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ENDOMORPHISM SEMIGROUPS OF SUMS OF RINGS

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Let $R\equiv\langle R, +, \cdot\rangle$ be the cartesian sum of the rings R_i, $i=1, 2, \ldots, n$ denoted by $R=\sum_{i=1}^n R_i$, and recall that R is a ring under the componentwise operations. It is well-known (e.g. [1], p. 212) that the endomorphisms of the group $\langle R, +\rangle$ form a ring $\text{Hom}_Z R$ (under function addition and composition) and moreover $\text{Hom}_Z R$ is isomorphic to the matrix ring $\mathscr{M}=\{[\sigma_{ij}] \mid \sigma_{ij} \in \text{Hom}_Z(R_i, R_j)\}$ under the usual matrix operations of addition and multiplication.

Let $\mathscr{S}$ denote the subset of $\mathscr{M}$ consisting of those matrices $A=[\sigma_{ij}]$ which represent ring endomorphisms of R. In this paper we characterize $\mathscr{S}$ (see Theorem 1) by finding necessary and sufficient conditions on the components σ_{ij} of A in order that $[\sigma_{ij}]$ correspond to a ring endomorphism. In other words, if End R denotes the semigroup of ring endomorphisms of R, we determine a matrix representation of this semigroup.

Recall from matrix theory that any automorphism of the ring of matrices (over a field) leaving the scalars fixed is an inner automorphism. This result has been generalized to a class of simple artinian rings (see [2] and [3]). In Section 2 of this paper we apply the matrix representation of End R to extend the result on simple rings to a class of semi-simple rings.

1. **Characterizations.** Let $\{R_\alpha\}_{\alpha=1}^n$ be a finite collection of rings. The cartesian sum R of these rings R_α will be denoted by $R=\sum_{\alpha=1}^n R_\alpha$. Clearly, if each R_α has a multiplicative identity e_α then $e=\langle e_1, e_2, \ldots, e_n\rangle$ is an identity for R. We also remark here, that if a ring S has an identity, then we do *not* require that an endomorphism ϕ of S preserve this identity.

THEOREM 1. *Let $\{R_\alpha\}$, $\alpha=1, \ldots, n$ be a collection of rings and $[\chi_{ij}]$ an $n\times n$ matrix where each $\chi_{ij}: R_i\to R_j$ satisfies the following properties:*

(1) *χ_{ij} is a ring morphism,*

(2) *$\forall x_i \in R, \forall x_k \in R_k, i\neq k \Rightarrow x_i\,\chi_{ij}\,x_k\,\chi_{kj}=0$ for all j.*

Then $[\chi_{ij}]$ determines a ring endomorphism of $R=\sum_{\alpha=1}^n R_\alpha$. Moreover every ring endomorphism ϕ of R determines such a matrix.

Proof. Define $\chi: R\to R$ by $\langle x_1, \ldots, x_n\rangle\,\chi=\langle x_1, \ldots, x_n\rangle[\chi_{ij}]$. Since each χ_{ij} is a morphism for the addition in R_i it is easy to show that χ is a morphism for the pointwise addition in R. Also, using condition (2) and the fact that each χ_{ij} is a morphism for the multiplication in R_i one finds that $x\,y\,\chi=x\,\chi\,y\,\chi$ for each x, y in R.

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Conversely let ϕ be a ring endomorphism of $R=\sum R_\alpha$. Let $\beta_i : R_i \to R$ be the insertion map $(x_i \to \langle 0, \ldots, 0, x_i, 0, \ldots, 0\rangle)$ and $\rho_j : R \to R_j$ $(\langle x_1, \ldots, x_j, \ldots, x_n\rangle \to x_j)$ the projection map. For all i, j, $\beta_i \phi \rho_j : R_i \to R_j$ is a ring morphism. Let $\phi_{ij} \equiv \beta_i \phi \rho_j$ and take $x_i \in R_i$, $x_k \in R_k$, $i \neq k$. Then $x_i \phi_{ij} x_k \phi_{kj} = (x_i \beta_i x_k \beta_k)\phi\rho_j = 0$. Thus the ϕ_{ij} satisfy conditions (1) and (2) and $[\phi_{ij}]$ is the desired matrix.

COROLLARY. *If each of the rings R_α in the above theorem has an identity e_α then condition* (2) *can be replaced by*

$$(2^*) \qquad e_i \chi_{ij} e_k \chi_{kj} = 0 \quad \text{for } i \neq k \quad \text{and all } j.$$

Moreover, for the identity $e = \langle e_1, \ldots, e_n\rangle$ of R and any ring endomorphism ϕ of R, $e\phi = e \Leftrightarrow \sum_{i=1}^n e_i \chi_{ij} = e_j$ for all j.

Proof. Since $x_i \chi_{ij} x_k \chi_{kj} = x_i \chi_{ij} e_i \chi_{ij} e_k \chi_{kj} x_j \chi_{kj}$, condition (2) is equivalent to condition (2*). The second statement of the corollary is immediate.

For the remainder of this paper we let $\mathscr{S}$ denote the collection of matrices $[\chi_{ij}]$ satisfying conditions (1) and (2) of Theorem 1 and let End R denote the semigroup of ring endomorphisms of R. If further, each R_i has multiplicative identity e_i, then $\mathscr{S}_1$ denotes the subset of $\mathscr{S}$ of matrices satisfying $\sum_{i=1}^n e_i \chi_{ij} = e_j$ and $\text{End}_1 R$ denotes the subsemigroup of identity preserving ring endomorphisms.

THEOREM 2. *For $R = \sum_{i=1}^n R_i$,* End $R \cong \mathscr{S}$ *and* $\text{End}_1 R \cong \mathscr{S}_1$.

Proof. Let $\phi \in \text{End } R$. From Theorem 1, we note that the correspondence $\phi \leftrightarrow [\phi_{ij}]$ is a bijection and that $\mathscr{S}$ and $\mathscr{S}_1$ are semigroups. If $\phi \to [\phi_{ij}]$ and $\theta \to [\theta_{ij}]$ then $\phi \circ \theta \to [(\phi \circ \theta)_{ij}] = [\beta_i \phi \circ \theta \rho_j] = [\beta_i \phi (\sum \rho_k \beta_k) \theta \rho_j]$ since $\sum \rho_k \beta_k = 1_R$. From this we obtain $[(\phi \circ \theta)_{ij}] = [\sum \beta_i \phi \rho_k \beta_k \theta \rho_j] = [\phi_{ij}][\theta_{ij}]$.

Thus, for any ring R which can be represented as a finite direct sum of rings, the above theorem characterizes those elements of $\text{Hom}_Z R$ which also preserve the multiplication in R.

We conclude this section with some remarks concerning the extension of the above results to an arbitrary family $\{R_\alpha\}(\alpha \in \Lambda)$ of rings. In this case the direct sum $R(\equiv \sum R_\alpha)$ is defined to be the collection of all vectors $\langle \ldots, r_\alpha, \ldots \rangle$ such that almost all components (all but a finite number) are zero and the ring operations are componentwise. In the analogous situation for abelian groups, Fuchs ([1], p. 212) uses row convergent matrices (a matrix $[\sigma_{\alpha\beta}](\alpha \in \Lambda, \beta \in \Omega)$ where $\sigma_{\alpha\beta} \in \text{Hom}_Z(R_\alpha, R_\beta)$ is said to be row convergent if for each row α and each $x_\alpha \in R_\alpha$, $x_\alpha \sigma_{\alpha\beta} = 0$ for almost all β).

In this setting, Theorems 1 and 2 become:

THEOREM 3. *Let $R = \sum_{\alpha \in \Lambda} R_\alpha$, $|\Lambda| = \mu$. The endomorphism semigroup* End R *of R is isomorphic to the semigroup of all μ by μ row convergent matrices $\mathscr{S}' = \{[\sigma_{\alpha\beta}]\}$ such that*

(1′) *$\sigma_{\alpha\beta} : R_\alpha \to R_\beta$ is a ring morphism,*

(2′) *for $x_\alpha \in R_\alpha$, $x_\gamma \in R_\gamma$, if $\gamma \neq \alpha$ then $x_\alpha \sigma_{\alpha\beta} x_\gamma \sigma_{\gamma\beta} = 0$ for all β.*

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2. **Main result.** In this section we use the above characterizations to extend the following known result (called the Noether-Skolem Theorem by Herstein, [3], p. 99) to a class of semisimple rings:

If R is a simple artinian ring, finite dimensional over its center $\mathscr{C}(R)$,
(*) then every ring automorphism ϕ which fixes the elements of $\mathscr{C}(R)$
is an inner automorphism.

We first note that the hypothesis of (*) can be (apparently) weakened to allow ϕ to be an endomorphism. For, any endomorphism ϕ of R satisfying the stated conditions is an injective linear transformation of the finite dimensional vector space R (over $\mathscr{C}(R)$) and hence is a surjective map.

Let R be any semisimple artinian ring. From Wedderburn Theory, $R=\sum_{\alpha=1}^{n} R_\alpha$ where R_α is a simple ring with identity e_α (and thus $e=\langle e_1, \ldots, e_n\rangle$ is an identity for R).

Lemma. *Let R be a semisimple artinian ring. If R is a finitely generated module over its center $\mathscr{C}(R)$ then R_α is a finite dimensional vector space over its center $\mathscr{C}(R_\alpha)$, for all α.*

Proof. If $G=\{g_1, g_2, \ldots, g_h\}$ is a generating set for R over $\mathscr{C}(R)$, then for $r\in R$, $r=\sum_{j=1}^{h} c_\alpha^j g_j$ where $c_j\in\mathscr{C}(R)$. Clearly $\mathscr{C}(R)=\sum\mathscr{C}(R_\alpha)$ and from the decompositions $r=\langle r_1, \ldots, r_\alpha, \ldots, r_n\rangle$, $g_j=\langle g_1^j, \ldots, g_\alpha^j, \ldots, g_n^j\rangle$ and $c_j=\langle c_1^j, \ldots, c_\alpha^j, \ldots, c_n\rangle$, $c_\alpha^j\in\mathscr{C}(R_\alpha)$ one obtains

$$r_\alpha=\sum_{j=1}^{h} c_\alpha^j g_\alpha^j \quad\text{where}\quad c_\alpha^j\in\mathscr{C}(R_\alpha) \quad\text{and}\quad g_\alpha^j\in R_\alpha.$$

Theorem 4. *Let R be a semisimple artinian ring, finitely generated as a module over $\mathscr{C}(R)$. Every endomorphism ϕ of R which fixes the elements of $\mathscr{C}(R)$ is an inner automorphism of R.*

Proof. Since ϕ fixes elements of $\mathscr{C}(R)$, $e\phi=e$ which implies that ϕ determines a matrix $[\phi_{ij}]$ of $\mathscr{S}_1$. Since $\langle 0, \ldots, 0, e_i, 0, \ldots, 0\rangle\in\mathscr{C}(R)$,

$$\langle 0, \ldots, 0, e_i, 0, \ldots, 0\rangle[\phi_{ij}] = \langle 0, \ldots, 0, e_i, 0, \ldots, 0\rangle$$

which implies that $e_i\,\phi_{ij}=e_i$ if $i=j$ and $e_i\,\phi_{ij}=0$ for $i\neq j$. Then, for any $x_i\in R_i$ and $i\neq j$, $x_i\,\phi_{ij}=x_i\,\phi_{ij}\,e_i\,\phi_{ij}=0$. Consequently $[\phi_{ij}]$ is diagonal,

$$[\phi_{ij}]=\begin{bmatrix}\phi_{11} & 0 & \cdots & & 0\\ 0 & \phi_{22} & 0 & \cdots & 0\\ \cdot & & & & \\ \cdot & & & & \\ \cdot & & & & \\ 0 & \cdots & 0 & & \phi_{nn}\end{bmatrix}$$

where ϕ_{ii} is a ring endomorphism of the simple ring R_i, fixing the elements of $\mathscr{C}(R_i)$. From the above lemma and the Noether-Skolem Theorem, there exists

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$b_i \in R_i$ such that $x_i \phi_{ii} = b_i^{-1} x_i b_i$. Hence for $x \in R$,

$$x = \langle x_1, \ldots, x_n \rangle, x\phi = \langle x_1, \ldots, x_n \rangle [\phi_{ij}] = \langle b_1^{-1} x_1 b_1, \ldots, b_n^{-1} x_n b_n \rangle = b^{-1} x b$$

where $b = \langle b_1, \ldots, b_n \rangle$. Since each ϕ_{ii} is an automorphism, so is ϕ.

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