O(g)$ for any $g \in G$, so that $C_G(g)$ is open as soon as $C_O(g)$ is so. Part (ii) follows.

(iii) Clear since the projection $G \rightarrow G/N$ is an open map.

(iv) Let H be a discrete subgroup of G such that $N_G(H)$ is open and let V be an open compact subgroup of $N_G(H)$. Consider $h \in H$. Since the V -conjugacy class of h is compact, thus finite, h is centralized by an open subgroup of V and so $h \in \text{QZ}(G)$ as claimed.

(v) Let $N \leq G$ be a discrete normal subgroup. We have $N \leq \text{QZ}(G)$ by (iv) and $\text{QZ}(G)/N \leq \text{QZ}(G/N)$ by (iii). Let now $g \in G$ such that $gN \in \text{QZ}(G/N)$. Thus there is $U \in \mathcal{B}(G)$ such that $[g, U] \subset N$. Since N is discrete, there exists also $V \in \mathcal{B}(G)$ with $V \cap N = 1$. Since the commutator map $x \mapsto [g, x]$ is continuous, we may, upon replacing U by a sufficiently small open subgroup, assume that $[g, U] \subset V$. We deduce that $[g, U] = 1$, whence $g \in \text{QZ}(G)$. This confirms that $\text{QZ}(G)/N \supseteq \text{QZ}(G/N)$, and the required assertion follows. \square

Given a general Hecke pair (G, U) , we define the **quasicentre of G with respect to U** , denoted by $\text{QZ}(G, U)$, to be the set of all elements of G that have finitely many U -conjugates. When (G, U) is a t.d.l.c. pair, this is equal to the quasicentre of G as defined previously and we adopt the notation $\text{QZ}(G)$ for the quasicentre of G with respect to U in the general case as well when there is no ambiguity about the choice of U . Subgroups H of G are understood to have the Hecke pair structure $(H, H \cap U)$, so $\text{QZ}(H)$ is the set of all elements of H that have finitely many $(H \cap U)$ -conjugates. Note that $\text{QZ}(H)$ is a normal subgroup of H , and agrees with the usual definition of the quasicentre for topological groups in the case that (G, U) is t.d.l.c. and H is closed. When H is bounded with respect to (G, U) , observe that $\text{QZ}(H)$ is just the union of all finite conjugacy classes of H .

Note that a version of Lemma 3.1(iv) also holds for Hecke pairs (G, U) :

LEMMA 3.2. *Let (G, U) be a Hecke pair and let $H \leq G$. Suppose that there is a finite index subgroup V of U such that $V \leq N_G(H)$ and $H \cap V = 1$. Then $H \leq \text{QZ}(G)$.*

Proof. Let $x \in H$. Then there is a finite index subgroup W of V such that $xWx^{-1} \leq V$, since V is commensurated by G . Given $w \in W$, we have $[x, w] \in H$, since H is normalized by W , and also $[x, w] \in xWx^{-1}W \subseteq V$. Thus $[x, w] \in H \cap V$, so $[x, w] = 1$. Hence $x \in C_G(W) \leq QZ(G)$. \square

The quasicentre has a useful characterization in the case that it is locally equivalent to the trivial group.

COROLLARY 3.3. *Let (G, U) be a Hecke pair. Suppose that $QZ(G) \in [1]$, that is, there exists a finite index subgroup V of U such that $QZ(G) \cap V = 1$. Then $QZ(G)$ is the unique largest locally normal representative of $[1]$. In particular, if $QZ(G) = 1$, then the trivial group is the only locally normal representative of $[1]$ in G .*

Proof. Follows immediately from Lemma 3.2. \square

Observe that if V is a finite index subgroup of U , then $QZ(V) = QZ(G) \cap V$. So in taking the local approach, to prove results about Hecke pairs (G, U) with $QZ(G) \in [1]$, we are usually free to assume that $QZ(G) = 1$.

3.3. Quasicentralizers. Generalizing the quasicentre, we introduce a notion of the centralizer of a subgroup that depends only on its local equivalence class.

DEFINITION 3.4. Let (G, U) be a Hecke pair, let H and K be subgroups of G and let \mathcal{C} be the class of subgroups of $K \cap U$ of finite index. The **quasicentralizer** of K in H is given by

$$QC_H(K) := \bigcup_{L \in \mathcal{C}} C_H(L).$$

Clearly the quasicentralizer $QC_H(K)$ contains the centralizer $C_H(K)$. Moreover, the quasicentralizer is a subgroup of G , since $C_H(L_1)C_H(L_2) \subseteq C_H(L_1 \cap L_2)$. Similarly we have $QC_H(K) = QC_H(L)$ for any finite index subgroup L of $K \cap U$, so that $QC_H(K)$ depends only on the local equivalence class $\alpha = [K]$ and we can write $QC_H(\alpha)$ without ambiguity. Some caution is required in saying that two subgroups ‘quasicommute’, as the relation A is contained in the quasicentralizer of B is not symmetric. We see that $QZ(H) = QC_H(H)$.

We emphasize that the quasicentralizer of a subgroup need not be closed, even in the context of profinite groups. For example, if $G = \prod_{\mathbb{N}} S_3$, then $QC_G(G)$ is the dense subgroup $\bigsqcup_{\mathbb{N}} S_3$. However, it is the case that $QC_G(H) = QC_G(\overline{H})$ in this context.

LEMMA 3.5. *Let (G, U) be a Hecke pair, such that G is a topological group and U is open in G . Let H be a subgroup of G . Then*

$$\text{QC}_G(H) = \text{QC}_G(\overline{H}).$$

Proof. It may be assumed that $H \leq U$ and hence, since U is an open subgroup and therefore closed, that $\overline{H} \leq U$. Hence, to compute the quasicentralizers, we may work with finite index subgroups of H and \overline{H} .

Let $K \leq \overline{H}$ have finite index. Then $K \cap H$ has finite index in H and $C_G(K \cap H) \geq C_G(K)$. Hence $\text{QC}_G(H) \geq \text{QC}_G(\overline{H})$. On the other hand, if L has finite index in H , then \overline{L} has finite index in \overline{H} and $C_G(L) = C_G(\overline{L})$. Hence $\text{QC}_G(H) \leq \text{QC}_G(\overline{H})$. \square

In the context of t.d.l.c. groups, the definition of the quasicentralizer can be reformulated as follows.

LEMMA 3.6. *Let (G, U) be a t.d.l.c. Hecke pair and H, K be subgroups of G . Then we have*

$$\text{QC}_H(K) := \bigcup_{V \in \mathcal{B}(G)} C_H(K \cap V).$$

Proof. In view of Lemma 3.5, we have $\text{QC}_H(K) = \text{QC}_H(\overline{K})$. Moreover, for any $V \in \mathcal{B}(G)$ we have $\overline{K \cap V} = \overline{K} \cap \overline{V}$ because V is open, which yields $C_H(K \cap V) = C_H(\overline{K \cap V}) = C_H(\overline{K} \cap \overline{V})$. We may therefore assume that K is closed.

For any $V \in \mathcal{B}(G)$, we have $C_H(K \cap V) \geq C_H(K \cap U \cap V)$, so that $\bigcup_{V \in \mathcal{B}(G)} C_H(K \cap V) = \bigcup_{V \in \mathcal{B}(G)} C_H(K \cap U \cap V)$. Given a finite index subgroup $L \leq K \cap U$, we have $C_H(L) = C_H(\overline{L})$. Moreover, since K is closed and since \overline{L} is an open subgroup of $K \cap U$ there exists $V \in \mathcal{B}(G)$ with $K \cap U \cap V = \overline{L}$. Denoting by \mathcal{C} be the class of subgroups of $K \cap U$ of finite index, we infer that

$$\text{QC}_H(K) = \bigcup_{L \in \mathcal{C}} C_H(L) = \bigcup_{V \in \mathcal{B}(G)} C_H(K \cap U \cap V)$$

as required. \square

The following observation is borrowed from [2, Proposition 2.6].

LEMMA 3.7. *Let G be a group, let U and V be subgroups of G , and let $W \leq U \cap V$. Suppose $\theta : U \rightarrow V$ is an (abstract) isomorphism such that $\theta(x) = x$ for all $x \in W$, and suppose also that $C_G(W \cap g^{-1}Wg) = 1$ for all $g \in U$. Then $U = V$ and $\theta = \text{id}_U$.*

Proof. Let $g \in U$ and let $x \in W \cap g^{-1}Wg$. Then

$$g x g^{-1} = \theta(g x g^{-1}) = \theta(g)\theta(x)\theta(g^{-1}) = \theta(g)x\theta(g^{-1}),$$

so $g^{-1}\theta(g)$ centralizes x . This shows that $g^{-1}\theta(g) \in C_G(W \cap g^{-1}Wg) = 1$. Thus $g = \theta(g)$. \square

In the following lemmas, we obtain some consequences of the hypothesis that a given subgroup H of G has trivial quasicentre.

LEMMA 3.8. *Let (G, U) be a Hecke pair and H be a subgroup of G such that $\text{QZ}(H) = 1$. Then:*

(i) $\text{QC}_G(H) \cap N_G(H) = C_G(H)$;

(ii) *Every finite locally normal subgroup of G has trivial intersection with H .*

Proof. For part (i), let $t \in \text{QC}_G(H) \cap N_G(H)$. Then conjugation by t induces an automorphism, θ_t , of H that fixes pointwise some finite index subgroup, K , of $H \cap U$. Since K is commensurated by H and $\text{QZ}(H, H \cap U) = 1$, we have $C_H(K \cap g^{-1}Kg) = 1$ for all $g \in H$ and so, by Lemma 3.7, $\theta_t = \text{id}_H$, that is, $t \in C_G(H)$. The reverse inclusion is clear.

For part (ii), let K be a finite locally normal subgroup of G . Then there is a subgroup V of G that is commensurate to U such that $K \leq V \leq N_G(K)$. Let $H' = H \cap V$ and let $K' = H \cap K$. Then K' is finite and normal in H' , and therefore contained in the quasicentre of H' . However, H' has trivial quasicentre, and so K' is trivial. \square

LEMMA 3.9. *Let (G, U) be a Hecke pair and let H and K be subgroups of G such that $\text{QZ}(H) = 1$. Suppose that $|H \cap K|$, $|H : N_H(K)|$ and $|K : N_K(H)|$ are all finite. Then $N_K(H) = C_K(H)$.*

Proof. Set $A = N_H(K)$ and $B = N_K(H)$, so that A and B normalize each other. We have $A \cap B \leq H \cap K$, so that $A \cap B$ is finite by hypothesis. Since $A \cap B$ is normal in A , we infer that $A \cap B$ is centralized by a finite index subgroup of A , hence by a finite index subgroup of H since $|H : A|$ is finite by hypothesis. Therefore, $A \cap B \leq \text{QZ}(H) = 1$. Since A and B normalize each other, it follows that they commute. Hence B centralizes a finite index subgroup of H , which implies that $B \leq \text{QC}_G(H)$. Since B also normalizes H , we deduce that $B \leq C_G(H)$ by Lemma 3.8(i). \square

LEMMA 3.10. *Let (G, U) be a Hecke pair and H be a subgroup of G such that $N_U(H)$ has finite index in U .*

- (i) Suppose (G, U) is t.d.l.c. (as defined in Section 2.1 above), H is closed and $\text{QZ}(H)$ is discrete and torsion-free. Then $\text{QC}_G(H)$ is closed and $H \cap \text{QC}_G(H)$ is discrete and torsion-free. In particular, $\text{QC}_G(H)$ cannot be dense in G unless H is discrete and torsion-free.
- (ii) Let V be a subgroup of G that is commensurate with U . If $\text{QZ}(H) \cap V = 1$, then $\text{QC}_G(H)$ is locally equivalent to $C_G(H \cap V)$. In particular, we have

$$\text{QC}_G(\text{QC}_G(H)) = \text{QC}_G(C_G(H \cap V)) = \text{QC}_G(C_V(H \cap V)).$$

Proof. (i) Let V be any open compact subgroup of G . Then $\text{QZ}(H \cap V) = 1$ because $\text{QZ}(H)$ is discrete and torsion-free and $\text{QZ}(H \cap V) \leq \text{QZ}(H)$. Therefore, by Lemma 3.8(i) and because $\text{QC}_G(H \cap V) = \text{QC}_G(H)$,

$$\text{QC}_G(H) \cap N_G(H \cap V) = C_G(H \cap V). \quad (3.1)$$

In particular, $\text{QC}_G(H) \cap (H \cap V) \leq Z(H \cap V) = 1$. Hence $H \cap \text{QC}_G(H)$ intersects trivially any open compact subgroup of G and is therefore a discrete and torsion-free. That $\text{QC}_G(H)$ is closed follows from (3.1) by noting that $N_G(H \cap V)$ is open because it contains $N_U(H) \cap V$ and that $C_G(H \cap V)$ is closed.

(ii) Let $W = N_V(H)$. Since $\text{QZ}(H) \cap V = 1$, it follows that $\text{QZ}(H \cap V) = 1$, and Lemma 3.8(i) implies that

$$\text{QC}_G(H \cap V) \cap W \leq \text{QC}_G(H \cap V) \cap N_G(H \cap V) = C_G(H \cap V).$$

On the other hand, the inclusion $C_G(H \cap V) \cap W \leq \text{QC}_G(H \cap V) \cap W$ holds because $C_G(H \cap V) \leq \text{QC}_G(H \cap V)$. Since $\text{QC}_G(H \cap V) = \text{QC}_G(H)$, we infer that $\text{QC}_G(H) \cap W = C_G(H \cap V) \cap W$ and hence that $[\text{QC}_G(H)] = [C_G(H \cap V)]$. \square

3.4. C-stability. Recall that one of our aims is to use the map $K \mapsto \text{QC}_G(K)$ to define a pseudocomplementation on $\mathcal{LN}(G, U)$, or at least on a subset of $\mathcal{LN}(G, U)$. Observe that any pseudocomplementation \perp on a lattice \mathcal{L} satisfies the following identity:

$$\alpha^\perp \wedge \alpha^{\perp 2} = 0 \quad \forall \alpha \in \mathcal{L}. \quad (3.2)$$

The following property applied to locally normal subgroups, closely related to the identity (3.2), will prove to be the key criterion for our purposes.

DEFINITION 3.11. Let (G, U) be a Hecke pair and let H be a subgroup of G . Say H is **C-stable** in G if

$$R(H) := \text{QC}_G(H) \cap \text{QC}_G(C_G(H)) \in [1].$$

We shall mostly be interested in the case that H is contained in a subgroup commensurate with U . The following shows that, in that case, C-stability could have equivalently been defined to mean that $\text{QC}_V(HC_G(H)) = 1$ for some finite index subgroup V of U . In the t.d.l.c. case, this is equivalent to requiring that $\text{QC}_G(HC_G(H))$ be discrete.

LEMMA 3.12. *Let (G, U) be a Hecke pair and let H be a bounded subgroup of G . Then*

$$R(H) = \text{QC}_G(H) \cap \text{QC}_G(C_G(H)) = \text{QC}_G(HC_G(H)).$$

Proof. Let $x \in \text{QC}_G(HC_G(H))$, so x centralizes some finite index subgroup K of $HC_G(H) \cap U$. Then $H \cap K$ has finite index in $H \cap U$ and $C_G(H) \cap K$ has finite index in $C_G(H) \cap U$ and x centralizes both subgroups. Hence $x \in \text{QC}_G(H)$ and $x \in \text{QC}_G(C_G(H))$.

Conversely, consider $x \in \text{QC}_G(H) \cap \text{QC}_G(C_G(H))$ and let V be a subgroup commensurate to U such that $H \leq V$. Then there are subgroups L of finite index in $H \cap U$ and M of finite index in $C_{U \cap V}(H)$ such that

$$x \in C_G(L) \cap C_G(M).$$

Hence x centralizes the group LM . Since L has finite index in H and M has finite index in $C_V(H)$, we thus have that x centralizes a finite index subgroup of $HC_V(H)$. Finally, the condition that $H \leq V$ implies that $HC_V(H) = HC_G(H) \cap V$, and the latter group is commensurate with $HC_G(H) \cap U$. Hence x centralizes a finite index subgroup of $HC_G(H) \cap U$ as required. \square

We also note the role of the quasicentre of (G, U) as a whole in C-stability. Given any subgroup H of G , then

$$\text{QC}_G(H) \cap \text{QC}_G(C_G(H)) \geq \text{QZ}(G),$$

and equality holds in the case $H = G$. So if $\text{QZ}(G) \in [1]$, then G is C-stable in itself, and moreover, every subgroup of G commensurate to U is C-stable in G . If instead $\text{QZ}(G) \notin [1]$, then G does not have any C-stable subgroups. So in proving properties of C-stable subgroups of a Hecke pair (G, U) , we may assume $\text{QZ}(G) \in [1]$.

In particular, the following is an immediate consequence of Corollary 3.3:

COROLLARY 3.13. *Let (G, U) be a Hecke pair and let H be a C -stable locally normal subgroup of G . Then*

$$R(H) = \mathrm{QC}_G(H) \cap \mathrm{QC}_G(C_G(H)) = \mathrm{QZ}(G).$$

The next lemma records the most basic properties of C -stable subgroups.

LEMMA 3.14. *Let (G, U) be a Hecke pair and let H be a C -stable subgroup of G .*

- (i) *Let K be a subgroup of H that is commensurate with $H \cap U$. Then K is C -stable in G .*
- (ii) *$\mathrm{QZ}(H) \in [1]$; indeed, $\mathrm{QZ}(H) \leq \mathrm{QC}_G(H) \cap \mathrm{QC}_G(C_G(H))$.*
- (iii) *If $H \cap U$ is finite, then $H \leq \mathrm{QC}_G(H) \cap \mathrm{QC}_G(C_G(H))$.*

Proof. Let V be a subgroup of G that is commensurate with U .

(i) Since K is commensurate with $H \cap U$, we have $\mathrm{QC}_G(K) = \mathrm{QC}_G(H)$. Since $K \leq H$, we have $C_G(K) \geq C_G(H)$, so that $\mathrm{QC}_G(C_G(K)) \leq \mathrm{QC}_G(C_G(H))$. Therefore,

$$\mathrm{QC}_G(K) \cap \mathrm{QC}_G(C_G(K)) \leq \mathrm{QC}_G(H) \cap \mathrm{QC}_G(C_G(H)) \in [1].$$

(ii) We have $\mathrm{QZ}(H) = \mathrm{QC}_H(H) \leq \mathrm{QC}_G(H)$. Moreover, any $g \in \mathrm{QZ}(H)$ belongs to H and therefore commutes with $C_G(H)$, so that $\mathrm{QZ}(H) \leq C_G(C_G(H)) \leq \mathrm{QC}_G(C_G(H))$. Therefore,

$$\mathrm{QZ}(H) \leq \mathrm{QC}_G(H) \cap \mathrm{QC}_G(C_G(H)) \in [1].$$

(iii) If $H \cap U$ is finite, then $\mathrm{QZ}(H) = H$, so $H \leq \mathrm{QC}_G(H)$. Clearly also H is contained in $\mathrm{QC}_G(C_G(H))$. \square

For each integer $n > 0$, we define inductively

$$C_H^{n+1}(K) = C_H(C_H^n(K)) \quad \text{and} \quad \mathrm{QC}_H^{n+1}(\alpha) = \mathrm{QC}_H(\mathrm{QC}_H^n(\alpha)),$$

where H and K are subgroups of G and α is either a subgroup, or a local equivalence class of subgroups. For convenience we also adopt the convention $C_H^0(K) = K = \mathrm{QC}_H^0(K)$.

The following is elementary, and will be used frequently without comment.

LEMMA 3.15. *Let G be a group and let H be a subgroup of G . Then*

$$C_G^2(H) \geq H \quad \text{and} \quad C_G^3(H) = C_G(H). \quad \square$$

The following lemma gives a useful relationship between n th centralizers and n th quasicentralizers in the context of C -stable subgroups.

LEMMA 3.16. *Let (G, U) be a Hecke pair, let H be a subgroup of G . As before, we set $R(H) = \text{QC}_G(H) \cap \text{QC}_G(C_G(H))$. Let $n \geq 0$ and let V be a subgroup of G commensurate with U .*

- (i) *If L is a subgroup of G that contains a finite index subgroup of U , then $R(C_L^{n+1}(H)) \leq R(C_L^n(H))$. In particular, if H is C -stable, then so is $C_L^n(H)$.*
- (ii) *Suppose that H is C -stable and locally normal in G . Let V be a subgroup of G commensurate with U such that $\text{QZ}(G) \cap V = 1$. Then we have*

$$\text{QC}_V(\text{QC}_G^n(H)) = C_V^{n+1}(H \cap V).$$

In particular, $\text{QC}_V(H) = C_V(H \cap V)$.

Proof. (i) Fix $n \geq 0$, let $H' = C_L^n(H)$ and let $K = C_L^{n+1}(H) = C_L(H')$. On the one hand $H' \cap L$ is locally equivalent to H' and contained in $C_G(K)$, so $\text{QC}_G(C_G(K)) \leq \text{QC}_G(H')$; on the other hand we also have $\text{QC}_G(K) = \text{QC}_G(C_G(H'))$. Hence

$$\text{QC}_G(K) \cap \text{QC}_G(C_G(K)) \leq \text{QC}_G(H') \cap \text{QC}_G(C_G(H')),$$

in other words, $R(C_L^{n+1}(H)) \leq R(C_L^n(H))$. Thus if H is C -stable, then so are the groups $C_L^n(H)$ for $n \geq 0$.

(ii) Without loss of generality $H \leq V$, since neither side of the desired equation is affected by replacing H by $H \cap V$, and $H \cap V$ retains the properties of being C -stable and locally normal in G . Thus $\text{QZ}(H) = 1$ by Lemma 3.14(ii) and Corollary 3.13.

First suppose that $n = 0$: we must show that $\text{QC}_V(H)$ commutes with H . Consider $x \in \text{QC}_V(H)$, so that $x \in C_V(K)$ for some finite index subgroup K of $H \cap U$. Then K has finite index in H and, replacing K by its core in H , we may assume that K is normal in H .

Moreover, K is C -stable by Lemma 3.14(i), and it follows from part (i) and Lemma 3.14(ii) that $\text{QZ}(C_V(K)) = 1$.

Now $C_V(K) \cap H = 1$ since $\text{QZ}(H) = 1$. Also, H normalizes $C_V(K)$ since $H \leq V$ and since H normalizes K . Furthermore H is locally normal, so that H

is normalized by some finite index subgroup W of $U \cap V$ such that W is normal in V . Hence H is normalized by $C_W(K)$. Thus

$$[H, C_W(K)] \leq H \cap C_W(K) \leq H \cap C_V(K) = 1,$$

so that H centralizes $C_W(K)$. As $C_W(K)$ is a subgroup of $C_V(K)$ of finite index, we see that H centralizes $C_V(K)$ by Lemma 3.8(i), and in particular H centralizes x . This concludes the proof for $n = 0$.

For the general case, note first that $C_V^n(H)$ is C -stable by part (i), has trivial quasicentre by Lemma 3.14(ii) and is also locally normal by Lemma 2.1(iii). Therefore, by repeated applications of Lemma 3.10(ii), recalling that $H \leq V$, we have the equation

$$QC_V^{n+1}(H) = QC_V(QC_V^n(H)) = QC_V(C_V^n(H)).$$

The desired conclusion follows, since by the base case $n = 0$, we have

$$QC_V(C_V^n(H)) = C_V(C_V^n(H)). \quad \square$$

We are now able to define a partial map $\alpha \mapsto [QC_G(\alpha)]$ on the abstract structure lattice.

COROLLARY 3.17. *Let (G, U) be a Hecke pair and let K be a C -stable locally normal subgroup of G . Suppose $QZ(G) \cap U = 1$. Then for every finite index subgroup L of $K \cap U$, we have*

$$QC_U(K) = C_U(L).$$

In particular we have $[C_G(K \cap U)] = [C_U(K \cap U)] = [QC_G(K)]$, so that the local equivalence class $[C_U(K)]$ depends only on $[K]$.

Proof. By Lemma 3.16(ii), we have $QC_U(K) = C_U(K \cap U)$.

Let now L be a finite index subgroup of $K \cap U$. By definition of the quasicentralizer, we have $QC_G(K) = QC_G(K \cap U) = QC_G(L) \geq C_G(L)$. Since $L \leq K \cap U$, we also have $C_G(L) \geq C_G(K \cap U)$. Intersecting with U and combining this with the observation above, we obtain

$$QC_U(K) = QC_U(L) \geq C_U(L) \geq C_U(K \cap U) = QC_U(K).$$

Therefore all those subgroups coincide, which yields the desired assertion. \square

3.5. Local C-stability. The previous section motivates the following definition.

DEFINITION 3.18. A Hecke pair (G, U) is called **locally C-stable** if all locally normal subgroups of G are C-stable in G .

We can now establish the following theorem, which characterizes locally C-stable Hecke pairs under the assumption that U is residually finite; in particular, we obtain a characterization of those t.d.l.c. Hecke pairs that are locally C-stable. We also obtain several stability properties of quasicentralizers; to avoid unnecessary complications, we state these under the assumption that G has trivial quasicentre. In view of Lemma 3.1(v) this hypothesis is harmless: indeed if G is locally C-stable, then $\text{QZ}(G)$ is discrete, so that $G/\text{QZ}(G)$ is locally C-stable and has trivial quasicentre.

THEOREM 3.19. *Let (G, U) be a Hecke pair such that U is residually finite.*

Then G is locally C-stable if and only if $\text{QZ}(G) \in [1]$ and every bounded abelian locally normal subgroup of G is contained in $\text{QZ}(G)$ (see Section 2.1 for the definition of the term bounded).

Moreover, if G is locally C-stable and $\text{QZ}(G) = 1$, then every locally normal subgroup H of G has the following properties:

- (i) $\text{QZ}(H) = 1$.
- (ii) $\text{QC}_G(H) = \text{C}_G(H)$. In particular $\text{C}_G(H)$ depends only on $[H]$.
- (iii) Given a locally normal subgroup K of G , then H and K commute if and only if $H \cap K = 1$.

The proof requires some properties of subnormal subgroups of residually finite groups, given in the following lemma.

LEMMA 3.20. *Let G be a group and let K be a normal subgroup of G .*

- (i) *If M is an abelian minimal normal subgroup of K , then the normal closure of M in G is abelian.*
- (ii) *If M is a finite perfect normal subgroup of K and G is residually finite, then $|G : \text{C}_G(M)|$ is finite.*

Proof. (i) Let $g \in G$. Since M is a minimal normal subgroup of K , then either $M = gMg^{-1}$, in which case $[M, gMg^{-1}] = 1$ since M is abelian, or

$M \cap gMg^{-1} = 1$, in which case $[M, gMg^{-1}] \leq M \cap gMg^{-1} = 1$ since M and gMg^{-1} are normal subgroups of K . In either case, we see that M commutes with gMg^{-1} , so every G -conjugate of M commutes with every G -conjugate of M . Hence the normal closure of M in G is abelian.

(ii) Since M is finite and G is residually finite, there is a normal subgroup H of G of finite index such that the quotient map $G \rightarrow G/H$ restricts to an injective map $M \rightarrow G/H$. In particular, $M \cap H = 1$. Now consider $[H, K]$: this is a subgroup of $H \cap K$ that is normal in G . Thus $[H, K]$ and M are subgroups of G that normalize each other and have trivial intersection, so $[H, K, M] = 1$, where $[H, K, M] = [[H, K], M]$. In particular, $[H, M, M] = 1$, since M is a subgroup of K ; equivalently $[M, H, M] = 1$. By the Three Subgroups Lemma (see [12, Theorem 10.3.5]), we have $[M, M, H] = 1$. But M is perfect, so $M = [M, M]$, and hence $[M, H] = [M, M, H] = 1$, demonstrating that $H \leq C_G(M)$. \square

Proof of Theorem 3.19. Suppose that G has an abelian locally normal subgroup L such that $L \not\leq \text{QZ}(G)$. Then $L \notin [1]$ by Lemma 3.2, and moreover

$$L \leq \text{QC}_G(L) \cap \text{QC}_G(C_G(L)),$$

so L is not C -stable in G .

From now on, we may suppose that $\text{QZ}(G) \in [1]$ and every nontrivial abelian bounded locally normal subgroup A of G is contained in $\text{QZ}(G)$; in particular, every such A is finite. If (G, U) is t.d.l.c., it is enough to consider compact locally normal subgroups, since the closure of an abelian locally normal subgroup is an abelian locally normal subgroup.

We now claim that for every bounded locally normal subgroup L of G , then $\text{QZ}(L) \leq \text{QZ}(G)$. If L is finite, then $L \leq [1]$ by the fact that U is residually finite, so $L \leq \text{QZ}(G)$ by Lemma 3.2. Given a bounded locally normal subgroup L of G , then the quasicentre of L is characteristic in L , so the quasicentre is itself bounded locally normal. Let V be a finite index subgroup of U such that $\text{QZ}(G) \cap V = 1$. To show $\text{QZ}(L) \leq \text{QZ}(G)$, it is enough to show that $\text{QZ}(L)$ is finite, and hence it is enough to show that $\text{QZ}(L \cap V)$ is finite. Thus we may assume that L is an infinite subgroup of V such that $L = \text{QZ}(L)$. By replacing V with $N_V(L)$, we may assume $L \triangleleft V$.

Since G has no infinite abelian bounded locally normal subgroup, the centre of L is finite. By ([1], Proposition 3 and Theorem 2), it follows that L is the union of its finite normal subgroups, so there exists a minimal finite normal subgroup M of L . Then M is characteristically simple, so either M is abelian or M is perfect. If M is abelian, then the normal closure N of M in V is a bounded locally normal subgroup of G , and N is abelian by Lemma 3.20(i), so $N = 1$; if M is perfect, then $|V : C_V(M)|$ is finite by Lemma 3.20(ii), so $M \leq \text{QZ}(G) = 1$. In

either case, we have a contradiction to the assumption that M is nontrivial. This contradiction proves the claim that every bounded locally normal subgroup L of G has quasicentre contained in $\text{QZ}(G)$.

We next claim that for any bounded locally normal subgroup L of G , we have

$$\text{QC}_G(\text{LC}_G(L)) \leq \text{QZ}(G).$$

This implies that L is C -stable in view of Lemma 3.12.

Let V be a finite index subgroup of U such that $\text{QZ}(G) \cap V = 1$ and $V \leq N_G(L)$, and let W be a subgroup of G commensurate to U such that $L \trianglelefteq W$. Then $L \cap V$ has finite index in L , so $L = \bigcup_{i=1}^n (L \cap V)l_i$ for some finite set of elements $\{l_1, \dots, l_n\}$ of L . Hence

$$(\text{LC}_G(L)) \cap W = \text{LC}_W(L) = \bigcup_{i=1}^n (L \cap V)C_W(L)l_i = \bigcup_{i=1}^n \bigcup_{j=1}^m (L \cap V)C_V(L)l_i w_j,$$

where $\{w_1, \dots, w_m\}$ is a finite subset of W . Consequently $\text{QC}_G(\text{LC}_G(L)) = \text{QC}_G(K)$, where $K = (L \cap V)C_V(L)$.

Notice that $C_V(K) \leq M \cap C_V(M)$, where $M = C_V(L)$. In particular, $C_V(K)$ is abelian, and moreover it is bounded and locally normal in G since $C_V(K) \trianglelefteq V$. It must thus be contained in $\text{QZ}(G)$ by hypothesis. Hence $C_V(K) \leq \text{QZ}(G) \cap V = 1$. On the other hand $\text{QZ}(K) = 1$ by the first part of the proof above. Therefore, by Lemma 3.8(i) we have $\text{QC}_V(K) = C_V(K) = 1$. In particular, the group generated by $\text{QC}_G(K)$ and V is a semidirect product $\text{QC}_G(K) \rtimes V$. Given $g \in \text{QC}_G(K)$, there is a finite index subgroup Y of V such that $YgYg^{-1}$ is contained in Y . We see that $ygy^{-1}g^{-1} \in \text{QC}_G(K) \cap V = 1$ for all $y \in Y$, so $g \in C_G(Y) \leq \text{QZ}(G)$. Thus $\text{QC}_G(K) \leq \text{QZ}(G)$, as desired, and the claim is proved.

It remains to prove that every locally normal subgroup H satisfies the assertions (i), (ii) and (iii), under the assumption that $\text{QZ}(G) = 1$.

Let $V = N_U(H)$. Then $H \cap V$ is bounded locally normal, so

$$\text{QZ}(H) \cap V = \text{QZ}(V) \leq \text{QZ}(G).$$

Hence $\text{QZ}(H)$ is normalized by V and has trivial intersection with V , so $\text{QZ}(H) \leq \text{QZ}(G)$ by Lemma 3.2, which proves (i).

Let $\tilde{H} = \text{QC}_G(H)$. Since $\text{QZ}(H) = 1$, Lemma 3.8(i) ensures that

$$\text{QC}_G(H) \cap N_G(H) = C_G(H),$$

in other words $N_{\tilde{H}}(H) = C_G(H)$. Now $N_{\tilde{H}}(H)$ contains a finite index subgroup L say of $\tilde{H} \cap U$, so

$$H \leq C_G(L) \leq \text{QC}_G(\tilde{H}).$$

Moreover, the definition $\tilde{H} = \text{QC}_G(H)$ implies that H normalizes \tilde{H} , and also that \tilde{H} is locally normal. Then $\text{QZ}(\tilde{H}) = 1$ by (i), and we infer from Lemma 3.8(i) that H centralizes \tilde{H} . Hence $\tilde{H} = C_G(H)$ and (ii) holds.

Let K be a locally normal subgroup of G . If $K \leq C_G(H)$, then certainly $H \cap K = 1$ by property (i). Conversely, suppose $H \cap K = 1$. Then there is a finite index subgroup W of U such that $H \cap W$ normalizes $K \cap W$, and hence $H \cap W \leq C_G(K \cap W)$ by Lemma 3.9. Hence $H \cap W \leq \text{QC}_G(K)$, but then by property (ii) we have $H \cap W \leq C_G(K)$. Hence by the same argument $K \leq \text{QC}_G(H) = C_G(H)$, proving (iii). \square

4. Direct decomposition of locally normal subgroups

4.1. Local decomposition lattices. Let (G, U) be a Hecke pair and let H be a bounded C -stable locally normal subgroup of G . In this section, we obtain Boolean algebras corresponding to the decomposition of finite index subgroups of H into direct factors.

LEMMA 4.1. *Let (G, U) be a Hecke pair; let H be a bounded locally normal subgroup of G with trivial quasicentre, let K be a subgroup of H of finite index and let L be a direct factor of K . Then L is C -stable as a subgroup of H . Moreover, $C_K(L)$ is the unique direct complement of L in K and $L = C_K^2(L)$.*

If H is C -stable in G , then L is also C -stable in G .

Proof. Let $K = L \times M$. Then $LC_H(L)$ is a finite index subgroup of H , so

$$\text{QC}_H(LC_H(L)) = \text{QZ}(H) = 1.$$

Thus L is C -stable as a subgroup of H by Lemma 3.12. In particular, L has trivial centre; we see that $C_K(L) = M$, and similarly $C_K(M) = L$. In addition $LC_G(L)$ contains K and also contains $C_G(K)$, so $KC_G(K) \leq LC_G(L)$.

If H is C -stable in G , then K is C -stable in G by Lemma 3.14(i), and thus by Lemma 3.12,

$$\text{QC}_V(LC_G(L)) \leq \text{QC}_V(KC_G(K)) = 1$$

for every subgroup V of G commensurate with U . In this case L is C -stable in G . \square

LEMMA 4.2. *Let G be a group such that $Z(G) = 1$, let K be a direct factor of G , and let H be a subgroup of G such that $H = C_G^2(H)$. Then $H = (H \cap K) \times C_H(K)$.*

Proof. We have $G = K \times C_G(K)$; let π_1 and π_2 be the projections onto K and $C_G(K)$ respectively associated with this decomposition. Then $H \leq \pi_1(H) \times \pi_2(H)$. Moreover, $\pi_1(H)$ and $\pi_2(H)$ are centralized by $C_G(H)$, so contained in H . Thus $H = \pi_1(H) \times \pi_2(H)$, which implies $\pi_1(H) = H \cap K$ and $\pi_2(H) = C_H(K)$. \square

DEFINITION 4.3. Given a group H and a subgroup K of H , say K is an **almost direct factor** of H if there is a finite index subgroup L of H such that K is a direct factor of L .

Let H be a bounded locally normal subgroup of G , such that the quasicentre of H is finite. Define the **local decomposition lattice** $\mathcal{LD}(G, U; H)$ of G at H to be the subset of $\mathcal{LN}(G, U)$ consisting of elements $[K]$ where K is locally normal in G and K is an almost direct factor of H .

In the case that G is a t.d.l.c. group and U is an open compact subgroup of G , we write $\mathcal{LD}(G; H)$ to mean $\mathcal{LD}(G, U; H)$.

LEMMA 4.4. *Let (G, U) be a Hecke pair, let H be a bounded locally normal subgroup of G , and let K be a locally normal subgroup of G that is commensurate with H . Then $\mathcal{LD}(G, U; H) = \mathcal{LD}(G, U; K)$ as subsets of $\mathcal{LN}(G, U)$.*

Proof. Without loss of generality, we may assume that K is a finite index subgroup of H .

It is clear from the definition that $\mathcal{LD}(G, U; K) \subseteq \mathcal{LD}(G, U; H)$. Conversely, suppose the finite index subgroup L of H decomposes as $L_1 \times L_2$ with L_1 and L_2 locally normal in G . Then $L_i \cap K$ is locally normal, and has finite index in L_i for $i \in \{1, 2\}$. Hence $(L_1 \cap K) \times (L_2 \cap K)$ has finite index in K , so $[L_1] = [L_1 \cap K]$ and $[L_2] = [L_2 \cap K]$ are contained in $\mathcal{LD}(G, U; K)$. \square

Consequently, it makes sense to write $\mathcal{LD}(G, U; \alpha)$ where α is any element of $\mathcal{LN}(G, U)$ that has a representative with trivial quasicentre, and this lattice admits an action of G_α by conjugation. The case $\alpha = [U]$ is of particular interest, and we define $\mathcal{LD}(G, U) := \mathcal{LD}(G, U; [U])$.

As the name suggests, the local decomposition lattice is a sublattice of $\mathcal{LN}(G, U)$, but more is true:

THEOREM 4.5. *Let (G, U) be a Hecke pair and let $\alpha \in \mathcal{LN}(G, U)$ have a bounded locally normal representative H with trivial quasicentre. Then $\mathcal{LD}(G, U; \alpha)$ is a sublattice of $\mathcal{LN}(G, U)$, and internally it is a Boolean algebra (relative to the maximum α), with complementation map $\perp_\alpha : \beta \mapsto [\text{QC}_H(\beta)]$. If (G, U)*

is t.d.l.c. and α has a closed representative, then $\mathcal{LD}(G; \alpha)$ is a sublattice of $\mathcal{LN}(G)$.

We prove the following separately, as it will be reused later.

LEMMA 4.6. *Let \mathcal{M} be a meet semilattice with a least element 0. Suppose $\perp: \mathcal{M} \rightarrow \mathcal{M}$ is an order-reversing involution such that for all $\alpha, \beta \in \mathcal{M}$, $\alpha \wedge \beta = 0$ if and only if $\beta \leq \alpha^\perp$. Then \mathcal{M} is a Boolean algebra, with minimum 0 and maximum 0^\perp , complementation given by \perp and join operation given by $\alpha \vee \beta = (\alpha^\perp \wedge \beta^\perp)^\perp$.*

Proof. Let $\gamma \in \mathcal{M}$. The existence of an order-reversing involution ensures that \mathcal{M} is a lattice, with maximum and join as given. For all $\alpha \in \mathcal{M}$ we have $\alpha \vee \alpha^\perp = (\alpha^\perp \wedge \alpha)^\perp = 0^\perp$ and clearly $\alpha \wedge \alpha^\perp = 0$, so α^\perp is a complement of α relative to the maximum element 0^\perp .

It remains to show that \mathcal{M} is distributive. We first show

$$\alpha \wedge \beta \leq \gamma \Leftrightarrow \alpha \leq \beta^\perp \vee \gamma, \tag{4.1}$$

for all $\alpha, \beta, \gamma \in \mathcal{M}$. Using the properties of \perp in the hypothesis, we have $\alpha \wedge \beta \leq \gamma$ if and only if $\alpha \wedge \beta \wedge \gamma^\perp = 0$, while $\alpha \leq \beta^\perp \vee \gamma$ if and only if $\delta = 0$ where $\delta = \alpha \wedge (\beta^\perp \vee \gamma)^\perp$. Now $\delta = \alpha \wedge \beta \wedge \gamma^\perp$, so (4.1) is proved.

Now let $\alpha, \beta, \gamma, \delta \in \mathcal{M}$. By (4.1) we have

$$(\alpha \vee \delta) \wedge \beta \leq \gamma \Leftrightarrow \alpha \vee \delta \leq \beta^\perp \vee \gamma \Leftrightarrow (\alpha \leq \beta^\perp \vee \gamma \text{ and } \delta \leq \beta^\perp \vee \gamma).$$

Using (4.1) again we have $\alpha \leq \beta^\perp \vee \gamma \Leftrightarrow \alpha \wedge \beta \leq \gamma$ and $\delta \leq \beta^\perp \vee \gamma \Leftrightarrow \delta \wedge \beta \leq \gamma$. Finally, the conjunction of the inequalities $\alpha \wedge \beta \leq \gamma$ and $\delta \wedge \beta \leq \gamma$ is given by the inequality $(\alpha \wedge \beta) \vee (\delta \wedge \beta) \leq \gamma$, proving the distributive law. \square

Proof of Theorem 4.5. Fix a bounded locally normal representative H of α , such that H has trivial quasicentre. Then every finite index subgroup of H has trivial quasicentre.

Let K and L be almost direct factors of H such that K and L are locally normal in G . Then $K C_H(K)$ and $L C_H(L)$ both have finite index in H , and $K \cap C_H(K) = L \cap C_H(L) = 1$. We may replace K with $C_H^2(K)$ and so assume $K = C_H^2(K)$, since this substitution does not affect the commensurability class of K or its status as an almost direct factor of H that is locally normal in G . Let $M = L C_H(L)$ and let $K_2 = K \cap M$. Then $K_2 = C_M(C_H(K))$. Since K is C -stable in H by Lemma 4.1, in fact $C_H(K) = C_H(K_2)$ by Corollary 3.17; in turn, $C_H(K)$ has the same centralizer in H as $C_M(K)$ for the same reason. So in fact $K_2 = C_M^2(K_2)$. It follows by Lemma 4.2 that $K_2 \cap L$ is a direct factor of K_2 . In turn, K_2 is a direct

factor of $K_2C_H(K)$, which has finite index in H . Thus $K_2 \cap L$ is an almost direct factor of H .

By a similar argument, there is a finite index subgroup R of $C_H(K)$ that factorizes as $(R \cap L) \times C_R(L)$, so there is a finite index subgroup of H that factorizes as

$$K_2 \times (R \cap L) \times C_R(L).$$

The first two factors are contained in K_2L , while the last factor centralizes K_2L ; in addition $K_2L \cap C_R(L) \leq \text{QZ}(H) = 1$. It follows that K_2L is an almost direct factor of H . Finally, note that K_2 and L are both bounded and locally normal, and this property is inherited by both their intersection and their product. We conclude that $[K_2 \cap L], [K_2L] \in \mathcal{LD}(G, U; \alpha)$, so $\mathcal{LD}(G, U; \alpha)$ is a sublattice of $\mathcal{LN}(G, U)$.

By Lemma 3.10, $[QC_H(K)]$ has a representative $C_H(K)$; this also represents an element of $\mathcal{LD}(G, U; \alpha)$, so \perp_α preserves $\mathcal{LD}(G, U; \alpha)$. One has $[K] \wedge \beta = 0$ for $\beta \in \mathcal{LD}(G, U; \alpha)$ whenever $\beta \leq [K]^{-\alpha}$, since $\text{QZ}(K) = 1$, and conversely $[K] \wedge \beta = 0$ implies by Lemma 3.9 that some representative of β centralizes a representative of $[K]$, so in this case $\beta \leq [K]^{-\alpha}$. It is also clear that \perp_α is an order-reversing involution on $\mathcal{LD}(G, U; \alpha)$. Thus $\mathcal{LD}(G, U; \alpha)$ is a Boolean algebra by Lemma 4.6.

Now suppose (G, U) is t.d.l.c. and that α has a closed representative. Then by Proposition 2.4, α has a compact locally normal representative H_2 , and moreover from the proof, H_2 can be taken to be the closure of a finite index subgroup of H . We see that $H \cap H_2$ has trivial quascentre. Since $H \cap H_2$ is a dense subgroup of H_2 of finite index, we have $\text{QC}_G(H_2) = \text{QC}_G(H \cap H_2)$. In particular, H_2 has trivial quascentre. We now follow the same argument as before, with H_2 in place of H , and see that every element of $\mathcal{LN}(G; \alpha)$ has a representative of the form $L = C_{H_2}^2(K)$, where K is locally normal. Since centralizers are closed, it follows that L is compact and locally normal in G , so $\mathcal{LN}(G; \alpha) \subseteq \mathcal{LN}(G)$; since $\mathcal{LN}(G; \alpha)$ is a sublattice of $\mathcal{LN}(G, U)$, it is also a sublattice of $\mathcal{LN}(G)$. \square

Proof of Theorem 1 (i). Since G has trivial quascentre, the quasicentralizer $\text{QC}_G(U)$ of any open compact subgroup of G is trivial. In particular, U is C -stable in G , so by Theorem 4.5, the lattice $\mathcal{LD}(G; [U]) = \mathcal{LD}(G)$ is a Boolean algebra. \square

In the case of a topological group G and a subgroup H , there naturally arises the question of how direct factors of H relate to direct factors of \overline{H} . Put another way, one can consider the **quasifactors** of a topological group, meaning the direct factors of its dense subgroups. If \overline{H} is compact and centreless, there is good control of this situation. As a result, we obtain a relationship between decomposition lattices of t.d.l.c. groups and their dense subgroups.

LEMMA 4.7. *Let G be a compact topological group such that $Z(G) = 1$ and let D be a dense subgroup of G . Let K be a direct factor of D . Then*

$$G = C_G(K) \times C_G^2(K),$$

and $K = C_G^2(K) \cap D$.

In particular, the map $K \mapsto C_G^2(K)$ defines an injective homomorphism from \mathcal{L}_D to \mathcal{L}_G , where \mathcal{L}_D is the lattice of direct factors of D and \mathcal{L}_G is the lattice of direct factors of G .

Proof. Since centralizers are closed, we have $C_G(D) = 1$.

Let $D = K \times L$. Then $Z(L) \leq C_G(D) = 1$, so $C_D(L) = K$. Similarly, $C_D(K) = L$. Thus D is contained in the set $E = C_G(K)C_G(L)$. We see that both $C_G(K)$ and $C_G(L)$ are closed in G , hence compact, so E is compact, hence closed. Since E is also dense, it follows that $E = G$. Each of the subgroups $C_G(K)$ and $C_G(L)$ is normalized by D , and hence by G , since the normalizer of a closed subgroup is closed. Moreover, $C_G(K) \cap C_G(L) = C_G(D) = 1$. Hence

$$G = C_G(K) \times C_G(L).$$

We see that $Z(C_G(K)) \leq Z(G) = 1$, so in fact $C_G(L) = C_G^2(K)$. It follows that

$$C_G^2(K) \cap D = C_D(L) = K.$$

The remaining conclusions are clear. □

PROPOSITION 4.8. *Let G be a t.d.l.c. group such that $\text{QZ}(G) = 1$ and let D be a dense subgroup of G . Let U be an open compact subgroup of G , and let R be a commensurated subgroup of D that is also a dense subgroup of U . Let H be a bounded locally normal subgroup of the Hecke pair (D, R) . Then $\text{QZ}(D) = 1$; moreover the following assertions hold.*

- (i) *If \overline{H} has trivial quasicentre, then H has trivial quasicentre. Moreover, the map $[K] \mapsto [C_{\overline{H}}^2(K)]$ defines an injective order-preserving map from $\mathcal{LD}(D, R; H)$ to $\mathcal{LD}(G, U, \overline{H})$.*
- (ii) *If H has trivial quasicentre, then $\mathcal{LD}(G, U; H)$ is embedded in $\mathcal{LD}(D, R; H)$ as a sublattice in a natural way. If H is closed in G , then this embedding is an isomorphism.*

Proof. We have $\text{QC}_G(U) \leq \text{QZ}(G) = 1$, so $\text{QC}_G(R) = 1$ by Lemma 3.5. In particular, every finite index subgroup of R has trivial centralizer in D . Thus $\text{QZ}(D) = 1$.

Since H is locally normal in (D, R) , without loss of generality we may assume $H \leq R$.

(i) We have $\text{QC}_G(H) = \text{QC}_G(\overline{H})$ by Lemma 3.5. Suppose $\text{QZ}(\overline{H}) = 1$. Then $\text{QZ}(H) = 1$.

Let K be an almost direct factor of H that is locally normal in (D, R) . Then $N_G(K)$ contains a finite index subgroup of R , so $N_G(\overline{K})$ contains a finite index subgroup of $\overline{R} = U$; thus \overline{K} is compact and locally normal in (G, U) . We see from Lemma 4.7 that the map $\phi: [K] \mapsto [C_H^2(K)]$ defines an order-preserving map from $\mathcal{LD}(D, R; H)$ to $\mathcal{LD}(G, U, \overline{H})$. Moreover, we have $K = M \cap C_H^2(K)$, where M is any finite index subgroup of H of which K is a direct factor, so ϕ is injective.

(ii) If K is an almost direct factor of H that is locally normal in (G, U) , then it is also locally normal in (D, R) , since $R \leq U$. We can therefore regard $\mathcal{LD}(G, U; H)$ as a subset of $\mathcal{LD}(D, R; H)$, consisting of those elements $\alpha \in \mathcal{LD}(D, R; H)$ such that there exists a representative K of α such that $N_G(K)$ contains a finite index subgroup of U . Moreover, $\mathcal{LD}(G, U; H)$ forms a sublattice of $\mathcal{LD}(D, R; H)$, since the meet and join operations for $\mathcal{LD}(G, U; H)$ and $\mathcal{LD}(D, R; H)$ are essentially the same (in both cases, the meet is the class of the intersection of representatives, and the join is the class of the product of suitably chosen representatives).

If H is closed in G , then every almost direct factor K of H has finite index in a closed almost direct factor $C_H^2(K)$, and we see that $C_H^2(K)$ is locally normal in (G, U) if and only if it is locally normal in (D, R) , in other words $\mathcal{LD}(D, R; H) = \mathcal{LD}(G, U; H)$. \square

It is clear that the relation ‘ A is a direct factor of B ’ is transitive. A similar transitivity applies to local decomposition lattices:

LEMMA 4.9. *Let (G, U) be a Hecke pair and let $\alpha, \beta \in \mathcal{LN}(G, U)$ have bounded locally normal representatives with trivial quasicentre. Then $\mathcal{LD}(G, U; \alpha) \subseteq \mathcal{LD}(G, U; \beta)$ if and only if $\alpha \in \mathcal{LD}(G, U; \beta)$.*

Proof. Suppose $\alpha \in \mathcal{LD}(G, U; \beta)$. Then there is a representative H of α and a representative K of β such that H is a direct factor of K . It follows that all almost direct factors of H are also almost direct factors of K . \square

4.2. Locally finitely decomposable t.d.l.c. groups. The following are natural properties to consider in the context of local decomposition lattices:

DEFINITION 4.10. Let H be a group. Say H is **finitely decomposable** if

$$H = H_1 \times \cdots \times H_n$$

for some $n \geq 1$, such that each H_i is **indecomposable**, that is, all of whose direct product decompositions are trivial. Given a t.d.l.c. group G , say G is **locally finitely decomposable** if every open compact subgroup of G is finitely decomposable.

PROPOSITION 4.11. *Let G be a group such that $Z(G) = 1$.*

- (i) *Suppose $G = G_1 \times \cdots \times G_n$ for some $n \geq 1$, such that each G_i is indecomposable. Then $\{G_1, \dots, G_n\}$ is precisely the set of indecomposable direct factors of G and every direct factor of G is a product of some subset of $\{G_1, \dots, G_n\}$. In particular, $\text{Aut}(G)$ acts on $\{G_1, \dots, G_n\}$ and there are exactly 2^n direct factors of G .*
- (ii) *Suppose that G is compact and not finitely decomposable. Then G is topologically isomorphic to an infinite Cartesian product of closed subgroups. In particular, G has uncountably many closed direct factors up to commensurability.*

Proof. For any direct factor H of G , then $C_G(H)$ is the complement of H and thus also a direct factor, so $H = C_G(C_G(H))$. Suppose $G = G_1 \times \cdots \times G_n$ for some $n \geq 1$, such that each G_i is indecomposable. Let K be a direct factor of G . Then $G = KC_G(K)$, and $K \cap C_G(K) \leq Z(G) = 1$. Let $I = \{i \mid G_i \not\leq C_G(K)\}$. Then $K \leq \prod_{i \in I} G_i$, since K is centralized by the remaining factors. Given $i \in I$, then $K \cap G_i$ is a direct factor of both K and G_i , by Lemma 4.2, so $K = G_i$ by the fact that K and G_i are indecomposable. Thus $K = \prod_{i \in I} G_i$.

Suppose G is compact and not finitely decomposable. We shall define a family $\{G_{i,j}\}_{1 \leq j \leq i}$ of nontrivial subgroups as follows, so that for all n , $G = G_{n,1} \times \cdots \times G_{n,n}$ and $G_{n,n}$ is not finitely decomposable.

Set $G_{1,1} = G$. Suppose we have $G = G_{n,1} \times \cdots \times G_{n,n}$ for some n . Choose some nontrivial direct decomposition $G_{n,n} = H_n \times K_n$, ensuring that K_n is not finitely decomposable (this is always possible, as by induction $G_{n,n}$ is not finitely decomposable). Now set $G_{n+1,n} = H_n$, $G_{n+1,n+1} = K_n$ and $G_{n+1,j} = G_{n,j}$ for $1 \leq j < n$.

Let $K = \bigcap_{j=1}^{\infty} G_{j,j}$. We claim now that

$$G \cong K \times \prod_{j=1}^{\infty} G_{j+1,j}.$$

The decomposition $G = G_{n,1} \times \cdots \times G_{n+1,n+1}$ gives a continuous epimorphism $G \rightarrow \prod_{j=1}^n G_{n+1,j}$ with kernel $G_{n+1,n+1}$, and we note from the construction that $\prod_{j=1}^n G_{n+1,j} = \prod_{j=1}^n G_{j+1,j}$. These epimorphisms are compatible with

each other as n varies so that they generate an inverse system, with a limit homomorphism $\phi : G \rightarrow \prod_{j=1}^{\infty} G_{j+1,j}$. We see that $\ker \phi$ is precisely K . In addition $\phi(C_G(K))$ contains $\bigoplus_{j=1}^{\infty} G_{j+1,j}$, since $G_{j+1,j}$ centralizes $G_{n+1,n+1}$ for all $1 \leq j \leq n$, so $\phi(C_G(K)) = \prod_{j=1}^{\infty} G_{j+1,j}$. In particular $G = KC_G(K)$: this implies $K \cap C_G(K) \leq Z(G) = 1$, so in fact $G = K \times C_G(K)$. The restriction of ϕ to $C_G(K)$ is consequently bijective, giving the desired isomorphism. We now obtain uncountably many direct factors $G_J = \prod_{j \in J} G_{j+1,j}$ by letting J range over all subsets of \mathbf{N} . For two such direct factors $\prod_{j \in J_1} G_{j+1,j}$ and $\prod_{j \in J_2} G_{j+1,j}$ to be commensurate, the symmetric difference of the sets J_1 and J_2 must be finite. This defines an equivalence relation on the power set of \mathbf{N} whose equivalence classes are countable, so there must be uncountably many equivalence classes. \square

COROLLARY 4.12. *Let G be a t.d.l.c. group with $\text{QZ}(G) = 1$.*

- (i) *G is locally finitely decomposable if and only if G has a basis of identity neighbourhoods consisting of open compact subgroups that are finitely decomposable.*
- (ii) *Suppose G is first-countable. Then G is locally finitely decomposable if and only if $\mathcal{LD}(G)$ is countable.*

Proof. (i) The necessity of the condition is obvious. Conversely, if G is not locally finitely decomposable, then G has some open compact subgroup V which is not finitely decomposable. Since G has trivial quasicentre, the group V must be centre-free, and is thus an infinite Cartesian product of closed subgroups by Proposition 4.11. Now, any open subgroup W of V contains infinitely many direct factors of V ; those factors are then also direct factors of W , and hence W cannot be finitely decomposable.

(ii) Suppose that G is locally finitely decomposable. Since $\mathcal{LD}(G) = \mathcal{LD}(U)$ for any open compact subgroup U , we may assume that G is compact. In this case G has only countably many open subgroups, each with only finitely many direct factors, and thus $\mathcal{LD}(G)$ is countable.

Conversely, suppose $\mathcal{LD}(G)$ is countable. Then given an open compact subgroup U of G , the direct factors of U fall into countably many commensurability classes. Thus U is finitely decomposable by Proposition 4.11. \square

PROPOSITION 4.13. *Let G be a t.d.l.c. group and let H be a C -stable compact locally normal subgroup that is locally finitely decomposable. Then $\mathcal{LD}(G; H) = \mathcal{LD}(H)$; indeed $\mathcal{LD}(G; H)$ contains $\mathcal{LD}(K)$ for every almost direct factor K of H .*

Proof. It is clear that $\mathcal{LD}(G; H) \subseteq \mathcal{LD}(H)$. Let L be an open subgroup of H , and let K be a direct factor of L . Then $N_G(L)$ is open in G by Lemma 2.1(ii), and it preserves the set X of indecomposable direct factors of L ; since X is finite, there is an open subgroup R of $N_G(L)$ that normalizes every indecomposable direct factor of L . In turn K is a product of indecomposable direct factors of L , so it is normalized by R . Hence K is locally normal, so represents an element of $\mathcal{LD}(G; H)$. Hence $\mathcal{LD}(G; H) = \mathcal{LD}(H)$, and $\mathcal{LD}(H)$ contains $\mathcal{LD}(H; K)$ by Lemma 4.9. In turn, K is locally finitely decomposable, because given an open subgroup M of K , the direct factors of M are also direct factors of $MC_H(K)$, which is an open subgroup of H . Hence $\mathcal{LD}(H; K) = \mathcal{LD}(K)$. \square

5. The centralizer lattice and weakly decomposable actions

5.1. The centralizer lattice. It would be useful to combine all the local decomposition lattices of (G, U) (which are not necessarily individually preserved by the action of G) into a single Boolean algebra admitting a G -action. There are two difficulties here: one is that the property of having trivial quasicentre may not be well behaved under intersections, and the other is that we need a unified notion of complementation. To avoid the first difficulty and overcome the second, we restrict our attention to locally C -stable Hecke pairs (G, U) in the sense of Section 3.5. Recall from Theorem 3.19 that under the assumption that U is residually finite, these are precisely the Hecke pairs with trivial quasicentre and no nontrivial abelian locally normal subgroups. The resulting lattice will not be a sublattice of $\mathcal{LN}(G, U)$ in general (because the join operation is different), but it arises from the structure lattice in a natural way.

DEFINITION 5.1. Let (G, U) be a locally C -stable Hecke pair. Define the map

$$\perp : \mathcal{LN}(G, U) \rightarrow \mathcal{LN}(G, U) : \alpha \mapsto [\text{QC}_G(\alpha)].$$

This is well defined by Theorem 3.19, since there exists a finite index subgroup V of U such that $\text{QZ}(V) = 1$, and we have

$$[\text{QC}_G(\alpha)] = [\text{QC}_V(\alpha)] = [C_G(K)]$$

for every locally normal representative K of α contained in V . The **centralizer lattice** $\mathcal{LC}(G, U)$ is defined to be the set $\{\alpha^\perp \mid \alpha \in \mathcal{LN}(G, U)\}$ together with the map \perp restricted to $\mathcal{LC}(G, U)$, the partial order inherited from $\mathcal{LN}(G, U)$ and the binary operations \wedge_c and \vee_c given by:

$$\begin{aligned} \alpha \wedge_c \beta &= \alpha \wedge \beta \\ \alpha \vee_c \beta &= (\alpha^\perp \wedge \beta^\perp)^\perp. \end{aligned}$$

In general, we write \vee instead of \vee_c in contexts where it is clear that we are working inside the centralizer lattice.

Given a t.d.l.c. group G , we define $\mathcal{LC}(G) := \mathcal{LC}(G, U)$, where U is some open compact subgroup of G . (The choice of U here is inconsequential.)

THEOREM 5.2. *Let (G, U) be a locally C -stable Hecke pair, such that U is residually finite.*

- (i) *The poset $\mathcal{LC}(G, U)$ is a Boolean algebra and $\perp^2 : \mathcal{LN}(G, U) \rightarrow \mathcal{LC}(G, U)$ is an idempotent surjective lattice homomorphism. If G is t.d.l.c., then*

$$\mathcal{LC}(G) = \{\alpha^\perp \mid \alpha \in \mathcal{LN}(G)\} \subseteq \mathcal{LN}(G).$$

- (ii) *Fix $\alpha \in \mathcal{LN}(G, U)$ and let θ be the map on $\mathcal{LN}(G, U)$ given by $\beta \mapsto \beta^{\perp^2} \wedge \alpha$. Then θ is an idempotent order-preserving map, and $\theta(\mathcal{LN}(G, U)) = \theta(\mathcal{LC}(G, U)) \supseteq \mathcal{LD}(G, U; \alpha)$. Moreover, $\theta^{-1}(\mathcal{LD}(G, U; \alpha)) \cap \mathcal{LC}(G, U)$ is a subalgebra of $\mathcal{LC}(G, U)$.*

We need the following basic group theoretic facts.

LEMMA 5.3. *Let G be a group such that every normal subgroup has trivial centre, in other words that G has no nontrivial abelian normal subgroups. Then the following equations are satisfied for any normal subgroups A and B of G .*

- (i) $C_G^2(A \cap B) = C_G^2(A) \cap C_G^2(B)$;
(ii) $C_A(B) = C_A(A \cap B)$;
(iii) $C_A(C_G(B)) = C_A(C_A(B))$.

Proof. (i) Let $K = C_G^2(A)$ and let $L = C_G^2(B)$. It is clear that

$$C_G(C_G(A \cap B)) \leq C_G(C_G(A)C_G(B)) = K \cap L,$$

so it suffices to show that $K \cap L$ centralizes $C_G(A \cap B)$. For this it is enough to show that $M = C_G(A \cap B) \cap K \cap L = 1$, since all subgroups under consideration are normal. We see that $M \cap A \cap B \leq Z(A \cap B) = 1$, so $M \cap A$ centralizes B , in other words $M \cap A \leq C_G(B)$. Then $M \cap A$ is centralized by L , so $M \cap A$ is central in M , so $M \cap A = 1$, and hence $M \leq C_G(A)$. Since K centralizes $C_G(A)$ and contains M , we see that M is central in $C_G(A)$ and hence trivial, as required.

(ii) Notice that $C_A(A \cap B)$ is normal in G and has trivial intersection with $A \cap B$ (since $Z(A \cap B) = 1$). Therefore, $[C_A(A \cap B), B] \subset C_A(A \cap B) \cap B = 1$, whence $C_A(A \cap B) \leq C_A(B)$. The reverse inclusion is clear.

(iii) This is a special case of (ii), using the normal subgroups A and $C_G(B)$ instead of A and B . □

Proof of Theorem 5.2. Since we are concerned only with properties of a local nature, we are free to replace G and U with a finite index subgroup V of U (which can be assumed to be open in the t.d.l.c. case) such that $\text{QZ}(V) = 1$. In other words, we may assume that G has trivial quasicentre.

(i) Let A be a bounded locally normal subgroup of G , let $\alpha = [A]$, and let V be a subgroup of G commensurate with U such that $A \trianglelefteq V$. By Lemma 3.16, $C_V^n(A)$ is then a representative of $\alpha^{\perp n}$. We deduce $\alpha^{\perp 2} \geq \alpha$ and $\alpha^{\perp 3} = \alpha^{\perp}$ from Lemma 3.15. In particular, \perp acts as an involution on $\mathcal{LC}(G, U)$, so $\perp^2(\mathcal{LN}(G, U)) = \mathcal{LC}(G, U)$. In the t.d.l.c. case, we note that $C_U(C_V(A))$ is a closed representative of $\alpha^{\perp 2}$, so $\alpha^{\perp 2} \in \mathcal{LN}(G)$, and since $\perp^3 = \perp$, it follows that $\perp(\mathcal{LN}(G)) = \perp(\mathcal{LN}(G, U)) =: \mathcal{LC}(G)$ and that $\mathcal{LC}(G)$ is contained in $\mathcal{LN}(G)$.

It is clear that \perp is order-reversing (so \perp^2 is order-preserving). Let $\alpha, \beta \in \mathcal{LN}(G, U)$ and choose representatives A and B of α, β respectively that are normal in a subgroup V that is commensurate with U . Then we have $C_V(AB) = C_V(A) \cap C_V(B)$, so that

$$(\alpha \vee \beta)^\perp = \alpha^\perp \wedge \beta^\perp. \tag{5.1}$$

This shows that $\mathcal{LC}(G, U)$ is a meet sublattice of $\mathcal{LN}(G, U)$. Since the only abelian normal subgroup of V is the trivial one, it also follows from Lemma 5.3(i) that \perp^2 is compatible with the meet operation.

By Lemma 3.9, $\alpha \wedge \beta = 0$ if and only if B centralizes A , which amounts to saying that $\beta \leq \alpha^\perp$. Hence $\mathcal{LC}(G, U)$ is a Boolean algebra by Lemma 4.6.

We see that \perp^2 is compatible with the join operations in passing from $\mathcal{LN}(G, U)$ to $\mathcal{LC}(G, U)$ as follows, using (5.1):

$$\alpha^{\perp 2} \vee_c \beta^{\perp 2} = (\alpha^{\perp 3} \wedge \beta^{\perp 3})^\perp = (\alpha^\perp \wedge \beta^\perp)^\perp = (\alpha \vee \beta)^{\perp 2}.$$

Hence \perp^2 is a lattice homomorphism.

(ii) It is clear that $\theta(\mathcal{LN}(G, U)) = \theta(\mathcal{LC}(G, U))$, since \perp^2 is a projection of $\mathcal{LN}(G, U)$ to $\mathcal{LC}(G, U)$. It is also clear that θ is order-preserving. Given $\beta \in \mathcal{LN}(G, U)$ we have

$$\theta^2(\beta) = (\beta^{\perp 2} \wedge \alpha)^{\perp 2} \wedge \alpha = \beta^{\perp 4} \wedge \alpha^{\perp 2} \wedge \alpha = \beta^{\perp 2} \wedge \alpha,$$

so $\theta^2 = \theta$.

Let A be a bounded locally normal representative of α and let K be a direct factor of A . Choose V commensurate with U such that both A and K are normal

in U . Then $K = C_A^2(K)$, so $K = C_A(C_V(K))$ by Lemma 5.3(iii). In other words, $[K] = \beta^{\perp^2} \wedge \alpha$ where $\beta = \beta^{\perp^2}$ is the element $[C_V^2(K)] = [QC_V^2(K)]$ of $\mathcal{LC}(G, U)$. Thus $\theta(\mathcal{LC}(G, U))$ contains $\mathcal{LD}(G, U; \alpha)$.

Let $\mathcal{L} = \theta^{-1}(\mathcal{LD}(G, U; \alpha)) \cap \mathcal{LC}(G, U)$. Then \mathcal{L} is closed under meets in $\mathcal{LC}(G, U)$, by the fact that \perp^2 is compatible with the meet operation. Furthermore \mathcal{L} is closed under complementation in $\mathcal{LC}(G, U)$, since the complement of $\beta \wedge \alpha$ in $\mathcal{LD}(G, U; \alpha)$ is $(\beta \wedge \alpha)^\perp \wedge \alpha$ by Theorem 4.5. Using Lemma 5.3(ii) we see that

$$(\beta \wedge \alpha)^\perp \wedge \alpha = [C_A(A \cap B)] = [C_A(B)] = \beta^\perp \wedge \alpha,$$

where A and B are representatives of α and β respectively that normalize each other. So \mathcal{L} is a subalgebra of $\mathcal{LC}(G, U)$. □

Theorem I (ii) now follows from Theorem 5.2 together with Theorem 3.19.

Let (G, U) be a locally C-stable Hecke pair such that $\text{QZ}(G) = 1$. Then each element α of the centralizer lattice has a canonical ‘global’ representative, namely $\text{QC}_G(\alpha^\perp)$. This subgroup is the unique largest locally normal representative of α . If (G, U) is t.d.l.c., then $\text{QC}_G(\alpha^\perp)$ is closed, but not necessarily compact.

We can consider the set $\mathcal{L} = \{\text{QC}_G(\alpha) \mid \alpha \in \mathcal{LN}(G, U)\}$, ordered by inclusion, as the **global centralizer lattice** of (G, U) . As a Boolean algebra, it is G -equivariantly isomorphic to $\mathcal{LC}(G, U)$. The elements of \mathcal{L} also have a more direct characterization using Theorem 3.19, as follows:

PROPOSITION 5.4. *Let (G, U) be a locally C-stable Hecke pair such that U is residually finite and $\text{QZ}(G) = 1$. Then the set of subgroups $\mathcal{L} := \{\text{QC}_G(\alpha) \mid \alpha \in \mathcal{LN}(G, U)\}$ enjoys the following properties:*

- (i) *If $H \in \mathcal{L}$, then $H = C_G^2(H)$ and H is locally normal. Indeed $H \cap R = C_R^2(H)$, where R is any locally normal subgroup of G .*
- (ii) *Let K be a subgroup of G such that $C_G(K)$ is locally normal. Then $C_G(K) \in \mathcal{L}$. If K is locally normal, in fact $C_G(K) = \text{QC}_G([K])$ and $C_G^2(K) = \text{QC}_G([K]^\perp)$.*

Proof. Let $\alpha \in \mathcal{LN}(G, U)$, let $H = \text{QC}_G(\alpha)$. Since α is an element of $\mathcal{LN}(G, U)$, then $|U : U_\alpha|$ is finite; in turn U_α normalizes H , since H is defined as a subgroup of G in terms of α . Hence $|U : N_U(H)|$ is finite, so that H is locally normal.

Let $L = \text{QC}_G(\alpha^\perp)$. Then $L = C_G(H)$ by Theorem 3.19. The same argument shows that $H = C_G(L)$, so $H = C_G^2(H)$. Now let $M = C_G(H)$, and suppose R is a locally normal subgroup of G . Then $R \cap M$ is locally normal and hence centreless,

since G is locally C -stable and $\text{QZ}(G) = 1$. Then $C_R(R \cap M) \geq H \cap R$, since M centralizes H . Moreover, we have

$$C_R(R \cap M) \cap M = C_R(R \cap M) \cap (R \cap M) \leq Z(R \cap M) = 1.$$

Thus $C_R(R \cap M)$ and M are locally normal subgroups that intersect trivially, so they commute by Theorem 3.19(iii), so that $C_R(R \cap M) = C_R(M)$, and hence $C_R(R \cap M) \leq C_G^2(H) = H$. This proves (i).

Now let K be a locally normal subgroup of G . Then $C_G(K) = \text{QC}_G([K])$ by Theorem 3.19, so in particular $C_G(K) \in \mathcal{L}$. Setting $L = C_G(K)$, the same argument shows that $C_G(L) = \text{QC}_G(\alpha^\perp)$. More generally, if K is a subgroup of G such that $C_G(K)$ is locally normal, then $C_G(K) = C_G^3(K)$, and in turn $C_G^3(K)$ is the centralizer of a locally normal subgroup $C_G^2(K)$ of G , so $C_G(K) \in \mathcal{L}$. This completes the proof of (ii). \square

As a consequence of Proposition 5.4 and Lemma 4.2, we obtain information about the decomposability of centralizers in G .

COROLLARY 5.5. *Let (G, U) be a locally C -stable Hecke pair such that U is residually finite and $\text{QZ}(G) = 1$. Let A and B be locally normal subgroups of G . Suppose that $B = K \times L$ for some subgroups K and L . Then*

$$C_B(A) = C_K(A) \times C_L(A).$$

Proof. Let $H = C_G(A)$ and let $R = C_B(A)$. Then $R = C_B^2(H)$ by Proposition 5.4; consequently $R \geq C_B^2(R)$, so in fact $R = C_B^2(R)$. Thus by Lemma 4.2, $R \cap K$ and $R \cap L$ are complementary direct factors of R , giving the required factorization. \square

Similar to Proposition 4.8, there is a relationship between the centralizer lattice of a t.d.l.c. group and that of its dense subgroups, viewed as Hecke pairs.

PROPOSITION 5.6. *Let G be a locally C -stable t.d.l.c. group such that $\text{QZ}(G) = 1$ and let D be a dense subgroup of G . Let U be an open compact subgroup of G , and let R be a commensurated subgroup of D that is also a dense subgroup of U .*

- (i) *The Hecke pair (D, R) is locally C -stable, with $\text{QZ}(D) = 1$.*
- (ii) *Let \mathcal{L} be the global centralizer lattice of (G, U) and let \mathcal{L}' be the global centralizer lattice of (D, R) . Then there is an injective D -equivariant order-preserving map from \mathcal{L}' to \mathcal{L} given by $K \mapsto C_G^2(K)$.*

Proof. Given a locally normal subgroup L of D , then \overline{L} and $C_G(L)$ are both closed and locally normal in G ; in particular, $C_G(L) \in \mathcal{L}$ by Proposition 5.4. Assuming L is nontrivial, then since \overline{L} cannot be finite or abelian, neither can L . In addition, we have $\text{QZ}(D) = 1$ by Proposition 4.8. This proves (i) in view of Theorem 3.19.

Given $K \in \mathcal{L}'$, then $C_G(K) \in \mathcal{L}$ as above, hence $C_G^2(K) \in \mathcal{L}$. Thus $\theta : K \mapsto C_G^2(K)$ is a map from \mathcal{L}' to \mathcal{L} . Clearly θ is D -equivariant and order-preserving. Since $K = C_D^2(K)$, we see that $K = \theta(K) \cap D$ as follows:

$$K \leq \theta(K) \cap D = C_G^2(K) \cap D \leq C_G(C_D(K)) \cap D = C_D^2(K) = K.$$

Thus θ is injective. This proves (ii). \square

EXAMPLE 5.7. In general, given a t.d.l.c. group G , the centralizer lattice of a dense subgroup of G does not determine the centralizer lattice of G . Let F be a nonabelian free group (with the discrete topology), and consider F as a Hecke pair (F, F) . It is clear that F is locally C -stable, so we can define the centralizer lattice $\mathcal{LC}(F, F)$. Since $C_F(x)$ is cyclic for all $x \in F \setminus \{1\}$, in fact $\mathcal{LC}(F, F) = \{0, \infty\}$. Likewise $\mathcal{LC}(\hat{F}_p) = \{0, \infty\}$, where \hat{F}_p is the pro- p completion of F . However, there are many pro- p groups P with dense free subgroups such that $\mathcal{LC}(P)$ is large. For instance, if P is a pro- p branch group, then by results of Wilson, P acts faithfully on $\mathcal{LC}(P)$, and P contains a dense nonabelian free subgroup. (See [26].)

The global centralizer lattice is a natural frame of reference for considering quasifactors of a locally C -stable t.d.l.c. group that has trivial quasicentre. In contrast to the compact case considered in Lemma 4.7, a closed quasifactor of a noncompact t.d.l.c. group G need not be a direct factor of G . (See [7, Appendix II])

PROPOSITION 5.8. *Let G be a locally C -stable t.d.l.c. group with $\text{QZ}(G) = 1$. Let K be a closed normal subgroup of G , and suppose $KC_G(K)$ is dense in G . Let $\alpha = [K]$. Then there is a dense subgroup of G of the form $\text{QC}_G(\alpha) \times \text{QC}_G(\alpha^\perp)$, where both direct factors are closed and normal in G . In addition, the following are equivalent:*

- (i) $G = K \times C_G(K)$;
- (ii) $G = \text{QC}_G(\alpha) \times \text{QC}_G(\alpha^\perp)$;
- (iii) $\alpha \in \mathcal{LD}(G)$;
- (iv) $\alpha^{\perp 2} \in \mathcal{LD}(G)$.

Proof. We have $QC_G(\alpha) = C_G(K)$ and $QC_G(\alpha^\perp) = C_G^2(K)$ by Proposition 5.4; in particular, both $QC_G(\alpha)$ and $QC_G(\alpha^\perp)$ are closed. Since K is normal, both $QC_G(\alpha)$ and $QC_G(\alpha^\perp)$ are normal in G . Since $Z(G) = 1$, we see that both K and $C_G(K)$ have trivial centre. This ensures that the subgroup generated by $QC_G(\alpha)$ and $QC_G(\alpha^\perp)$ is a direct product.

Let U be an open compact subgroup of G and let L be a direct factor of G . Then $L = C_G^2(L)$, so $[L] = [L]^{\perp^2}$. Moreover, the topology of G is generated as a product topology by the topologies of L and $C_G(L)$. In particular

$$(L \cap U) \times C_U(L)$$

is open in G , hence has finite index in U . Thus $L \cap U$ is an almost direct factor of U , so $\alpha = [L \cap U] \in \mathcal{LD}(G)$. Thus (i) implies (iii) and (ii) implies (iv).

On the other hand, let L be a closed normal subgroup of G such that $LC_G(L)$ is dense and $[L] \in \mathcal{LD}(G)$.

$$(L \cap U) \times C_U(L \cap U)$$

is open in U , since it is a closed subgroup of finite index, and moreover $C_U(L \cap U) = C_U(L)$ by Theorem 3.19. Hence $LC_G(L)$ contains an open subgroup of G . Since $LC_G(L)$ is also dense, we conclude that $G = L \times C_G(L)$. So (iii) implies (i) and (iv) implies (ii).

It is clear that (i) implies (ii), since $K \leq QC_G(\alpha^\perp)$ and $C_G(K) = QC_G(\alpha)$. Conversely if (ii) holds, then $K \times C_G(K)$ is a closed subgroup of $G = QC_G(\alpha) \times QC_G(\alpha^\perp)$, since K is closed in $QC_G(\alpha^\perp)$ and $C_G(K) = QC_G(\alpha)$. Since $K \times C_G(K)$ is also dense, it follows that (i) holds. This completes the proof that (i), (ii), (iii) and (iv) are equivalent. \square

COROLLARY 5.9. *Let G be a locally C -stable t.d.l.c. group such that $\mathcal{LC}(G) = \mathcal{LD}(G)$ and such that $QZ(G) = 1$. Then every closed quasifactor of G is a direct factor of G .*

5.2. Weakly decomposable actions. Let \mathcal{A} be a Boolean algebra. Then by the Stone representation theorem, \mathcal{A} defines a **profinite space** (that is, a compact totally disconnected space), the **Stone space** $\mathfrak{S}(\mathcal{A})$ of \mathcal{A} , whose points are the ultrafilters of \mathcal{A} , with clopen subsets corresponding to elements of \mathcal{A} . Conversely, given a profinite space \mathfrak{X} , the set $\mathcal{A}(\mathfrak{X})$ of clopen subsets of \mathfrak{X} forms a Boolean algebra. This correspondence produces a natural isomorphism between $\text{Aut}(\mathcal{A})$ and $\text{Aut}(\mathfrak{S}(\mathcal{A}))$. In practice we often find it convenient to abuse notation and treat elements of \mathcal{A} as subsets of $\mathfrak{S}(\mathcal{A})$, identifying $\alpha \in \mathcal{A}$ with

$$\mathfrak{S}(\alpha) := \{p \in \mathfrak{S}(\mathcal{A}) \mid \alpha \in p\} \in \mathcal{A}(\mathfrak{S}(\mathcal{A})).$$

Given $\alpha \in \mathcal{A}$ and $\mathfrak{p} \in \mathfrak{S}(\mathcal{A})$, the expressions ' $\alpha \in \mathfrak{p}$ ' and ' $\mathfrak{p} \in \alpha$ ' can therefore be taken to be synonymous. A **partition** \mathcal{P} of $\alpha \in \mathcal{A}$ is a finite subset of \mathcal{A} such that the join of \mathcal{P} is α and the meet of any two distinct elements of \mathcal{P} is 0; a partition of \mathcal{A} is just a partition of ∞ in \mathcal{A} .

DEFINITION 5.10. Let G be a topological group acting on a set Ω . Say that the action of G on Ω is **smooth** if every point stabilizer is open; this implies that for every open compact subgroup U of G , the orbits of U on Ω are all finite. Note that if the action is smooth, then G is orbit-equivalent to any of its dense subgroups.

The following fact shows the relevance of smooth actions for t.d.l.c. groups:

LEMMA 5.11. *Let G be a t.d.l.c. group, let \mathfrak{X} be a profinite space and \mathcal{A} a Boolean algebra.*

- (i) *If G acts on \mathfrak{X} by homeomorphisms, and if the G -action is continuous with respect to the topology of uniform convergence, then the corresponding G -action on the Boolean algebra of clopen subsets of \mathfrak{X} is smooth.*
- (ii) *If G acts on \mathcal{A} by automorphisms, and if the G -action is smooth, then the corresponding G -action on the Stone space $\mathfrak{S}(\mathcal{A})$ is continuous with respect to the topology of uniform convergence. In particular, the action map $(g, x) \mapsto gx$ is a continuous map from $G \times \mathfrak{S}(\mathcal{A})$ to $\mathfrak{S}(\mathcal{A})$.*

Proof. Let $U \in \mathcal{B}(G)$ and let $\mathfrak{X} = \mathfrak{X}_1 \cup \dots \cup \mathfrak{X}_n$ be a clopen partition of $\mathfrak{S}(\mathcal{A})$.

If G acts on \mathfrak{X} by homeomorphisms, and if the G -action is continuous with respect to the topology of uniform convergence, then for any net (g_α) converging to the identity in G , for all i and each $x \in \mathfrak{X}_i$, we have $g_\alpha(x) \in \mathfrak{X}_i$ for all sufficiently large α . Therefore, g_α eventually belongs to the common stabilizer $H = \bigcap_{i=1}^n G_{\mathfrak{X}_i}$, which is thus an open subgroup of G . It follows that $H \cap U$ is open in G , and so it is therefore $U_{\mathfrak{X}_i} = U \cap G_{\mathfrak{X}_i}$. Thus (i) holds.

Assume now that G acts smoothly on \mathcal{A} by automorphisms and set $\mathfrak{X} = \mathfrak{S}(\mathcal{A})$. Since the action is smooth, the common stabilizer $V = \bigcap_{i=1}^n U_{\mathfrak{X}_i}$ is open in U . Thus any net (g_α) converging to the identity in G eventually belongs to V and thus stabilizes each \mathfrak{X}_i . Thus (g_α) uniformly converges to the identity in the space of homeomorphisms of \mathfrak{X} . In particular the action of G on $\mathfrak{S}(\mathcal{A})$ is jointly continuous (see [16, Ch. 7]). This proves (ii). \square

As before, let G be a group acting on a set Ω . Given $X \subseteq \Omega$, the **rigid stabilizer** of X is the subgroup

$$\text{rist}_G(X) := \{g \in G \mid gx = x \text{ for all } x \in \Omega \setminus X\}.$$

Similarly, if G acts on a Boolean algebra \mathcal{A} , then given $\alpha \in \mathcal{A}$ we define

$$\text{rist}_G(\alpha) := \{g \in G \mid g\beta = \beta \text{ for all } \beta \leq \alpha^\perp\}.$$

(Equivalently, one could define $\text{rist}_G(\alpha)$ to be the rigid stabilizer of the subset of $\mathfrak{S}(\mathcal{A})$ corresponding to α .)

Notice that if \mathcal{A} is a subalgebra of $\mathcal{LC}(G)$ or of $\mathcal{LD}(G)$, then $\text{QC}_G(\alpha) \leq \text{rist}_G(\alpha^\perp)$ for all $\alpha \in \mathcal{A}$.

LEMMA 5.12. *Let G be a t.d.l.c. group acting smoothly on a Boolean algebra \mathcal{A} and let $v \subseteq \mathfrak{S}(\mathcal{A})$. Then $\text{rist}_G(v)$ is closed.*

Proof. It suffices to show that G_p is closed for all $p \in \mathfrak{S}(\mathcal{A})$. This is immediate from Lemma 5.11. \square

Related to the notion of rigid stabilizers is that of **(weakly, locally) decomposable** actions. The centralizer and local decomposition lattices account for all faithful actions of this kind of a t.d.l.c. group (if any such actions exist).

DEFINITION 5.13. Let (G, U) be a Hecke pair such that G acts on a Boolean algebra \mathcal{A} , with kernel K . Say the action is **weakly decomposable** if, for every $\alpha \in \mathcal{A} \setminus \{0\}$, the U -orbit of α is finite and $\text{rist}_G(\alpha)/K$ is nontrivial. Note that the fact that U has finite orbits ensures that UK/K is residually finite. Moreover, taking $\alpha = \infty$, we see that G/K must be nontrivial. Say the action is **locally decomposable** if it is weakly decomposable and if, for all $\alpha \in \mathcal{A}$, the product $\text{rist}_U(\alpha)\text{rist}_U(\alpha^\perp)$ has finite index in U . Given an action of G by homeomorphisms on a profinite space \mathfrak{X} , we say the action is weakly or locally decomposable if the action of G on the algebra of clopen subsets of \mathfrak{X} is weakly or locally decomposable respectively.

Say a Hecke pair (G, U) is **faithful weakly decomposable** if $\text{QZ}(G) = 1$ and G admits a faithful weakly decomposable action on a Boolean algebra. (We require that $\text{QZ}(G) = 1$ to avoid degenerate situations, such as Hecke pairs of the form $(G, 1)$.)

A property closely related to the weakly decomposable property has been considered by Möller and Vonk [19], referred to in [19] as property H. In particular, the authors give a criterion for obtaining nondiscrete topologically simple t.d.l.c. groups, of a kind that we study further in [8]. As can easily be seen from the definitions, Property H and weakly decomposable are related as follows:

PROPOSITION 5.14. *Let G be a group of automorphisms of a thick locally finite tree T , let v be a vertex of T and let $U = G_v$. Then (G, U) is a Hecke pair, and*

the action of G on T has property H in the sense of [19] if and only if the action of G on the space of ends of T is weakly decomposable.

Here are some basic properties of weakly decomposable actions. We state them for actions on profinite spaces; of course these can be translated via the Stone correspondence into corresponding statements for actions on Boolean algebras.

LEMMA 5.15. *Let (G, U) be a Hecke pair such that G acts by homeomorphisms on a profinite space \mathfrak{X} . Assume that the action is weakly decomposable with kernel K . Then:*

- (i) \mathfrak{X} has no isolated point.
- (ii) If $\text{QZ}(G/K) = 1$, then for every nonempty clopen set $\alpha \subseteq \mathfrak{X}$, we have $[\text{rist}_G(\alpha)] > [K]$.
- (iii) Suppose $\text{QZ}(G/K) = 1$. Then for any two clopen subsets α, β , we have $\alpha \neq \beta$ if and only if $[\text{rist}_G(\alpha)] \neq [\text{rist}_G(\beta)]$.

Proof. There is no loss of generality in replacing (G, U) by the Hecke pair $(G/K, UK/K)$, so we may assume that the action is faithful. Since the action is weakly decomposable, it is clear that $|\mathfrak{X}| > 1$.

(i) If a point $x \in \mathfrak{X}$ is isolated, its complement is a nonempty clopen set whose rigid stabilizer is trivial, contradicting the assumption that the action is weakly decomposable.

(ii) Let $\alpha \subseteq \mathfrak{X}$ be a nonempty clopen subset. Then G_α contains a finite index subgroup V of U , and contains $\text{rist}_G(\alpha)$ as a normal subgroup; by assumption $\text{rist}_G(\alpha)$ is nontrivial. Since $\text{QZ}(G) = 1$, it follows from Lemma 3.2 that $\text{rist}_U(\alpha)$ is infinite, so $[\text{rist}_U(\alpha)] > 0$.

(iii) Assume that $\alpha \neq \beta$. Upon swapping α and β , we may assume that $\alpha \not\leq \beta$. Then $\alpha \wedge \beta^\perp \neq 0$ so that $\text{rist}_U(\alpha \wedge \beta^\perp)$ is an infinite bounded locally normal subgroup contained in $\text{rist}_U(\alpha)$. On the other hand, $\text{rist}_U(\alpha \wedge \beta^\perp) \cap \text{rist}_U(\beta) = 1$. Hence $[\text{rist}_G(\alpha)] \neq [\text{rist}_G(\beta)]$. \square

5.3. Faithful weakly decomposable groups. We now study the rigid stabilizers of the G -action on the centralizer and local decomposition lattices. Of particular interest is the case of faithful actions of G .

In general, if a group G acts on a Boolean algebra \mathcal{A} with an invariant subalgebra \mathcal{A}' , then $\text{rist}_G(\alpha)$ could be larger when defined in terms of the action on \mathcal{A}' than it is when defined in terms of the action on \mathcal{A} . However, this ambiguity does not arise in the case of faithful actions on subalgebras of the centralizer

lattice: the rigid stabilizers turn out to be exactly the corresponding elements of the global centralizer lattice discussed earlier.

PROPOSITION 5.16. *Let (G, U) be a locally C -stable Hecke pair and let \mathcal{A} be a G -invariant subalgebra of $\mathcal{LC}(G, U)$ on which G acts faithfully. Then $\text{QZ}(G) = 1$. Moreover, the following hold.*

- (i) *For each $\alpha \in \mathcal{A}$, we have $\text{rist}_G(\alpha) = \text{QC}_G^2(\alpha) = C_G(\text{QC}_G(\alpha)) = \text{QC}_G(\alpha^\perp)$.*
- (ii) *If G is a topological group and U is open in G , then the action of G on \mathcal{A} is smooth.*
- (iii) *For each $\alpha \in \mathcal{A}$, we have $\alpha = [\text{rist}_G(\alpha)]$.*
- (iv) *The action of G on \mathcal{A} is weakly decomposable, and if \mathcal{A} is a subalgebra of $\mathcal{LD}(G, U)$ the action is locally decomposable.*

Proof. Observe that for any Hecke pair (G, U) , the quasicentre of G fixes every element of $\mathcal{LN}(G, U)$ by conjugation. Thus to have a faithful action on a subalgebra of $\mathcal{LC}(G, U)$, we must have $\text{QZ}(G) = 1$. Note also that U has finite orbits on \mathcal{A} .

(i) We see that $\text{QC}_G(\alpha^\perp) \leq \text{rist}_G(\alpha)$ and $\text{QC}_G(\alpha) \leq \text{rist}_G(\alpha^\perp)$. In particular, $\text{rist}_G(\alpha)$ centralizes $\text{QC}_G(\alpha)$, so $\text{rist}_G(\alpha) \leq C_G(\text{QC}_G(\alpha))$. By Proposition 5.4, in fact

$$C_G(\text{QC}_G(\alpha)) = \text{QC}_G^2(\alpha) = \text{QC}_G(\alpha^\perp).$$

Hence $\text{rist}_G(\alpha) = C_G(\text{QC}_G(\alpha)) = \text{QC}_G^2(\alpha)$. This proves (i).

(ii) For each $\alpha \in \mathcal{A}$, then $\text{rist}_G(\alpha)$ is closed by (i), so $G_\alpha = N_G(\text{rist}_G(\alpha))$ is closed. Since moreover $|U : U_\alpha|$ is finite, we conclude that U_α is open in U , and hence G_α is open in G .

(iii) Follows from (i) and the fact that $\alpha = \alpha^{\perp\perp}$, see Theorem 5.2(i).

(iv) It is clear from (iii) that the action is weakly decomposable. If \mathcal{A} is a subalgebra of $\mathcal{LD}(G, U)$, note that for all $\alpha \in \mathcal{A}$ there are representatives K of α and L of α^\perp such that KL has finite index in U . Consequently $\text{rist}_G(\alpha)\text{rist}_G(\alpha^\perp)$ contains a finite index subgroup of U . □

COROLLARY 5.17. *Let (G, U) be a locally C -stable Hecke pair and let \mathcal{A} be a G -invariant subalgebra of $\mathcal{LC}(G, U)$ on which G acts faithfully. Let K be a locally normal subgroup of G , and let $\alpha = [K]^{\perp 2}$. Suppose $\alpha \in \mathcal{A}$. Then the set of points in $\mathfrak{S}(\mathcal{A})$ moved by K forms an open dense subset of α .*

Proof. Let v be the set of points in $\mathfrak{S}(\mathcal{A})$ moved by K . We have $K \leq \text{rist}_G(\alpha)$ by Propositions 5.4 and 5.16(i), which ensures that $v \subseteq \alpha$. Suppose v is not dense in

α . Then ν is contained in a proper clopen subset β of α . But then $K \leq \text{rist}_G(\beta)$, so $[K] \leq [\text{rist}_G(\beta)] = \beta < \alpha$, a contradiction. \square

A Hecke pair (G, U) with trivial quasicentre that admits a faithful weakly decomposable action is locally C-stable. Moreover, the faithful locally or weakly decomposable actions are controlled by $\mathcal{LD}(G, U)$ and $\mathcal{LC}(G, U)$ respectively.

THEOREM 5.18. *Let (G, U) be a Hecke pair.*

- (i) *Suppose that G has a faithful weakly decomposable action on some Boolean algebra \mathcal{A} . Then the following are equivalent:*
- (a) $\text{QZ}(G) = 1$;
 - (b) *The action of U is weakly decomposable, and every nontrivial normal subgroup of G has nontrivial intersection with U ;*
 - (c) *G is locally C-stable, and \mathcal{A} is G -equivariantly isomorphic to a subalgebra of $\mathcal{LC}(G, U)$ (indeed, the set of rigid stabilizers of \mathcal{A} form a subalgebra of the global centralizer lattice).*
- (ii) *Every G -equivalence class of faithful locally decomposable actions of G (if there are any) occurs as the action of G on a subalgebra of $\mathcal{LD}(G, U)$.*

We begin the proof of Theorem 5.18 with two lemmas.

LEMMA 5.19. *Let A, N be subgroups of a group G . Assume that N is normal, and contains an element $x \in N$ such that $[A, xAx^{-1}] = 1$. Then N contains $[A, A]$.*

Proof. Let $a, b, c \in A$. Since a and b commute with xcx^{-1} , we have $[a, b] = [a, bxcx^{-1}]$. Setting $c = b^{-1}$, we infer that $[a, b] = [a, [b, x]]$, which belongs to N since N is normal. \square

LEMMA 5.20. *Let U be a residually finite group acting faithfully on an atomless Boolean algebra \mathcal{A} . Let K be a nontrivial normal subgroup of U and let $\alpha \in \mathcal{A} \setminus \{0\}$ be such that $g\alpha \wedge \alpha = 0$ for some $g \in K$.*

- (i) *If K is abelian, then $\text{rist}_U(\alpha)$ is abelian.*
- (ii) *If K is finite, then $\text{rist}_U(\alpha)$ is finite-by-abelian (and hence centre-by-finite).*

In either case, there exists $\beta \in \mathcal{A}$ such that $0 < \beta \leq \alpha$ and $\text{rist}_U(\beta) = 1$, so the action of U on \mathcal{A} is not weakly decomposable.

Proof. We see that $K \cap \text{rist}_U(\alpha)$ and $K \cap \text{rist}_U(g\alpha)$ are conjugate subgroups of K , and also that $\text{rist}_U(\alpha) \cap \text{rist}_U(g\alpha) = 1$ since U acts faithfully on \mathcal{A} . Moreover, by Lemma 5.19, $K \cap \text{rist}_U(\alpha)$ contains $R = [\text{rist}_U(\alpha), \text{rist}_U(\alpha)]$.

Suppose K is abelian. Then the conjugation action of K on its subgroups is trivial, so $K \cap \text{rist}_U(\alpha) = K \cap \text{rist}_U(g\alpha)$; since $\text{rist}_U(\alpha) \cap \text{rist}_U(g\alpha) = 1$, it follows that $K \cap \text{rist}_U(\alpha) = \text{rist}_K(\alpha) = 1$. Thus $R = [\text{rist}_U(\alpha), \text{rist}_U(\alpha)]$ is trivial and $\text{rist}_U(\alpha)$ is abelian.

Suppose instead that K is finite. In particular, $R = [\text{rist}_U(\alpha), \text{rist}_U(\alpha)]$ is finite, so $\text{rist}_U(\alpha)$ has a finite index subgroup Z such that $Z \cap R = 1$. We see that $[\text{rist}_U(\alpha), Z] \leq Z \cap R = 1$, so Z is central in $\text{rist}_U(\alpha)$ and hence $\text{rist}_U(\alpha)$ is centre-by-finite.

Since \mathcal{A} is atomless, there is a strictly descending sequence $(\alpha_i)_{i \in \mathbb{N}}$ of nonzero elements of \mathcal{A} such that $\alpha_0 = \alpha$. If $\text{rist}_U(\alpha)$ is finite, then there exist $i, j \in \mathbb{N}$ with $i < j$ such that $\text{rist}_U(\alpha_i) = \text{rist}_U(\alpha_j)$, and thus $\text{rist}_U(\beta)$ is trivial, where $\beta = \alpha_i \wedge \alpha_j^\perp$. Thus we may suppose that $\text{rist}_U(\alpha)$ is infinite. Let Z be a central subgroup of $\text{rist}_U(\alpha)$ of finite index. Since U acts faithfully on \mathcal{A} , there must be some $\beta \in \mathcal{A} \setminus \{0\}$ and $z \in Z$ such that $z\beta \wedge \beta = 0$; it follows that $\beta \leq \alpha$, since $\text{rist}_U(\alpha)$ fixes α^\perp pointwise as a subspace of $\mathfrak{S}(\mathcal{A})$. Now $\text{rist}_U(\beta)$ and $\text{rist}_U(z\beta)$ are subgroups of $\text{rist}_U(\alpha)$ that are Z -conjugate, so $\text{rist}_U(\beta) = \text{rist}_U(z\beta)$, and hence $\text{rist}_U(\beta) = 1$ since $z\beta \wedge \beta = 0$. □

Proof of Theorem 5.18. (i) Let \mathcal{A} be a Boolean algebra equipped with a faithful weakly decomposable action of G . Then \mathcal{A} is atomless by Lemma 5.15. We also see that U is residually finite, since U has finite orbits on \mathcal{A} .

Suppose $\text{QZ}(G) = 1$. Then given a nontrivial subgroup K of G such that $|U : N_U(K)|$ is finite, then $K \cap U$ is infinite. In particular, $\text{rist}_U(\alpha)$ is infinite for every $\alpha \in \mathcal{A} \setminus \{0\}$, so the action of V on \mathcal{A} is weakly decomposable, for every subgroup V of G commensurate with U . Hence (a) implies (b). Moreover, by Lemma 5.20, V has no nontrivial finite or abelian normal subgroups. Hence G is locally C -stable, by Theorem 3.19.

Let $\alpha \in \mathcal{A}$ and let $\beta = \alpha^\perp$. Then $\text{rist}_G(\alpha)$ and $\text{rist}_G(\beta)$ commute. Let $x \in C_G(\text{rist}_G(\beta))$. We claim that $x \in \text{rist}_G(\alpha)$, in other words, the set v of points moved by x is a subset of α . If not, then $v \cap \beta$ contains some $\gamma \in \mathcal{A}$, since v is an open subset of $\mathfrak{S}(\mathcal{A})$. Indeed there is some $\delta \in \mathcal{A}$ such that $0 < \delta \leq \gamma$ and $x\delta \neq \delta$. The fact that the action is weakly decomposable ensures that $\text{rist}_G(\delta)$ and $\text{rist}_G(x\delta)$ are not locally equivalent by Lemma 5.15(iii), so x does not commensurate $\text{rist}_G(\delta)$, and in particular $x \notin \text{QC}_G(\text{rist}_G(\delta))$. Since $\text{rist}_G(\delta) \leq \text{rist}_G(\beta)$, we have a contradiction. This shows that

$$\text{rist}_G(\alpha) = C_G(\text{rist}_G(\alpha^\perp)). \tag{5.2}$$

By Proposition 5.4, $\text{rist}_G(\alpha)$ is an element of the global centralizer lattice of G .

We therefore have a map

$$\theta : \mathcal{A} \rightarrow \mathcal{LC}(G, U); \theta(\alpha) = [\text{rist}_G(\alpha)].$$

By Lemma 5.15, θ is injective, and (5.2) shows that θ is compatible with complementation; θ is also compatible with meets. Hence θ is a homomorphism of Boolean algebras.

From the above argument, we conclude that (a) implies (b) and (c).

In the other direction, it is clear that (c) implies (a). Suppose instead that (b) holds. By Lemma 5.20, U has no nontrivial virtually abelian normal subgroups. This implies that the quasicentre of U is trivial, in other words, $\text{QZ}(G) \cap U = 1$. Since every nontrivial normal subgroup of G intersects U nontrivially, we conclude that $\text{QZ}(G) = 1$. Hence (b) implies (a). This completes the proof of (i).

(ii) If the action of G on \mathcal{A} is locally decomposable, then $\text{rist}_U(\alpha) \times \text{rist}_U(\beta)$ has finite index in U , so $\text{rist}_U(\alpha)$ is an almost direct factor of U for all $\alpha \in \mathcal{A}$. So in this case $\theta(\mathcal{A})$ is contained in $\mathcal{LD}(G, U)$. \square

COROLLARY 5.21. *Let G be a t.d.l.c. group with $\text{QZ}(G) = 1$ and let U be an open compact subgroup of G . Then every faithful weakly decomposable action of G (if there are any) on a profinite space is continuous with respect to the topology of uniform convergence.*

Proof. By Theorem 5.18, G is locally C-stable and by Proposition 5.16, the action of G on $\mathcal{LC}(G)$ is smooth. The conclusion now follows from Theorem 5.18 and Lemma 5.11. \square

Proof of Theorem II. (i) This follows from Proposition 5.16.

(ii) By Theorem 5.18 (i), there is a G -equivariant embedding ρ of the algebra \mathcal{A} of clopen subsets of \mathfrak{X} into $\mathcal{LC}(G)$. This embedding ρ then corresponds, via the contravariant functor provided by the Stone representation theorem, to a G -equivariant quotient map ρ^* from $\mathfrak{S}(\mathcal{LC}(G))$ to \mathfrak{X} . \square

Another consequence of Theorem 5.18 is that if (G, U) is locally C-stable and has a faithful action on its centralizer lattice, then both properties are inherited by subgroups with large normalizers.

PROPOSITION 5.22. *Let (G, U) be a faithful weakly decomposable Hecke pair. Let H be a locally normal subgroup of G . Then the Hecke pair $(H, H \cap U)$ is locally C-stable and H has a faithful weakly decomposable action on a principal ideal of $\mathcal{LC}(G, U)$; this ideal is H -equivariantly isomorphic to a subalgebra of $\mathcal{LC}(H, H \cap U)$.*

Proof. The existence of a faithful weakly decomposable action ensures that U is residually finite. By Proposition 5.16 and Theorem 5.18, (G, U) is locally C-stable and has a faithful weakly decomposable action on $\mathcal{A} = \mathcal{LC}(G, U)$. Let $\alpha = [H]^{\perp 2}$ and let \mathcal{I} be the ideal of \mathcal{A} generated by α . By Corollary 5.17, the set of points in $\mathfrak{S}(\mathcal{A})$ moved by H forms an open dense subset of α . Since G acts faithfully on \mathcal{A} , it follows that H acts faithfully on \mathcal{I} . Since \mathcal{I} is a principal ideal, it is a Boolean algebra in its own right. We have $\text{QZ}(H) := \text{QZ}(H, H \cap U) = 1$ by Theorem 3.19. Given $0 < \beta \in \mathcal{I}$, then $\text{rist}_G(\beta) = \text{QC}_G(\beta^\perp)$. Since $\beta^\perp > \alpha^{\perp 3} = \alpha^\perp$, we see that $[\text{rist}_G(\beta)] > [\text{QC}_G(H)]$. If $\text{rist}_H(\beta) = 1$, then $0 = [\text{rist}_G(\beta) \cap H] = [\text{rist}_G(\beta)] \wedge [H] = \beta \wedge [H]$. Hence $[H] \leq \beta^\perp$ so that $\alpha \leq \beta^{\perp 3} = \beta^\perp$. Since $\beta \leq \alpha$, we obtain $\beta = 0$, a contradiction. Thus the action of H on \mathcal{I} is weakly decomposable. Hence by Theorem 5.18, H is locally C-stable and \mathcal{I} is H -equivariantly isomorphic to a subalgebra of $\mathcal{LC}(H, H \cap U)$. \square

COROLLARY 5.23. *Let (G, U) be a faithful weakly decomposable Hecke pair and let H be a subgroup of G . Suppose there is a sequence of subgroups*

$$H = H_1 \leq H_2 \leq \dots \leq H_n = G$$

such that

$$|H_{i+1} \cap U : N_{H_{i+1}}(H_i) \cap U|$$

is finite for $1 \leq i < n$. Then either $H = 1$ or $(H, H \cap U)$ is faithful weakly decomposable. In particular, $\text{QZ}(H) = 1$.

Proof. Follows from Proposition 5.22 by induction. \square

5.4. Topologically free actions. For groups acting on topological spaces, there is a natural notion of actions that are in some sense the opposite of weakly decomposable:

DEFINITION 5.24. Let G be a group acting faithfully on a topological space X . The action is **locally faithful** at $x \in X$ if for every neighbourhood O of x , the pointwise stabilizer of O in G is trivial. The action is **topologically free** if it is locally faithful at every point in X .

Let (G, U) be a Hecke pair and let G act by homeomorphisms on a profinite space \mathfrak{X} , such that U has finite orbits on the set of clopen subsets of \mathfrak{X} . Clearly, if the action of G on \mathfrak{X} is weakly decomposable, then it is not locally faithful at any point. In general, the action of (G, U) on \mathfrak{X} may be neither weakly decomposable nor locally faithful at any point, even if the action is minimal. For example,

given a (faithful, minimal) weakly decomposable action of (G, U) on \mathfrak{X} , then $(G \times \mathbf{Z}/2\mathbf{Z}, U)$ has a (faithful, minimal) action on the disjoint union of two copies of \mathfrak{X} that is not weakly decomposable, but has no locally faithful point: this is achieved by letting G act identically on each copy of \mathfrak{X} and $\mathbf{Z}/2\mathbf{Z}$ swap the two copies of \mathfrak{X} . If G admits a noninner automorphism ϕ of order 2, then one could construct a similar example of an action of (H, U) , where this time $H = G \rtimes \langle \phi \rangle$ has trivial quasicentre with respect to U . As a concrete example, let T be a regular rooted tree, let G be the derived subgroup of $\text{Aut}(T)$, and let \mathfrak{X} be the set of ends of T . Then (G, G) is a Hecke pair; the G -action on \mathfrak{X} is faithful, minimal and weakly decomposable, and the G -orbit of any clopen subset is finite. The outer automorphism group $\text{Out}(G)$ contains (and is in fact equal to) $\text{Aut}(T)/G$, which is an infinite direct product of cyclic groups of order 2. In particular $\text{Out}(G)$ contains many involutions, which can be lifted to involutions in $\text{Aut}(G)$.

However, with a stronger restriction on the group structure of U , we can ensure the existence of locally faithful points. This property may be regarded as the antithesis of the faithful weakly decomposable property. The following definition and proposition are inspired by [20], which uses similar methods to comprehend the actions of free groups on rooted trees.

DEFINITION 5.25. Let U be group. Say U has the **virtually normal intersection property** (VNIP) if for any two nontrivial subgroups K, L of U such that $N_U(K)$ and $N_U(L)$ have finite index in U , the intersection $K \cap L$ is infinite.

If (G, U) is locally C -stable, one sees that U has VNIP if and only if $\mathcal{LC}(G, U)$ is trivial.

PROPOSITION 5.26. Let (G, U) be a Hecke pair, let \mathfrak{X} be a profinite space on which G acts faithfully by homeomorphisms, let \mathcal{A} be the Boolean algebra of clopen subsets of \mathfrak{X} , and suppose that U has finite orbits on \mathcal{A} . Suppose that $\text{QZ}(G) = 1$ and that U has VNIP. Then the set of points in \mathfrak{X} at which G acts locally faithfully is nonempty and closed. In particular, if the action of G on \mathfrak{X} is minimal, then it is topologically free.

Proof. Let \mathfrak{Y} be the set of points at which G acts locally faithfully. We see from the definition that $\mathfrak{X} \setminus \mathfrak{Y}$ is open and hence \mathfrak{Y} is closed, since if G does not act locally faithfully at some $\mathfrak{p} \in \mathfrak{X}$, then there is an open neighbourhood O of \mathfrak{p} such that the pointwise stabilizer of O is nontrivial, and hence G does not act locally faithfully at any point in O .

Suppose $\mathfrak{N} = \emptyset$. Then for each $p \in \mathfrak{X}$, we can find $\alpha_p \in \mathcal{A}$ such that $p \in \alpha_p$ and $\text{rist}_G(\alpha_p^\perp)$ is nontrivial. Since U has finite orbits on \mathcal{A} , the normalizer $V = N_U(\text{rist}_G(\alpha_p^\perp))$ has finite index in U ; the fact that $\text{QZ}(G) = 1$ then forces $\text{rist}_V(\alpha_p^\perp)$ to be infinite. Since every point in \mathfrak{X} is covered by some α_p , by compactness we have $\bigvee_{i=1}^n \alpha_{p_i} = \infty$ for some $p_1, \dots, p_n \in \mathfrak{X}$. Since U has VNIP, the intersection $K = \bigcap_{i=1}^n \text{rist}_U(\alpha_{p_i}^\perp)$ is nontrivial. But $\bigvee_{i=1}^n \alpha_{p_i} = \infty$, in other words $\bigwedge_{i=1}^n \alpha_{p_i}^\perp = 0$, so K acts trivially on \mathcal{A} , contradicting the hypothesis that G acts faithfully on \mathcal{A} . Hence \mathfrak{N} must be nonempty.

Clearly \mathfrak{N} is G -invariant, so if the action of G on \mathfrak{X} is minimal, then $\mathfrak{N} = \mathfrak{X}$. □

6. Some radicals of locally compact groups

6.1. The quasihypercentre. In previous sections, we found it useful to assume that the groups under consideration have trivial quasicycle. Of course, there is no reason in general for a t.d.l.c. group to have this property; more interesting, however, is the question of which locally compact groups G have a nontrivial Hausdorff quotient G/K such that $\text{QZ}(G/K) = 1$.

To address this issue, we need to introduce the following definitions.

DEFINITION 6.1. Let G be a locally compact group. A subgroup K of G is **quasihypercentral** in G if, whenever N is a closed normal subgroup of G such that $K \not\leq N$, then KN/N has nontrivial intersection with $\text{QZ}(G/N)$.

The **quasihypercentre** of G , denoted by $\text{QZ}^\infty(G)$, is the intersection of all closed normal subgroups N of G such that $\text{QZ}(G/N) = 1$.

By Lemma 3.1, we have $\text{QZ}(G) \leq \text{QZ}^\infty(G)$. If moreover $\text{QZ}(G)$ is discrete, then Lemma 3.1 ensures that $\text{QZ}(G/\text{QZ}(G)) = 1$ so that $\text{QZ}(G) = \text{QZ}^\infty(G)$ in this case.

The main properties of the quasihypercentre are collected in the following.

THEOREM 6.2. *Let G be a locally compact group. Then:*

- (i) $\text{QZ}^\infty(G)$ is a closed characteristic subgroup.
- (ii) The quotient $G/\text{QZ}^\infty(G)$ has trivial quasicycle. In particular $\text{QZ}^\infty(G)$ is the unique minimal closed normal subgroup of G with this property.
- (iii) $\text{QZ}^\infty(G)$ is the unique largest quasihypercentral subgroup of G .

Proof of Theorem 6.2. (i) is clear by the definition of the quasihypercentre.

(ii) Set $H = \text{QZ}^\infty(G)$ and let \mathcal{N} be the set of all closed normal subgroups N of G such that $\text{QZ}(G/N) = 1$. Let $x \in G$, and suppose $xH \in \text{QZ}(G/H)$. For any $N \in \mathcal{N}$, the group G/N is a quotient of G/H , so that $xN \in \text{QZ}(G/N)$ by Lemma 3.1(iii). Since $\text{QZ}(G/N) = 1$ by the definition of \mathcal{N} , we infer that $x \in N$. Thus $x \in \bigcap_{N \in \mathcal{N}} N = H$. In other words G/H has trivial quasiceentre, as desired.

(iii) Since $\text{QZ}(G/H) = 1$ by part (ii), it follows that every quasihypercentral subgroup of G is contained in H . It remains to show that H itself is quasihypercentral. Let M be a closed normal subgroup of G not containing H . Then $N = H \cap M$ does not contain H . Therefore, by the definition of H , the quasiceentre of G/N is nontrivial and, hence, there is some element x of G such that $x \notin N$ and $xN \in \text{QZ}(G/N)$. Since $\text{QZ}(G/H) = 1$ by part (ii), in fact we have $x \in H \setminus N = H \setminus M$. Since $N \leq M$, the group G/M is a quotient of G/N , and we deduce from Lemma 3.1(iii) that x is an element of H whose image in G/M is nontrivial and contained in $\text{QZ}(G/M)$. We conclude that H is quasihypercentral in G , as desired. \square

Recall that the quasiceentre $\text{QZ}(G)$ is characteristic but need not be closed. Theorem 6.2 ensures that if $\text{QZ}(G)$ is nontrivial, then $\text{QZ}^\infty(G)$ is a nontrivial closed characteristic subgroup (that could be the whole of G).

6.2. Relation with the quasihypercentre of subgroups. We now specialize to t.d.l.c. groups. The goal is to establish the following stability properties of the quasihypercentre by passage to subgroups.

PROPOSITION 6.3. *Let G be a t.d.l.c. group and H be a subgroup of G .*

- (i) *If H is closed and normal in G , then $\text{QZ}^\infty(H) \supseteq \text{QZ}^\infty(G) \cap H$.*
- (ii) *If H is open in G , then $\text{QZ}^\infty(H) = \text{QZ}^\infty(G) \cap H$.*

The proof of that proposition requires some preparations and will be given at the end of the subsection. We shall need additional definitions.

Recall that given two normal subgroups $K \supseteq L$ of an abstract group G , the **centralizer of K modulo L** is defined by

$$C_G(K/L) = \{g \in G \mid [g, K] \subset L\},$$

where we define $[u, g] = ugu^{-1}g^{-1}$. The centralizer $C_G(K/L)$ is a normal subgroup of G , since it coincides with the preimage in G of the centralizer $C_{G/L}(K/L)$. We shall need to quasify this notion as follows. (Recall that $\mathcal{B}(G)$ denotes the set of open compact subgroups of G .)

DEFINITION 6.4. Let G be a t.d.l.c. group and let K and L be closed subgroups. The **quasicentralizer of K modulo L** is defined by

$$\mathrm{QC}_G(K/L) = \{g \in G \mid \text{there is } U \in \mathcal{B}(G) \text{ such that } [g, U \cap K] \subset L.\}$$

The following shows that $\mathrm{QC}_G(K/L)$ is a normal subgroup of G provided that L is normal in K and that K and L are both ‘quasinormal’ in G :

LEMMA 6.5. *Let G be a t.d.l.c. group and let K and L be nontrivial subgroups of G such that $L \trianglelefteq K$. Assume that $gKg^{-1} \in [K]$ and $gLg^{-1} \in [L]$ for all $g \in G$. Then:*

- (i) $\mathrm{QC}_G(K/L) = \mathrm{QC}_G(K'/L')$ for all $K' \in [K]$ and $L' \in [L]$.
- (ii) $\mathrm{QC}_G(K/L)$ is a nontrivial normal subgroup of G .

Proof. (i) It is clear from the definition that $\mathrm{QC}_G(K/L)$ is invariant under replacing K with a subgroup of G that is locally equivalent to K .

Let $M = V \cap L$ for some open compact subgroup V of G . Certainly $\mathrm{QC}_G(K/M) \subseteq \mathrm{QC}_G(K/L)$. On the other hand, given $g \in \mathrm{QC}_G(K/L)$ and an open compact subgroup U of G such that $[u, g] \in L$ for all $u \in U \cap K$, there is an open subgroup W of V such that $WgWg^{-1}$ is a subset of V . Thus for all $w \in W \cap K$ we have $[w, g] \in V \cap L = M$, so $\mathrm{QC}_G(K/M) = \mathrm{QC}_G(K/L)$. Hence $\mathrm{QC}_G(K/L)$ is invariant under replacing L with a subgroup of G that is locally equivalent to L . This proves (i).

(ii) We see that $L \leq \mathrm{QC}_G(K/L)$, since $[K, L] \leq L$, so $\mathrm{QC}_G(K/L)$ is nontrivial.

Let $g, h \in \mathrm{QC}_G(K/L)$ and let U and V be open compact subgroups of G such that $[u, g] \in L$ for all $u \in U \cap K$ and $[v, h] \in L$ for all $v \in V \cap K$. Then for $u \in U \cap V$ we have

$$[u, g^{-1}h] = ug^{-1}hu^{-1}h^{-1}g = g^{-1}[g, u][u, h]g \in g^{-1}Lg.$$

Since $g^{-1}Lg \in [L]$ by hypothesis, we deduce from (i) that $g^{-1}h \in \mathrm{QC}_G(K/g^{-1}Lg) = \mathrm{QC}_G(K/L)$, so $\mathrm{QC}_G(K/L)$ is a subgroup of G . By definition, we have

$$g\mathrm{QC}_G(K/L)g^{-1} = \mathrm{QC}_G(gKg^{-1}/gLg^{-1})$$

for all $g \in G$. Therefore, $\mathrm{QC}_G(K/L)$ is normal in G by (i). \square

We note the following facts separately as they will be used again later.

LEMMA 6.6. *Let G be a t.d.l.c. group. Suppose that to every open compact subgroup U we have assigned a closed normal subgroup $R(U)$, such that the following statements hold:*

- (a) Given two open compact subgroups U and V of G , then $R(U)$ is commensurate with $R(V)$. If $V = gUg^{-1}$ then $R(V) = gR(U)g^{-1}$.
- (b) Given an open compact subgroup U of G , then the quotient $U/R(U)$ has trivial quasicycle.

Let $U \in \mathcal{B}(G)$ and set $K = \text{QC}_G(U/R(U))$. Then:

- (i) $K = \text{QC}_G(V/R(V))$ for any $V \in \mathcal{B}(G)$.
- (ii) $K \cap V = R(V)$ for every $V \in \mathcal{B}(G)$.
- (iii) K is a closed normal subgroup of G .
- (iv) $\text{QZ}(G/K) = 1$.

Proof. (i) Condition (a) ensures that the set $\{R(U) \mid U \in \mathcal{B}(G)\}$ lies in a single commensurability class of subgroups of G , so that $R(U)$ is a compact locally normal subgroup of G such that $[R(U)]$ does not depend on U . It follows from Lemma 6.5 that K is normal in G and does not depend on the choice of U . Thus (i) holds.

(ii) By the definition of K , we see that $K \cap U$ consists of those elements of U whose image in $U/R(U)$ is quasicentral. By condition (b), we thus have $K \cap U = R(U)$. Hence (ii) follows from (i).

(iii) We have already noted that K is normal in G as a consequence of Lemma 6.5. Since $K \cap U = R(U)$ is closed in U by (ii), it follows that K is closed in G .

(iv) Let $x \in G$ such that $xK \in \text{QZ}(G/K)$; then x centralizes VK/K for some open compact subgroup V of G , but this ensures in turn that $x \in \text{QC}_G(V/R(V)) = K$ by (ii), so xK is trivial. Hence $\text{QZ}(G/K) = 1$. \square

Proof of Proposition 6.3. (i) We see that the quasihypercentre of a t.d.l.c. group is a characteristic subgroup, so $\text{QZ}^\infty(H)$ is normal in G . Let $K = C_G(H/\text{QZ}^\infty(H))$ be the centralizer of H modulo $\text{QZ}^\infty(H)$. Since H and $\text{QZ}^\infty(H)$ are closed normal subgroups of G , so is K . Moreover, $K \cap H = \text{QZ}^\infty(H)$, since $H/\text{QZ}^\infty(H)$ has trivial quasicycle. Consequently the quotient G/K of G has a normal subgroup $HK/K \cong H/\text{QZ}^\infty(H)$ whose quasicycle is trivial by Theorem 6.2. Since

$$\text{QZ}(G/K) \cap HK/K \leq \text{QZ}(HK/K) = 1,$$

we infer that $\text{QZ}(G/K)$ is a normal subgroup of G/K with trivial intersection with HK/K . Therefore, HK/K and $\text{QZ}(G/K)$ commute. On the other hand the

definition of K implies that HK/K has trivial centralizer in G/K . It follows that $\text{QZ}(G/K) = 1$. Therefore, $K \geq \text{QZ}^\infty(G)$, and thus $\text{QZ}^\infty(H) = K \cap H \geq \text{QZ}^\infty(G) \cap H$.

(ii) Let $Q = \text{QZ}^\infty(G)$. Note that $H/(Q \cap H) \cong HQ/Q$, and HQ/Q is an open subgroup of G/Q ; thus $\text{QZ}(HQ/Q) \leq \text{QZ}(G/Q) = 1$, so $H/(Q \cap H)$ has trivial quasiceentre. By Theorem 6.2, we conclude that $Q \cap H \geq \text{QZ}^\infty(H)$.

Combining this with part (i), we see that $\text{QZ}^\infty(O) = \text{QZ}^\infty(G) \cap O$ in the case that O is an open normal subgroup of G .

Let now $V \leq U$ be compact open subgroups of G . Let O be the core of V in U , so that O is an open normal subgroup of both V and U . By the preceding paragraph, we have

$$\text{QZ}^\infty(U) \cap O = \text{QZ}^\infty(O) = \text{QZ}^\infty(V) \cap O.$$

In particular $\text{QZ}^\infty(U)$ and $\text{QZ}^\infty(V)$ are commensurate. This implies that for all $U, V \in \mathcal{B}(G)$, the groups $\text{QZ}^\infty(U)$ and $\text{QZ}^\infty(V)$ are commensurate.

Define $R(U) := \text{QZ}^\infty(U)$ for every profinite group U . We have just checked that condition (a) from Lemma 6.6 is satisfied, and so is condition (b) by Theorem 6.2. Hence, given $V \in \mathcal{B}(G)$, the group $K = \text{QC}_G(V/\text{QZ}^\infty(V))$ is a closed normal subgroup of G that does not depend on the choice of V . Moreover, Lemma 6.6 ensures that $\text{QZ}(G/K) = 1$, so $K \geq Q$. Also $K \cap V = \text{QZ}^\infty(V) \leq Q \cap V$, where the latter containment was observed in the first paragraph above. It follows that Q contains an open subgroup of K , so K/Q is a discrete normal subgroup of G/Q ; since $\text{QZ}(G/Q) = 1$, this forces $K = Q$ by Lemma 3.1.

This shows that $\text{QZ}^\infty(G) = \text{QC}_G(V/\text{QZ}^\infty(V))$ for any open compact subgroup V of G . In particular, if H is any open subgroup of G , we may replace V by $V \cap H$ and deduce that

$$\text{QZ}^\infty(G) \cap H = \text{QC}_G(V/\text{QZ}^\infty(V)) \cap H = \text{QC}_H(V/\text{QZ}^\infty(V)) = \text{QZ}^\infty(H).$$

This confirms (ii). □

REMARK 6.7. We have shown that the local equivalence class of the quasihypercentre is unchanged by passing to an open subgroup. In particular for any open subgroup H of G and any $V \in \mathcal{B}(H)$, the group $\text{QZ}^\infty(H)$ is locally equivalent to $\text{QZ}^\infty(V)$, so we have

$$\text{QZ}^\infty(G) = \text{QC}_G(V/\text{QZ}^\infty(V)) = \text{QC}_G(H/\text{QZ}^\infty(H)).$$

REMARK 6.8. In general, the quasihypercentre of an arbitrary closed subgroup of a t.d.l.c. group bears no relation to that of the group itself. For instance, any profinite group H can be embedded as a closed subgroup of the Cartesian product

G of its finite continuous images, in which case $\text{QZ}^\infty(G) = G$ since $\text{QZ}(G)$ is dense in G . On the other hand H can be embedded in a semidirect product $K \rtimes H$ (where K is an infinite Cartesian product of copies of the free profinite group on 2 generators, for instance, and H permutes the terms of this product) of such a form that $\text{QZ}^\infty(K \rtimes H) = \text{QZ}(K \rtimes H) = 1$.

The following question is natural.

QUESTION 6.9. *If H is a closed normal subgroup of G contained in $\text{QZ}^\infty(G)$, does it follow that H is quasihypercentral in G ?*

We can at least say that H is quasihypercentral in itself; in other words, given any proper closed normal subgroup K of H , then $\text{QZ}(H/K)$ is nontrivial.

6.3. Other radicals. We have seen that locally compact groups admit a unique largest quotient with trivial quasicentre. In a similar manner, for some classes of profinite group there is a unique largest quotient that has no nontrivial compact locally normal subgroups in the given class. In particular, a motivation for eliminating virtually abelian locally normal subgroups in this way is the centralizer lattice described in Section 5.1.

DEFINITION 6.10. Let \mathcal{C} be a class of profinite groups. Say \mathcal{C} is **stable** if the following conditions hold:

- (a) \mathcal{C} contains all abelian profinite groups, as well as all finite simple groups;
- (b) Given a group $U \in \mathcal{C}$ and a closed normal subgroup K of U , then $K \in \mathcal{C}$ and $U/K \in \mathcal{C}$;
- (c) Given a profinite group U that is a (possibly nondirect) product of finitely many closed normal \mathcal{C} -subgroups, then $U \in \mathcal{C}$;
- (d) \mathcal{C} is stable under isomorphisms, that is, it contains the whole isomorphism class of each of its elements.

Given a profinite group U , write $\mathcal{C}(U)$ for the closed subgroup generated by all normal \mathcal{C} -subgroups of U .

Given a stable class \mathcal{C} , a t.d.l.c. group G is **\mathcal{C} -semisimple** if $\text{QZ}(G) = 1$ and the only locally normal subgroup of G belonging to \mathcal{C} is the identity subgroup. For a profinite group U , a subgroup K of U is called **\mathcal{C} -regular** in U if for every closed normal subgroup L of U not containing K , the image of K in U/L contains a nontrivial locally normal \mathcal{C} -subgroup of U/L . For a t.d.l.c. group G and closed

subgroup H , say H is \mathcal{C} -**regular** in G if $U \cap H$ is \mathcal{C} -regular in U for all open compact subgroups U of G .

Write $[A]$ for the smallest stable class; note that by Fitting's Theorem, all groups in this class are virtually nilpotent. However, the class of $[A]$ -regular t.d.l.c. groups is larger than just the locally $[A]$ groups; for example, all t.d.l.c. groups with virtually soluble open compact subgroups are $[A]$ -regular. In a recent paper [24, Theorem 1.11], Wesolek has shown that every second-countable $[A]$ -regular t.d.l.c. group can be constructed from profinite groups and countable discrete groups by a (possibly transfinite) sequence of extensions and countable increasing unions. On the other hand, we shall see in Proposition 6.17 below that a t.d.l.c. group is $[A]$ -semisimple if and only if it is locally \mathcal{C} -stable with trivial quasicentre.

Another example of a stable class (within the class of profinite groups) is the class of profinite groups that are topologically generated by pronilpotent normal subgroups and quasisimple subnormal subgroups; for this class \mathcal{C} , then $\mathcal{C}(U)$ is the **generalized pro-Fitting subgroup** of U . The regular and semisimple classes associated to this class are discussed in [22].

Here is the main theorem for this section.

THEOREM 6.11. *Let G be a t.d.l.c. group and let \mathcal{C} be a stable class of topological groups. Then G has a uniquely defined closed normal subgroup $R_{\mathcal{C}}(G)$, the **regular radical** of G , that is characterized by either of the following properties:*

- (i) $R_{\mathcal{C}}(G)$ is the unique largest locally normal subgroup of G that is \mathcal{C} -regular in G ;
- (ii) $G/R_{\mathcal{C}}(G)$ is \mathcal{C} -semisimple, and given any closed normal subgroup N of G such that G/N is \mathcal{C} -semisimple, then $N \geq R_{\mathcal{C}}(G)$.

The proof will be given at the end of the section. The following classical fact is well known.

LEMMA 6.12 (Dietzmann's Lemma). *Let G be a group and Σ be a finite subset consisting of torsion elements. If Σ is invariant under conjugation, then $\langle \Sigma \rangle$ is finite.*

Proof. See [11] or [18, p. 154]. □

For the remainder of this section, \mathcal{C} is a stable class of profinite groups. We need to check that the local and global definitions of \mathcal{C} -regularity are not in conflict; this will be ensured by part (v) of the following lemma.

LEMMA 6.13. *Let U be a profinite group.*

- (i) *Suppose that $\mathcal{C}(U) = 1$. Then U is \mathcal{C} -semisimple.*
- (ii) *Let \mathcal{N} be the set of all closed normal subgroups N of U such that U/N is \mathcal{C} -semisimple, and define $K = \mathbf{R}(U)$ to be the intersection of \mathcal{N} . Then U/K is \mathcal{C} -semisimple.*
- (iii) *Let N be a nontrivial closed normal subgroup of U such that N does not contain any nontrivial closed normal \mathcal{C} -subgroups of U . Then $U/C_U(N)$ is a nontrivial \mathcal{C} -semisimple image of U .*
- (iv) *Let $V \geq K \geq L$ be closed normal subgroups of U and suppose K is \mathcal{C} -regular in U . Then L is \mathcal{C} -regular in U and K is \mathcal{C} -regular in V .*
- (v) *Let K be a closed normal subgroup of U , let V be an open subgroup of U and let $L \leq K$ be a closed normal subgroup of U that has finite index in K . Then the following are equivalent: (a) K is \mathcal{C} -regular in U ; (b) L is \mathcal{C} -regular in U ; (c) $K \cap V$ is \mathcal{C} -regular in V .*

Proof. (i) Let H be a nontrivial finite locally normal subgroup of U . Then by Dietzmann's Lemma, H is contained in a finite nontrivial normal subgroup K of U ; there is then a characteristic subgroup M of K that is a direct power of a finite simple group, so that M is a normal \mathcal{C} -subgroup of U . Thus the condition $\mathcal{C}(U) = 1$ ensures that every nontrivial locally normal subgroup of U is infinite.

Let H be an infinite locally normal \mathcal{C} -subgroup of U , and let V be a closed normal subgroup of U of finite index that is contained in $N_U(H)$. By intersecting with V , we may assume $H \leq V$. Let K be the smallest closed normal subgroup of U containing H . Then K is topologically generated by the set of conjugates of H in U , which is a finite set of normal subgroups of K (since $K \leq V$), so K is a normal \mathcal{C} -subgroup of U , contradicting the assumption that $\mathcal{C}(U) = 1$.

It remains to show that $\mathbf{QZ}(U) = 1$. If not, then there is some open normal subgroup V of U such that $C_U(V) > 1$. Then $C_U(V)$ is normal in U , hence infinite, so that $Z(V) = C_U(V) \cap V$ is infinite as well. But then $Z(V)$ is a nontrivial abelian normal subgroup of U , so $1 < Z(V) \leq \mathcal{C}(U)$, a contradiction.

(ii) We may assume $K = 1$. By (i) it suffices to show that $\mathcal{C}(U) = 1$. Let H be a nontrivial closed normal subgroup of U such that $H \in \mathcal{C}$. Then there is some $N \in \mathcal{N}$ such that $H \not\leq N$. But then the image of H in U/N is a nontrivial normal \mathcal{C} -subgroup, contradicting the hypothesis that U/N is \mathcal{C} -semisimple.

(iii) Let $K = C_U(N)$. We must have $Z(N) = 1$, since otherwise $Z(N)$ would be a nontrivial normal \mathcal{C} -subgroup of U ; thus $K \cap N = 1$. As a consequence, U/K has a normal subgroup NK/K isomorphic to N , and in particular $U/K \neq$

1; moreover, $C_{U/K}(NK/K) = 1$. Let H/K be a nontrivial normal \mathcal{C} -subgroup of U/K . Then $H/K \cap NK/K$ is nontrivial since NK/K has trivial centralizer. Since $H > K$, we then have $L > 1$ where $L = H \cap N$. Now L is isomorphic to $(H \cap NK)/K$, so L is a normal \mathcal{C} -subgroup of U contained in N , a contradiction. Thus no such subgroup H/K of U/K exists, so $\mathcal{C}(U/K) = 1$; by part (i), it follows that U/K is \mathcal{C} -semisimple.

(iv) If L is not \mathcal{C} -regular in U , then there is some closed normal subgroup M of U not containing L such that LM/M does not contain any nontrivial closed normal \mathcal{C} -subgroups of U/M . It follows by part (iii) that there is a closed normal subgroup R of U containing M but not L such that U/R is \mathcal{C} -semisimple. But then every \mathcal{C} -regular closed normal subgroup of U must be contained in R , so K is not \mathcal{C} -regular in U . This confirms that L is \mathcal{C} -regular in U as soon as K is so.

Let N be a closed normal \mathcal{C} -subgroup of V that does not contain K , and let R be the core of N in U . Then KR/R contains a nontrivial locally normal \mathcal{C} -subgroup S/R of U/R . Since $S > R$, there is some $u \in U$ such that $S \not\leq uNu^{-1}$. Since $S \leq KR \trianglelefteq U$, we infer that $u^{-1}Su \leq KR \trianglelefteq V$. Thus $u^{-1}SuN/N$ is a nontrivial locally normal \mathcal{C} -subgroup of V/N that is contained in KN/N . This confirms that K is \mathcal{C} -regular in V .

(v) We have already seen in (iv) that (a) implies (b). In the other direction, suppose that (b) holds and let M be a closed normal subgroup of U that does not contain K . If KM/M is finite, then it certainly contains a nontrivial normal \mathcal{C} -subgroup of U/M . If KM/M is infinite, then LM/M is also infinite and so contains a nontrivial normal \mathcal{C} -subgroup of U/M . Thus (a) and (b) are equivalent.

Given the equivalence of (a) and (b), from now on we may assume $K \leq V$. Suppose K is \mathcal{C} -regular in one of U and V . Then K is \mathcal{C} -regular in the core W of V in U by (iv). Let Y be either U or V . Given a closed normal \mathcal{C} -subgroup R of Y not containing K , then $R \cap W$ is a normal \mathcal{C} -subgroup of W not containing K . It follows that $K(R \cap W)/(R \cap W)$ contains a nontrivial locally normal \mathcal{C} -subgroup $S/(R \cap W)$ of $W/(R \cap W)$, so KR/R contains the nontrivial locally normal \mathcal{C} -subgroup SR/R of W/R ; note that SR/R is also locally normal in Y/R . Thus K is \mathcal{C} -regular in Y , so (a) and (c) are equivalent. \square

Proof of Theorem 6.11. Assume for the moment that G is profinite.

Let \mathcal{N} be the set of all closed normal subgroups of G such that G/N is \mathcal{C} -semisimple, and let K be the intersection of \mathcal{N} . Then K satisfies property (ii) of the theorem by Lemma 6.13 (ii). It is clear from the construction that K contains every subgroup of G that is \mathcal{C} -regular in G .

Let N be a closed normal subgroup of G not containing K . If KN/N is abelian, then it is a locally normal \mathcal{C} -subgroup of G/N . If KN/N is nonabelian, let $L = C_G(KN/N)$ be the centralizer of KN modulo N . Note that L does not contain

K , so by the definition of K , the quotient G/L cannot be \mathcal{C} -semisimple. It then follows by Lemma 6.13(iii) that KN/N contains a nontrivial closed normal \mathcal{C} -subgroup of G/N . Thus K is \mathcal{C} -regular in G .

We have proved the theorem in the profinite case. Now let G be an arbitrary t.d.l.c. group. We have already obtained a \mathcal{C} -regular radical for the compact subgroups of G . Moreover, the \mathcal{C} -regular radicals of the open compact subgroups satisfy the hypotheses of Lemma 6.6: this is clear from Lemma 6.13(v) and the fact that the \mathcal{C} -regular radical is the unique largest \mathcal{C} -regular normal subgroup. Let U be an open compact subgroup of G and define $R = R_{\mathcal{C}}(G) := \text{QC}_G(U/R_{\mathcal{C}}(U))$. Then by Lemma 6.6, R is a closed normal subgroup of G that does not depend on the choice of U , such that $\text{QZ}(G/R) = 1$ and $R \cap U = R_{\mathcal{C}}(U)$.

The open compact subgroups of G/R are of the form UR/R for U an open compact subgroup of G ; UR/R is isomorphic to $U/U \cap R = U/R_{\mathcal{C}}(U)$, so UR/R is \mathcal{C} -semisimple. Hence G/R is \mathcal{C} -semisimple; in particular $R_{\mathcal{C}}(G/R) = 1$.

Let N be a closed normal subgroup of G such that G/N is \mathcal{C} -semisimple. Then by the profinite case of the theorem, we have $N \geq R_{\mathcal{C}}(U)$ for every open compact subgroup U of G ; this ensures that RN/N is a discrete normal subgroup of G/N . But G/N has trivial quasicentre, so all discrete normal subgroups are trivial. Thus $N \geq R$. We have now shown that property (ii) holds, which is clearly a characterization of the \mathcal{C} -regular radical.

Given an open compact subgroup V of G , then the $V \cap R$ is \mathcal{C} -regular in V by the profinite case of the theorem. Hence R is \mathcal{C} -regular in G . The fact that G/R is \mathcal{C} -semisimple then ensures that R is the largest \mathcal{C} -regular locally normal subgroup of G : given $K \leq G$ such that $K \not\leq R$ and K is locally normal in G , then KR/R is nondiscrete (since $\text{QZ}(G/R) = 1$), so $KR/R \cap UR/R$ is nontrivial for every open compact subgroup U of G , but clearly KR/R does not contain any nontrivial locally normal \mathcal{C} -subgroup of UR/R . \square

Theorem III now follows immediately from Theorems 6.2 and 6.11.

Theorem 6.11 also has the following consequence:

COROLLARY 6.14. *Let G be a t.d.l.c. group. Then at least one of the following holds:*

- (i) G is $[A]$ -regular;
- (ii) there is a closed characteristic normal subgroup R of G such that G/R is nondiscrete and $[A]$ -semisimple.

In particular, consider a topologically characteristically simple t.d.l.c. group G . Then G is either $[A]$ -regular or $[A]$ -semisimple.

As in the case of the quasihypercentre (see Proposition 6.3), the \mathcal{C} -regular radical enjoys stability properties with respect to certain closed subgroups.

PROPOSITION 6.15. *Let G be a t.d.l.c. group and let \mathcal{C} be a stable class of profinite groups.*

- (i) *Let H be a closed normal subgroup of G . Then $R_{\mathcal{C}}(H) \geq R_{\mathcal{C}}(G) \cap H$.*
- (ii) *Let H be an open subgroup of G . Then $R_{\mathcal{C}}(H) = R_{\mathcal{C}}(G) \cap H$.*

Proof. (i) Let U be an open compact subgroup of G . Then $R_{\mathcal{C}}(U)$ is \mathcal{C} -regular in U , so $R_{\mathcal{C}}(U) \cap H$ is \mathcal{C} -regular in $U \cap H$ by Lemma 6.13(iv). Thus $R_{\mathcal{C}}(U) \cap H \leq R_{\mathcal{C}}(U \cap H)$. At the global level, we have $QC_H(G/(R_{\mathcal{C}}(U) \cap H)) = QC_H(G/R_{\mathcal{C}}(U))$ since H is normal in G , so $[g, h] \in H$ whenever $g \in G$ and $h \in H$. Hence we have

$$\begin{aligned} R_{\mathcal{C}}(H) &= QC_H(G/R_{\mathcal{C}}(U \cap H)) \\ &\geq QC_H(G/(R_{\mathcal{C}}(U) \cap H)) \\ &= QC_H(G/R_{\mathcal{C}}(U)) \\ &= R_{\mathcal{C}}(G) \cap H. \end{aligned}$$

(ii) We can obtain both $R_{\mathcal{C}}(H)$ and $R_{\mathcal{C}}(G) \cap H$ as $QC_H(U/R_{\mathcal{C}}(U))$, where U is an open compact subgroup of G that is contained in H . \square

We finish this section by linking it with our previous discussion of locally \mathcal{C} -stable t.d.l.c. groups.

Note the following variant of Lemma 3.7, giving a condition under which a locally normal subgroup has trivial centralizer in G .

LEMMA 6.16. *Let (G, U) be a Hecke pair such that $QZ(G) = 1$ and let L be a locally normal subgroup of G such that $C_U(L)$ is finite. Then:*

- (i) $C_G(L) = 1$.
- (ii) *Let H and K be subgroups of G containing L , such that $H \cap K$ contains a finite index subgroup V of U . Suppose $\theta : H \rightarrow K$ is an isomorphism such that $\theta(x) = x$ for all $x \in L$ and such that $\theta(V)$ is commensurate with V . Then $H = K$ and $\theta = \text{id}_H$.*

Proof. Observe that $C_U(L)$ is finite and normalized by a finite index subgroup of U , so in fact $C_U(L) \leq QZ(U) = 1$.

We now prove (ii). We may assume $V \leq N_G(L)$. Choose a finite index subgroup W of V such that $W\theta(W)$ is contained in V . Let $g \in W$ and let $x \in L$. Then

$$g x g^{-1} = \theta(g x g^{-1}) = \theta(g)\theta(x)\theta(g^{-1}) = \theta(g)x\theta(g^{-1}),$$

so $g^{-1}\theta(g)$ centralizes x . This shows that $g^{-1}\theta(g) \in C_V(L) = 1$. Thus $g = \theta(g)$, that is, θ fixes W pointwise.

We may now apply Lemma 3.7, since $C_G(W \cap g^{-1}Wg) \leq \text{QZ}(G) = 1$ for all $g \in G$. Thus $H = K$ and $\theta = \text{id}_H$. Part (ii) follows.

By applying (ii) to the automorphisms of G induced by conjugation by elements of $C_G(L)$, we infer that $C_G(L) \leq Z(G) = 1$, thereby confirming (i). \square

PROPOSITION 6.17. *Let G be a t.d.l.c. group such that $\text{QZ}(G) = 1$. The following are equivalent:*

- (i) G is locally C -stable;
- (ii) For every compact locally normal subgroup H of G and open compact subgroup U of G we have $C_G(\text{HC}_U(H)) = 1$;
- (iii) G has no nontrivial abelian locally normal subgroups;
- (iv) G is $[A]$ -semisimple.

Proof. Suppose G is locally C -stable. Then (ii) follows from Lemma 6.16(i).

Suppose (ii) holds and let H be an abelian locally normal subgroup of G . Let U be any open compact subgroup of G and let $K = H \cap U$. Then $K \leq C_G(\text{KC}_U(K)) = 1$, so H is discrete. Since $\text{QZ}(G) = 1$, we conclude that H is trivial.

Suppose (iii) holds. If G is not $[A]$ -semisimple, then there is an open compact subgroup U of G such that U has a nontrivial normal $[A]$ -subgroup H , by Lemma 6.13(i). But we see by hypothesis that U has no nontrivial finite or abelian locally normal subgroups, so by the construction of $[A]$, there is no way to obtain a normal $[A]$ -subgroup. Thus G is $[A]$ -semisimple.

Now suppose that G is $[A]$ -semisimple. In particular, $\text{QZ}(G) = 1$ and there are no nontrivial compact abelian locally normal subgroups. Then G is locally C -stable by Theorem 3.19. \square

The following illustration of the above discussion also provides a justification for the choice of terminology.

PROPOSITION 6.18. *Let G be a p -adic Lie group with Lie algebra $\mathfrak{g} = \text{Lie}(G)$. Let also $R = R_{[A]}(G)$ be the $[A]$ -regular radical of G .*

Then $\text{Lie}(R)$ coincides with the solvable radical \mathfrak{r} of \mathfrak{g} , and R is the largest closed normal subgroup of G with Lie algebra \mathfrak{r} . It coincides with the kernel of the G -action on $\mathfrak{g}/\mathfrak{r}$.

In particular R is locally soluble, and G is $[A]$ -semisimple if and only if its Lie algebra \mathfrak{g} is semisimple.

Proof. By Proposition 2.6, every ideal of \mathfrak{g} is the Lie algebra of a compact locally normal subgroup of G . In particular G has a compact locally normal subgroup K whose Lie algebra is the soluble radical \mathfrak{r} of \mathfrak{g} . By [5, Ch. III, Section 9, Corollary to Proposition 6], the group K has a soluble open subgroup, say K_0 . By Lemma 2.1 there is a compact open subgroup U of G containing K_0 as a normal subgroup.

By Theorem 6.11 and Proposition 6.15(ii), we have $K_0 = \text{Lie}(K_0) \leq R_{[A]}(U) = R \cap U$.

Since U/K_0 is a compact p -adic Lie group whose Lie algebra $\mathfrak{g}/\mathfrak{r}$ is a direct product of simple (nonabelian) Lie algebras, it follows from Proposition 2.6 that U is virtually a direct product of hereditarily just finite profinite groups that are not virtually abelian. In particular U has trivial quasicycle (see [2, Proposition 5.1]) and no nontrivial abelian locally normal subgroup. Therefore, U is $[A]$ -semisimple by Proposition 6.17, so that $K_0 = \text{Lie}(K_0) \leq R_{[A]}(U) = R \cap U$. In particular K_0 is an open subgroup of R that is soluble, and we have $\mathfrak{r} = \text{Lie}(K_0) = \text{Lie}(R)$.

Any closed normal subgroup of G with Lie algebra \mathfrak{r} is locally solvable by [5, Ch. III, Section 9, Corollary to Proposition 6], hence $[A]$ -regular, and thus contained in R by Theorem 6.11. Thus R is the largest such subgroup. It follows that R is the kernel of the G -action on $\mathfrak{g}/\mathfrak{r}$ (a short and complete argument showing this can be found in the proof of [10, Lemma 6.4]).

The remaining assertions are now clear. □

The $[A]$ -regular radical has been used recently to analyse the global structure of t.d.l.c. group admitting an open subgroup which has a continuous faithful finite-dimensional representation over a local field: by [9, Theorem R.1], the $[A]$ -regular radical of such a group is the largest closed normal subgroup that is locally soluble (that is, that has a soluble open subgroup).

6.4. Canonical completions of Hecke pairs. The radical theory developed so far in this section is specific to t.d.l.c. groups, as the topology plays an important role. However, results about t.d.l.c. groups naturally have consequences for Hecke pairs in general, due to the existence of a universal ‘completion’ for Hecke pairs that generalizes the well-known profinite completion of a group. A construction is given in [4] and a proof of universality in [21].

THEOREM 6.19. *Let (G, U) be a Hecke pair, such that G is a topological group (possibly discrete) and U is closed in G . Then there is an essentially unique group homomorphism $\theta : G \rightarrow \hat{G}_U$, the **localized profinite completion of a Hecke pair***

of G at U , where \hat{G}_U is a topological group, such that the following properties hold:

- (i) *The image of θ is dense, the restriction $\theta|_U$ of θ to U is continuous, and $\overline{\theta(U)}$ is an open profinite subgroup of \hat{G}_U .*
- (ii) *Let R be a topological group and let $\phi : G \rightarrow R$ be a group homomorphism. Suppose that $\phi|_U$ is continuous and $\overline{\phi(U)}$ is profinite. Then there is a unique continuous homomorphism $\psi : \hat{G}_U \rightarrow R$ such that $\phi = \psi\theta$.*

By composing with the appropriate quotient map, we also have canonical homomorphisms with related universal properties for t.d.l.c. groups with trivial quasicentre, respectively locally C -stable t.d.l.c. groups with trivial quasicentre, that follow immediately from the results we have so far.

COROLLARY 6.20. *Let (G, U) be a Hecke pair, such that G is a topological group (possibly discrete) and U is closed in G . Let*

$$\theta_1 : G \rightarrow \hat{G}_U/\text{QZ}^\infty(\hat{G}_U) \quad \text{and} \quad \theta_2 : G \rightarrow \hat{G}_U/\mathbf{R}_{[A]}(\hat{G}_U)$$

be the canonical homomorphisms obtained from Theorem 6.19, Theorem 6.2 and Theorem 6.11. Set $G_1 = \hat{G}_U/\text{QZ}^\infty(\hat{G}_U)$ and $G_2 = \hat{G}_U/\mathbf{R}_{[A]}(\hat{G}_U)$. Then the homomorphisms θ_i are characterized by the following properties:

- (i) *The image of θ_i is dense, the restriction $\theta_i|_U$ of θ_i to U is continuous, and $\overline{\theta_i(U)}$ is an open profinite subgroup of G_i . Moreover, $\text{QZ}(G_1) = \text{QZ}(G_2) = 1$, and G_2 is locally C -stable.*
- (ii) *Let R be a t.d.l.c. group and let $\phi : G \rightarrow R$ be a group homomorphism. Suppose that $\phi|_U$ is continuous and $\overline{\phi(U)}$ is profinite. If $\text{QZ}(R) = 1$, then there is a unique continuous homomorphism $\psi_1 : G_1 \rightarrow R$ such that $\phi = \psi_1\theta_1$. If in addition R is locally C -stable, then there is a unique continuous homomorphism $\psi_2 : G_2 \rightarrow R$ such that $\phi = \psi_2\theta_2$.*

Let (G, U) be a Hecke pair, equipped with the discrete topology, such that U is residually finite. Two questions arise at this point:

QUESTION 6.21. *If (G, U) has trivial quasicentre, is $\theta_1 : G \rightarrow \hat{G}_U/\text{QZ}^\infty(\hat{G}_U)$ necessarily injective? Equivalently, does (G, U) necessarily admit a dense embedding into a t.d.l.c. group with trivial quasicentre?*

QUESTION 6.22. *If (G, U) is locally C -stable with trivial quasicentre, is $\theta_2 : G \rightarrow \hat{G}_U/\mathbf{R}_{[A]}(\hat{G}_U)$ necessarily injective? Equivalently, does (G, U) necessarily*

admit a dense embedding into a locally C -stable t.d.l.c. group with trivial quasiconcentre?

Both questions are of a local nature: if (G, U) has trivial quasiconcentre, then by Lemma 3.2, every nontrivial normal subgroup of G has infinite intersection with U . Thus θ_i is injective for $i \in \{1, 2\}$ if and only if $\theta_i|_U$ is injective. The questions can thus be restated as follows:

QUESTION 6.23. *Let U be a residually finite group with trivial quasiconcentre. Can U be densely embedded in a profinite group P , such that P has trivial quasiconcentre?*

QUESTION 6.24. *Let U be a residually finite group with no nontrivial virtually abelian normal subgroups. Can U be densely embedded in a profinite group P , such that P has no nontrivial virtually abelian normal subgroups?*

7. Restrictions on the group topology

7.1. The structure lattice and refinements of the group topology. In this section, we investigate the structural significance of fixed points in the action of a t.d.l.c. group G on its structure lattice. Of course, if G has a closed normal subgroup K that is neither discrete nor open, then $[K]$ is a nontrivial fixed point in $\mathcal{LN}(G)$. Especially interesting, however, is the question of whether a compactly generated, topologically simple t.d.l.c. group can have nontrivial fixed points in $\mathcal{LN}(G)$. We shall return to this question in the second paper [8], elaborating on the results of the present chapter. Most results we shall obtain here are for t.d.l.c. groups that are first-countable or σ -compact; it is known (see [15]) that a compactly generated (or even σ -compactly generated) t.d.l.c. group without nontrivial compact normal subgroups is second-countable (and hence also first-countable and σ -compact).

Our first goal is to prove Theorem IV, giving a general connection between the structure lattice and group topologies for first-countable t.d.l.c. groups.

DEFINITION 7.1. Let G be a t.d.l.c. group and let μ be a commensurability class (or subset of a commensurability class) of compact subgroups of G . The **localization** of G at μ is the t.d.l.c. group H , unique up to isomorphism, that admits a continuous embedding $\phi : H \rightarrow G$, such that for some (any) $K \in \mu$, $\phi(H) = \text{Comm}_G(K)$ and $\phi^{-1}(K)$ is an open compact subgroup of H . This group topology on H is called the **localized topology** at μ ; it is denoted by $\mathcal{T}_{(\mu)}$.

Thus, as an abstract group, the localization of G at μ coincides with $\text{Comm}_G(K)$ so that the localization of G at μ is nothing but $(\text{Comm}_G(K), \mathcal{T}_{(\mu)})$.

Starting with a t.d.l.c. group G , one sees that the t.d.l.c. groups with the same group structure as G but a finer topology are precisely the groups of the form $(G, \mathcal{T}_{(\mu)})$, where μ is a G -invariant commensurability class of compact subgroups of G . If G is first-countable, we can in fact classify these topological refinements of G using the structure lattice: this is the purpose of Theorem IV.

First we recall a result of Caprace and Monod [7]. Recall that a t.d.l.c. group G is **residually discrete** if the intersection of all open normal subgroups of G is trivial.

PROPOSITION 7.2 [7, Corollary 4.1]. *Let G be a compactly generated t.d.l.c. group. Then the following are equivalent:*

- (i) G is residually discrete;
- (ii) G has a base of neighbourhoods of the identity consisting of subsets that are invariant under conjugation;
- (iii) G has a base of neighbourhoods of the identity consisting of open compact normal subgroups.

COROLLARY 7.3. *Let G be a residually discrete locally compact group and let $x \in G$. Then the open subgroups of G normalized by x form a base of neighbourhoods of the identity.*

LEMMA 7.4. *Let U be a first-countable profinite group and let K be a closed subgroup of U . Then U commensurates K if and only if U normalizes an open subgroup of K .*

Proof. Certainly, if U normalizes an open subgroup of K then U commensurates K . Conversely, suppose U commensurates K . Let μ be the commensurability class of K . Then $U_{(\mu)}$ is residually discrete, so by Corollary 7.3

$$U = \bigcup \{N_U(L) \mid L \leq_o K\}.$$

Now K is first-countable, so it has countably many open subgroups; thus we have expressed U as a union of countably many subgroups, each of which is closed for the given topology on U . By the Baire Category Theorem, at least one of these, say $N_U(L)$, must be open in U . Thus L has finitely many conjugates in U , and we

have $U = \text{Comm}_U(L)$ since L is commensurate with K . Hence $\bigcap_{u \in U} uLu^{-1}$ is a closed subgroup of L of finite index and thus an open subgroup of K . \square

Proof of Theorem IV. Let \mathcal{T} be the given topology of G and let \mathcal{T}' be a refinement of \mathcal{T} compatible with the group structure. By van Dantzig's theorem, under \mathcal{T}' there is a subgroup K of G that is open and compact; it follows that K is compact under \mathcal{T} . Moreover, the set of such subgroups forms a commensurability class μ that is invariant under conjugation in G . In particular, given a subgroup U of G that is open and compact under \mathcal{T} , then by Lemma 7.4, U normalizes a subgroup $L \in \mu$. Thus μ corresponds to an element $\alpha = [L]$ of $\mathcal{LN}(G)$; we have $G = G_\alpha$ since μ is G -invariant, and see that the topology \mathcal{T}' is exactly the topology $\mathcal{T}_{(\alpha)}$.

Conversely, let $\alpha \in \mathcal{LN}(G)$ that is fixed by G . Then G admits a locally compact topological refinement by declaring those representatives of α that are closed in \mathcal{T} to be open, in other words, the group $(G, \mathcal{T}_{(\alpha)})$, which by definition has the same underlying group structure as G . \square

Note that none of the group topologies in Theorem IV are σ -compact, except possibly the original topology of G , since a closed subgroup of a profinite group is either open or of uncountable index.

7.2. A group topology derived from the centralizer lattice. Let \mathcal{A} be a Boolean algebra, and equip $\text{Aut}(\mathcal{A})$ with the coarsest group topology such that the stabilizer of α is open, for all $\alpha \in \mathcal{A}$. Then $\text{Aut}(\mathcal{A})$ is a totally disconnected group. Given a group G acting faithfully on \mathcal{A} , write $\overline{G}_{\mathcal{A}}$ for the closure of G in $\text{Aut}(\mathcal{A})$, equipped with the subspace topology.

Of particular interest is the case in which an infinite commensurated subgroup U of G (perhaps G itself) has finite orbits. In this case, we see that U is residually finite and $\overline{U}_{\mathcal{A}}$ is profinite, so the group $\langle G, \overline{U}_{\mathcal{A}} \rangle$ can be equipped with a t.d.l.c. topology that extends the topology of $\overline{U}_{\mathcal{A}}$.

THEOREM 7.5. *Let (G, U) be a Hecke pair with trivial quasicentre and let \mathcal{A} be a Boolean algebra on which G has a faithful, weakly decomposable action. Let E be the subgroup of $\text{Aut}(\mathcal{A})$ generated by G and $\overline{U}_{\mathcal{A}}$. Then E commensurates $\overline{U}_{\mathcal{A}}$, so that E can be equipped with a t.d.l.c. group topology by extending the topology of $\overline{U}_{\mathcal{A}}$ (so that $\overline{U}_{\mathcal{A}}$ is embedded as an open subgroup). Moreover, the following holds:*

- (i) *As a t.d.l.c. group, E is locally C -stable and faithful weakly decomposable.*
- (ii) *As a locally compact topological group, E is characterized as follows:*

Given a locally compact group L and an injective homomorphism $\theta : G \rightarrow L$ with dense image such that $\overline{\theta(U)}$ is an open subgroup of L , then there exists a unique continuous surjective homomorphism $\phi : L \rightarrow E$ such that the composition $\phi\theta$ is the embedding of G in E .

(iii) If $|G : U|$ is countable, the following characterization also applies:

Given a topological group L that is a Baire space and an injective homomorphism $\theta : G \rightarrow L$ such that $L = \langle \theta(G), \overline{\theta(U)} \rangle$, then there exists a unique continuous homomorphism $\phi : L \rightarrow E$ such that the composition $\phi\theta$ is the embedding of G in E .

Proof. Let $V = \overline{U_{\mathcal{A}}}$. Given $g \in G$, we see that gUg^{-1} contains a finite index subgroup of U , which ensures that gVg^{-1} contains an open subgroup of V . Thus E can be equipped with a t.d.l.c. group topology by extending the topology of $\overline{U_{\mathcal{A}}}$.

Certainly E acts faithfully on \mathcal{A} , and the action of E on \mathcal{A} is weakly decomposable with respect to V . Given $x \in \text{QZ}(E)$, then x centralizes a finite index subgroup of U , and hence $x\text{rist}_U(\alpha)x^{-1}$ is commensurate with $\text{rist}_U(\alpha)$, for all $\alpha \in \mathcal{A}$. This implies that x acts trivially on \mathcal{A} , so $x = 1$. Thus E is locally C -stable by Theorem 5.18.

Let us suppose we have a homomorphism $\theta : G \rightarrow L$ as in either (ii) or (iii), with (G, U) and \mathcal{A} satisfying the necessary hypotheses. We prove (ii) and (iii) together, except where it is necessary to divide into cases.

Given $\alpha \in \mathcal{A}$, let $R_\alpha = \theta(\text{rist}_G(\alpha))$ and let $K_\alpha = C_L(R_{\alpha^\perp})$. Then K_α is a closed subgroup of L , for all $\alpha \in \mathcal{A}$. Moreover, we have $R_\alpha \leq K_\alpha$, but $K_\alpha \cap R_{\alpha^\perp}$ is trivial. This ensures that $K_\alpha \cap \theta(G) = R_\alpha$; in particular, given $\alpha, \beta \in \mathcal{A}$ such that $\alpha \neq \beta$, then $R_\alpha \neq R_\beta$ since the action of G is weakly decomposable, so $K_\alpha \neq K_\beta$. Now set $N_\alpha = N_L(K_\alpha)$. Then N_α is closed in L , since K_α is closed. Moreover, $N_\alpha \cap \theta(G) = N_{\theta(G)}(R_\alpha) = \theta(N_G(\text{rist}_G(\alpha)))$. We see that $N_\alpha \cap \theta(U)$ has finite index in $\theta(U)$, so $N_\alpha \cap P$ has finite index in P , where $P = \overline{\theta(U)}$.

In case (ii), it follows that $N_\alpha \cap P$ is open in P , and hence N_α is open in L , since P is open in L . In case (iii), N_α is a closed subgroup of L of countable index, and hence open by the Baire category theorem. In either case, we conclude that N_α is open in L , as is any finite intersection of subgroups N_α . Therefore, the set $\{K_\alpha \mid \alpha \in \mathcal{A}\}$ ordered by inclusion forms a Boolean algebra \mathcal{A}' on which L acts smoothly, so that $\theta(U)$ is orbit-equivalent to P on the finite subsets of \mathcal{A}' . The natural map $K_\alpha \mapsto \alpha$ induces a continuous homomorphism $\psi : L \rightarrow \text{Aut}(\mathcal{A})$ such that $\psi\theta$ is the given action of G on \mathcal{A} . In case (ii), $\psi(P)$ has a dense subgroup in common with $\overline{U_{\mathcal{A}}}$, and both groups are compact, so $\psi(P) = \overline{U_{\mathcal{A}}}$. In case (iii), $U_{\mathcal{A}}$ is dense in $\psi(P)$, so $\psi(P) \leq \overline{U_{\mathcal{A}}}$. In either case, $\psi(L)$ is generated by $\psi\theta(G)$

and $\psi(P)$, so $\psi(L) \leq E$, and hence ψ restricts to a continuous homomorphism $\phi : L \rightarrow E$ with the required properties (in particular, ϕ is surjective in case (ii)). Moreover, in both cases the required properties specify ϕ uniquely, since the condition on $\phi\theta$ ensures that ϕ is uniquely specified on a dense subset of L . \square

We see from Theorem 7.5(i) that the embedding $G \rightarrow E$ is not dependent on the choice of \mathcal{A} , only on G and the commensurability class of U . Thus if (G, U) is any faithful weakly decomposable Hecke pair, we may define E to be the **centralizer lattice completion** (\mathcal{LC} -completion) of (G, U) , denoted $\hat{G}_{\mathcal{LC}}$. For example, if G is a branch group, regarded as a Hecke pair (G, G) , then $\hat{G}_{\mathcal{LC}}$ is the **congruence completion** of G in the sense of [3].

Theorem 7.5 is of interest even if (G, U) is t.d.l.c., as it has implications for the topologies that are compatible with the group structure of G . In this case, $G = \hat{G}_{\mathcal{LC}}$ as topological groups. Notice that if G is second-countable, then $|G : U|$ is countable.

In particular, we can prove a slightly more general form of Theorem V.

COROLLARY 7.6. *Let G be a t.d.l.c. group that is faithful weakly decomposable. Let L be a topological group and let $\theta : L \rightarrow G$ be an abstract group isomorphism, such that one of the following conditions holds:*

- (i) *L is a t.d.l.c. group, and there exists an open compact subgroup U of L such that $\theta(U)$ is open and compact in G .*
- (ii) *G is second-countable and the topology of L is a Baire space.*

Then θ is continuous.

In particular, if G is second-countable, then the topology of G is the unique σ -compact locally compact topology compatible with the group structure (in particular, it is preserved by every automorphism of G as an abstract group), and the set of all locally compact group topologies of G is in natural bijection with the set of fixed points of G acting on $\mathcal{LN}(G)$ by conjugation.

Proof. By applying Theorem 7.5 to the group isomorphism $\theta^{-1} : G \rightarrow L$, we see that there exists a continuous group homomorphism ϕ such that $\phi\theta^{-1} = \text{id}_G$. Clearly $\phi = \theta$.

In particular, if G is second-countable, then the topology of G is the coarsest group topology on G that forms a Baire space, so in particular it is the coarsest locally compact Hausdorff group topology on G . The remaining assertions follow from Theorem IV. \square

7.3. Subgroups abstractly generated by compact locally normal subgroups.

Let G be a locally compact group. Recall that the group $\text{Aut}(G)$ of homeomorphic automorphisms carries a natural Hausdorff group topology, called the **Braconnier topology** (see [7, Appendix I]). Basic identity neighbourhoods in $\text{Aut}(G)$ are provided by the sets

$$\mathfrak{A}(K, U) = \{ \alpha \in \text{Aut}(G) \mid \forall x \in K, \alpha(x) \in Ux \text{ and } \alpha^{-1}(x) \in Ux \},$$

where K runs over compact subsets of G and U runs over identity neighbourhoods of G . In particular, for each compact open subgroup $U < G$, the stabilizer $\{ \alpha \in \text{Aut}(G) \mid \alpha(U) = U \}$ is an identity neighbourhood in $\text{Aut}(G)$.

In what follows, automorphism groups of locally compact groups will always be implicitly endowed with the Braconnier topology.

LEMMA 7.7. *Let G be a t.d.l.c. group, let I be a directed set, and let $(U_n)_{n \in I}$ be a net of open compact subgroups in G forming a base of identity neighbourhoods, such that $U_i < U_j$ whenever $i > j$. Let $(\alpha_n)_{n \in I}$ be a net in $\text{Aut}(G)$ such that $\lim_{n \in I} \alpha_n(x) = x$ for all $x \in G$.*

If for each $m \in I$, there is some $n_0 \in I$ such that $\alpha_n(U_m) = U_m$ for all $n \geq n_0$, then $(\alpha_n)_{n \in I}$ converges to the identity with respect to the Braconnier topology.

Proof. Let $K \subset G$ be a compact subset and $U < G$ an identity neighbourhood. Choose $U_m \subset U$. We can find a finite subset $\{x_1, \dots, x_k\} \subset K$ such that $K \subset \bigcup_{i=1}^k U_m x_i$. Since (α_n) converges pointwise to the identity, so does the sequence (α_n^{-1}) . Hence we may find n_0 such that $\alpha_n(x_i) \in U_m x_i$ and $\alpha_n^{-1}(x_i) \in U_m x_i$ for all $n \geq n_0$.

Let now $x \in K$. Choose i such that $U_m x = U_m x_i$. Thus we have

$$\alpha_n(x) \in \alpha_n(U_m x) = \alpha_n(U_m x_i) = U_m \alpha_n(x_i) = U_m x_i = U_m x.$$

Similarly, we get $\alpha_n^{-1}(x) \in U_m x$. This shows that α_n belongs to $\mathfrak{A}(K, U_m) \subset \mathfrak{A}(K, U)$ for all $n \geq n_0$. Thus (α_n) converges to the identity with respect to the Braconnier topology. \square

DEFINITION 7.8. Let G be a t.d.l.c. group and let $\alpha \in \mathcal{LN}(G)$. Define the **compact locally normal closure** $\text{LN}(G, \alpha)$ of α in G to be the group generated by all compact locally normal representatives of α .

We remark that $\text{LN}(G, \alpha)$ is a (not necessarily closed) subgroup of G with open normalizer, and if α is fixed by G then $\text{LN}(G, \alpha)$ is normal in G . If G is locally C-stable with trivial quasicycle and $\alpha \in \mathcal{LC}(G)$, then it follows from

Proposition 5.4 that $\text{LN}(G, \alpha)$ is an open subgroup of $\text{QC}_G(\alpha^\perp)$, which in turn is a closed subgroup of G_α , and the normalizer of $\text{LN}(G, \alpha)$ is exactly G_α .

PROPOSITION 7.9. *Let G be a t.d.l.c. group and let α be a fixed point of G in $\mathcal{LN}(G)$. Let $D = \text{LN}(G, \alpha)$. Then the natural homomorphism $G \rightarrow \text{Aut}(D, \mathcal{T}_{(\alpha)})$ induced by the G -action by conjugation is continuous.*

Proof. Since G commensurates the open compact subgroups of $(D, \mathcal{T}_{(\alpha)})$, it follows that the homomorphism $\varphi: G \rightarrow \text{Aut}(D, \mathcal{T}_{(\alpha)})$ induced by the conjugation action indeed takes its values in the group of homeomorphic automorphisms of $(D, \mathcal{T}_{(\alpha)})$. Let $(g_n)_{n \in I} \subset G$ be a net converging to the identity in G .

Let us first show that $\varphi(g_n)$ converges pointwise to the trivial automorphism. Let $x \in D$. Let L_1, \dots, L_k be compact locally normal representatives of α such that $x \in L_1 L_2 \dots L_k$. Then the group $\bigcap_{i=1}^k N_G(L_i)$ is open in G , and therefore contains g_n for all sufficiently large n . In particular, for such a large n , we have $\varphi(g_n).x = g_n x g_n^{-1} \in L_1 L_2 \dots L_k$. The latter product being compact with respect to the localized topology $\mathcal{T}_{(\alpha)}$, it follows that the net of commutators $(g_n x g_n^{-1} x^{-1})_{n \in I}$ is bounded in $(D, \mathcal{T}_{(\alpha)})$. Therefore, all we need to show is that 1 is its only accumulation point.

Suppose for a contradiction that some subnet $([g_{f(n)}, x])_{n \in I'}$ converges to $z \neq 1$. Choose a compact locally normal representative L of α which is sufficiently small so that $z \notin L$. We have $[g_{f(n)}, x] \in Lz$ for all large n . In particular $[g_{f(n)}, x] \in Oz$, where O is any open compact subgroup of G containing L . Since $z \notin L$, we may choose O small enough so that it does not contain z . Then O and Oz are disjoint so that $[g_{f(n)}, x] \notin O$ for all large n . Since O is an open subgroup of G , we deduce that $([g_{f(n)}, x])_{n \in I'}$ does not converge to the identity in G , which is absurd.

Let now L be a compact locally normal representative of α and let $(U_n)_{n \in I}$ be as in Lemma 7.7; we may assume $U_n \leq N_G(L)$ for all $n \in I$. Then the groups $L_m = L \cap U_m$ form a basis of identity neighbourhoods in $(D, \mathcal{T}_{(\alpha)})$. Clearly, for each $m \geq 0$, we have $g_n \in N_G(L_m)$ for all sufficiently large n . Lemma 7.7 now implies that $\varphi(g_n)$ converges to the identity with respect to the Braconnier topology. \square

COROLLARY 7.10. *Let G be a t.d.l.c. group, and let α be a fixed point of G in $\mathcal{LN}(G)$. Let D be an abstract normal subgroup of G generated by compact locally normal representatives of α , and let $H = \Gamma.D$ where Γ is a discrete normal subgroup of G . Then the natural homomorphism $G \rightarrow \text{Aut}(H, \mathcal{T}_{(\alpha)})$ induced by the G -action by conjugation is continuous.*

Proof. Note that Γ is a discrete normal subgroup of $(H, \mathcal{T}_{(\alpha)})$, and we have $H = \Gamma.D$. Note also that D is an open subgroup of $\text{LN}(G, \alpha)$ with respect to the

localized topology $\mathcal{T}_{(\alpha)}$, so the Braconnier topology on $\text{Aut}(D, \mathcal{T}_{(\alpha)})$ is compatible with the Braconnier topology on $\text{Aut}(\text{LN}(G, \alpha), \mathcal{T}_{(\alpha)})$.

Let (g_n) be a net tending to the identity in G ; let $K \subset H$ be $\mathcal{T}_{(\alpha)}$ -compact and $V \leq H$ be a $\mathcal{T}_{(\alpha)}$ -compact open subgroup. We need to show that $[g_n, K] \cup [g_n^{-1}, K]$ is contained in V for all sufficiently large n . Since $H = \Gamma.D$, it suffices to deal with the case when $K = \gamma C$ with $\gamma \in \Gamma$ and $C \subset D$ is $\mathcal{T}_{(\alpha)}$ -compact. For n large enough, we have $[g_n, \gamma] = 1$ since Γ is contained in the quasicentre of G . Therefore, $[g_n, \gamma C] = \gamma [g_n, C] \gamma^{-1}$. Observe that $W = \gamma^{-1} V \gamma$ is a compact open subgroup of $(H, \mathcal{T}_{(\alpha)})$. By Proposition 7.9, we have $[g_n, C] \subset W$ for all sufficiently large n . Hence $[g_n, \gamma C] \subset V$ for all large n . The argument with g_n^{-1} instead of g_n is similar. \square

In general, a compact subgroup of a t.d.l.c. group that is commensurate with a locally normal subgroup need not itself be locally normal. For example, if G has trivial quasicentre, then the only finite locally normal subgroup is the trivial one, but G may still have nontrivial finite subgroups. Proposition 7.9 leads to an interesting criterion for compact subgroups to be locally normal.

PROPOSITION 7.11. *Let G be a t.d.l.c. group and let $\alpha \in \mathcal{LN}(G)$. Let L be a compact subgroup of G such that $[L] = \alpha$. Then L is locally normal in G if and only if $L \leq \text{LN}(G, \alpha)$.*

Proof. If L is locally normal, then $L \leq \text{LN}(G, \alpha)$ by definition.

Assume conversely that $L \leq \text{LN}(G, \alpha)$. Since G_α is open, it follows that a subgroup of G contained in G_α is locally normal in G if and only if it is locally normal in G_α . Moreover, every locally normal representative of α is contained in G_α . Thus we have $\text{LN}(G, \alpha) = \text{LN}(G_\alpha, \alpha)$, and there is no loss of generality in assuming that $G = G_\alpha$.

Let D be the group $\text{LN}(G, \alpha)$ equipped with the localized topology $\mathcal{T}_{(\alpha)}$. Note that L is compact with respect to $\mathcal{T}_{(\alpha)}$. Thus L is an open compact subgroup of D . In particular the collection $\mathfrak{A}(L, L)$ of automorphisms of D which stabilize L is an identity neighbourhood with respect to the Braconnier topology on $\text{Aut}(D)$. Since the natural map $G \rightarrow \text{Aut}(D)$ is continuous by Proposition 7.9, there must be an open subgroup of G that normalizes L . \square

If U is an open compact subgroup of G that fixes α , there is a useful description of $\text{LN}(G, \alpha) \cap U$.

PROPOSITION 7.12. *Let G be a t.d.l.c. group, let $\alpha \in \mathcal{LN}(G)$ and let U be an open compact subgroup of G such that U fixes α . Let L be a closed subgroup of*

U such that $[L] \leq \alpha$. Then $L \leq \text{LN}(G, \alpha)$ if and only if there is a closed normal subgroup M of U such that $L \leq M$ and $[M] = \alpha$.

Proof. By Lemma 7.4, there is a normal subgroup N of U such that $[N] = \alpha$; by definition, $N \leq \text{LN}(G, \alpha)$.

Let L be a closed subgroup of U such that $[L] \leq \alpha$. Suppose $L \leq \text{LN}(G, \alpha)$. By replacing L with LN we may assume $N \leq L$. By Proposition 7.11, L is locally normal in G . In particular, L/N is a finite locally normal subgroup of U/N , so $N_{U/N}(L/N)$ has finite index in U/N , and hence $C_{U/N}(L/N)$ has finite index in U/N . Now by Dietzmann's Lemma, it follows that the normal closure M/N of L/N in U/N is finite and hence closed, so M is a closed normal subgroup of U such that $[M] = \alpha$. \square

COROLLARY 7.13. *Let U be a profinite group and let L be a closed subgroup. Assume that L is commensurated and locally normal in U . Then the abstract normal closure of L in U is closed, and contains L as a finite index subgroup.*

Proof. Let N be the abstract normal closure of L . Since $\alpha = [L]$ is fixed by G by hypothesis, we have $N \leq \text{LN}(G, \alpha)$. The previous proposition therefore implies that α has a representative M that is normal in U and contains L . Thus M is closed, the index of L in M is finite and we have $L \leq N \leq M$. Therefore, N is closed and contains L as a finite index subgroup. \square

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