Canad. Math. Bull. Vol. 21 (2), 1978

THE REPRESENTATION OF (C, k) SUMMABLE SERIES IN FOURIER FORM

BY
G. E. CROSS

1. Introduction. Several non-absolutely convergent integrals have been defined which generalize the Perron integral. The most significant of these integrals from the point of view of application to trigonometric series are the P^n- and $\mathcal{P}^n$-integrals of R. D. James [10] and [11]. The theorems relating the P^n-integral to trigonometric series state essentially that if the series

(1.1) $$a_0/2+\sum (a_n \cos nx+b_n \sin nx)\equiv \sum a_n(x)$$

is summable $(C, n-2)$ on $[0, 2\pi]$ to a finite function $f(x)$ and if a slightly weaker condition than $(C, n-2)$ summability holds on the conjugate series

(1.2) $$\sum (a_n \sin nx - b_n \cos nx)\equiv -\sum b_n(x)$$

then $f(x)$, $f(x)\cos nx$, $f(x)\sin nx$ are P^n-integrable on $[0, 2\pi]$ and the coefficients can be written in Fourier form using the integral.

In the case of the $\mathcal{P}^n$-integral, as in the case of the $C_{n-1}P$-integral of Burkill [4], it is necessary to posit summability $(C, n-2)$ of both series (1.1) and (1.2) [6].

In the original formulation of the P^n-integral there was an error which has now been corrected in two different ways ([7] and [12]) so that the original theorems by James on trigonometrical series remain valid in terms of the revised integral.

The definition of the P^n- and $\mathcal{P}^n$-major and minor functions and the proof of uniqueness of the integrals on an interval $[a, b]$ involve in an essential way the idea of a set of n points including the end points of the interval (we shall call it a "basis") at each point of which it is posited that the major and minor functions vanish.

One of the main theorems in the development of the theory of the P^n- and $\mathcal{P}^n$-integrals states that if a function is integrable with respect to a basis $\{\alpha_i\}_{i=1}^n$ on an interval $[a, b]$, then it is integrable with respect to any other basis $\{\beta_i\}_{i=1}^n$ in $[a, b]$. Thus if a function f is $\mathcal{P}^n$- or P^n-integrable on $[a, b]$ it is integrable with respect to a basis which includes a and b but the other $(n-2)$ points of which are taken arbitrarily close to a or b. Thus the property of integrability does not depend intrinsically on the basis.

Received by the editors July 6, 1977 and, in revised form, October 7, 1977.

Bullen [3] has simplified James' definition by eliminating the concept of a basis from the theory. He replaced the $2n$ conditions $Q(a_i)=q(a_i)=0$, $i=1,2,\ldots,n$, on the major and minor functions by the $2n$ conditions $Q_{(k)}(a_1)=q_{(k)}(a_1)=0$, $0\le k\le n-1$. The resulting integral is less general than the unsymmetric $\mathcal{P}^n$-integral ([3], Theorem 12(b)) and like the $\mathcal{P}^n$-integral does not give a satisfactory representation theorem for trigonometrical series.

The present paper combines the approaches of [3] and [7] to obtain a symmetric P_n^*-integral, simpler and more natural than the original P^n-integral, in terms of which a strong representation theorem for trigonometrical series still holds. The result is similar to that which holds for convergent series in terms of the *SCP*-integral [5] and for (C, n) summable series in terms of the $SC_{n+1}P$-integral [9] in the sense that the definite integral in the representation takes the form $\int_\alpha^{\alpha+2\pi}$ where α belongs to a set of full measure in $[0, 2\pi]$.

2. Definitions and Preliminaries. Let $F(x)$ be a real valued function defined on the bounded interval $[a, b]$. If there exist constants $\alpha_1, \alpha_2, \ldots, \alpha_r$ which depend on x_0 only and not on h, such that

$$F(x_0+h)-F(x_0)=\sum_{k=1}^{r}\alpha_k\frac{h^k}{k!}+o(h^r),\quad \text{as}\quad h\to 0, \tag{2.1}$$

then α_k, $1\le k\le r$, is called *the Peano derivative of order* k of F at x_0 and is denoted by $F_{(k)}(x_0)$. If F possesses derivatives $F_{(k)}(x_0)$, $1\le k\le r-1$, we write

$$\frac{h^r}{r!}\gamma_r(F; x_0, h)=F(x_0+h)-F(x_0)=\sum_{k=1}^{r-1}\frac{h^k}{k!}F_{(k)}(x_0). \tag{2.2}$$

We define

$$\bar{F}_{(r)}(x_0)=\limsup_{h\to 0}\gamma_r(F; x_0, h),$$

$$\underline{F}_{(r)}(x_0)=\liminf_{h\to 0}\gamma_r(F; x_0, h)$$

By restricting h to be positive (or negative) in (2.1) we can define one-sided Peano derivatives, which we write as $F_{(k)+}(x_0)$ (or $F_{(k)-}(x_0)$).

If there exist constants $\beta_0, \beta_2, \ldots, \beta_{2r}$ which depend on x_0 only, and not on h, such that

$$\frac{F(x_0+h)+F(x_0-h)}{2}=\sum_{k=0}^{r}\beta_{2k}\frac{h^{2k}}{(2k)!}+o(h^{2r}),\quad \text{as}\quad h\to 0,$$

then β_{2k}, $0\le k\le r$ is called *the de la Vallee Poussin derivative of order* $2k$ *of* F *at* x_0 and is denoted by $D_{2k}F(x_0)$.

If F has derivatives $D_{2k}F(x_0)$, $0\le k\le r-1$, we write

$$\frac{h^{2r}}{(2r)!}\theta_{2r}(F; x_0, h)=\frac{F(x_0+h)+F(x_0-h)}{2}-\sum_{k=0}^{r-1}\frac{h^{2k}}{(2k)!}D_{2k}F(x_0),$$

and define

$$\bar{D}^{2r}F(x_0) = \limsup_{h\to 0} \theta_{2r}(F; x_0, h)$$

$$\underline{D}^{2r}F(x_0) = \liminf_{h\to 0} \theta_{2r}(F; x_0, h).$$

All the above symbols are defined similarly for odd-numbered indices (see, for example, [10], pp. 163–164).

If $F_{(r)}(x_0)$ exists, so does $D^{(r)}F(x_0)$ and $F_{(r)}(x_0) = D^{(r)}F(x_0)$.

We denote the ordinary derivative of $F(x)$ at x_0 of order k by $F^{(k)}(x_0)$.

The function F will be said to satisfy condition $A_n^*(n \geq 2)$ in $[a, b]$ if it is continuous in $[a, b]$, if, for $1 \leq k \leq n-2$, each $F_{(k)}(x)$ exists and is finite in (a, b) and if

$$\lim_{h\to 0} h\theta_n(f; x, h) = 0 \tag{2.3}$$

for all $x \in (a, b) - E$, where E is countable.

When a function F satisfies condition (2.3) at a point x, F is said to be *n − smooth* at x.

Theorem 2.1. *If F satisfies condition $A_{2m}^*(A_{2m+1}^*)$ in $[a, b]$, then $F_{(2k)}(x) = D_{2k}F(x)(F_{(2k+1)}(x) = D_{(2k+1)}(x))$ does not have an ordinary discontinuity in (a, b) for $0 \leq k \leq m-1$.*

Proof. This is Lemma 8.1 [10].

Note: Condition A_{2m}^* is a stronger form of James' condition A_{2m}, [10], in that it replaces the requirement that $D_{2k}F(x)$ exist and be finite for $1 \leq k \leq m-1$ by the same condition on the Peano derivatives. Theorem 2.1 then shows that A_{2m}^* also implies James' condition B_{2m-2}, [10].

We shall make extensive use of the theory of n-convex functions in the following. For the definition and properties of n-convex functions we refer the reader to [2].

Theorem 2.2. *If F satisfies condition A_n^*, $(n \geq 2)$, in $[a, b]$ and*

(a) $\bar{D}^nF(x) \geq 0$, $x \in (a, b) - E$, $|E| = 0$,

(b) $\bar{D}^nF(x) > -\infty$, $x \in (a, b) - S$, S a scattered set,

(c) $\limsup_{h\to 0} h\theta_n(F; x, h) \geq 0 \geq \liminf_{h\to 0} h\theta_n(F; x, h), \qquad x \in S,$

then F is n-convex.

Proof. In [2], Theorem 16, Bullen proves a similar result which implies this theorem. In place of condition A_n^* he uses a condition C_n which is just A_n together with B_{n-2}, but as was noted above these are implied by A_n^*.

3. The P_n^*-integral. The $\mathcal{P}_n$-integral, as originally defined [10] and as revised [3], does not give as strong a theorem on trigonometrical series as the P^n-integral because the $\mathcal{P}_n$-major and minor functions are required to possess $(n-1)$st Peano derivatives *everywhere* on (a, b) or $[a, b]$, the interval of integration, while it is known only that the sum function of the series obtained by formally integrating a $(C, n-2)$ summable series term-by-term n times possesses an $(n-1)$st Peano derivative almost everywhere. We are thus led to a definition of an nth order integral which relaxes the condition on the $(n-1)$st derivative. It was the same motivation in the case of convergence that led Burkill [5] to modify the definition of the *CP*-integral to obtain the *SCP*-integral.

DEFINITION 3.1. The functions $Q(x)$ and $q(x)$ are called P_n^*-major and minor functions, respectively, of $f(x)$ on $[a, b]$ if

(3.1) $\qquad Q(x)$ and $q(x)$ satisfy condition A_n^* on $[a, b]$;

(3.2) $$Q_{(k)}(a+) = q_{(k)}(a+) = 0; \qquad 0 \le k \le n-1;$$

(3.3) $$\underline{D}^n Q(x) \ge f(x) \ge \bar{D}^n q(x), \quad \text{in} \quad [a, b]-E, \qquad |E| = 0;$$

(3.4) $$\underline{D}^n Q(x) > -\infty, \qquad \bar{D}^n q(x) < +\infty, \qquad x \in [a, b]-S, \ S \text{ a scattered set};$$

(3.5) $$\limsup_{h\to 0} h\theta_n(Q; x, h) \ge 0 \ge \liminf_{h\to 0} h\theta_n(Q; x, h)$$
$$\limsup_{h\to 0} h\theta_n(q; x, h) \ge 0 \ge \liminf_{h\to 0} h\theta_n(q; x, h) \quad \text{for} \quad x \in S.$$

THEOREM 3.1. *For every pair* $Q(x)-q(x)$, *satisfying* (3.1)–(3.5) *the difference* $Q(x)-q(x)$ *is n-convex in* $[a, b]$.

Proof. This is the Lemma of [7].

THEOREM 3.2. *For every pair* $Q(x)$, $q(x)$ *satisfying* (3.1)–(3.5) *the functions* $Q_{(r)}(x)-q_{(r)}(x)$, $0 \le r \le n-2$, $\{Q(x)-q(x)\}_{(n-1)+}$ *and* $\{Q(x)-q(x)\}_{(n-1)-}$ *are monotonic increasing on* $[a, b]$. *In particular* $Q(x)-q(x) \ge 0$.

Proof. Since $M(x) \equiv Q(x)-q(x)$ is n-convex in $[a, b]$ it follows that $M^{(r)}(x)$ exists and is continuous on $[a, b]$, $1 \le r \le n-2$, $M_{(n-1)-}(x)$, $M_{(n-1)+}(x)$ exist and are monotonic increasing on $[a, b]$, and $M_{(n-1)-}(x) = (M^{n-2}(x))'_-$, $M_{(n-1)+}(x) = (M^{n-2}(x))'_+$ (Theorem 7, [2]). We have then $M_{(n-1)+}(x) = (M^{n-2}(x))'_+ \ge (M^{(n-2)}(a))'_+ = M_{(n-1)+}(a) = 0$, $x \in [a, b]$, and so $M^{n-2}(x)$ is monotonic increasing in $[a, b]$ (see, e.g. [13], p. 354, Example IV). But then $(M^{(n-3)}(x))' = M^{(n-2)}(x) \ge M^{(n-2)}(a+) = 0$, on $[a, b]$ which shows that $M^{(n-3)}(x)$ is monotonic increasing on $[a, b]$, i.e. $M^{(n-3)}(x) \ge 0$. Continuing in this way we show that the derivatives of $M(x) = Q(x)-q(x)$ have the properties stated in the theorem.

https://doi.org/10.4153/CMB-1978-026-1 Published online by Cambridge University Press

DEFINITION 3.2. If corresponding to $\varepsilon > 0$ there exists a pair $Q(x)$, $q(x)$ satisfying the conditions (3.1)–(3.5) and such that

$$Q(b-) - q(b-) < \varepsilon,$$

then f is said to be P_n^*-integrable over $[a, b]$.

THEOREM 3.3. *If f is P_n^*-integrable over $[a, b]$ then it is P_n^*-integrable over $[a, x]$ for each $x \in [a, b]$.*

Proof. Obvious.

THEOREM 3.4. *If f is P_n^*-integrable over $[a, b]$ there is a function $F(x)$ which is the* inf *of all major functions of $f(x)$ and the* sup *of all minor functions.*

Proof. This follows in the usual way.

DEFINITION 3.3. If $f(x)$ is P_n^*-integrable over $[a, b]$ the P_n^*-integral of $f(x)$ over $[a, x]$, $x \in [a, b]$, is defined to be $F(x)$ where $F(x)$ is the function of Theorem 3.4. We write

$$F(x) = P_n^* \int_a^x f(t)\, dt, \qquad x \in [a, b].$$

The proof of the following theorem is straightforward, (see [3], [7], and [10]).

THEOREM 3.5. *If $f(x)$ is P_n^*-integrable and $F(x)$ is the function of* Definition 3.3, *then*

(i) *$F(x)$ is continuous on $[a, b]$;*

(ii) *For every major and minor function $Q(x)$ and $q(x)$ the differences $Q(x) - F(x)$ and $F(x) - q(x)$ are n-convex in $[a, b]$;*

(iii) *$F(x)$ possesses derivatives $F_{(k)}(x)$, $1 \le k \le n-2$;*

(iv) *$F(x)$ is n smooth in (a, b).*

We do not have the power of proving integrability on sub-intervals and additivity of the integral on abutting intervals but this is not surprising since additivity on abutting intervals is closely connected with the existence of the $(n-1)$st one-sided derivatives of $F(x)$ and $Q(x)$ (see [8]).

It is easy to prove that the unsymmetric P^n-integral of [3] is included in the P_n^*-integral.

The relationship between the P_n^*-integral and the symmetric P^n-integral of [7] is described in the following theorem:

THEOREM 3.6. *If $f(x)$ is P_n^*-integrable on $[a, b]$ then $f(x)$ is P^n-integrable on $[a, b]$ with respect to any basis $a = \alpha_1 < \alpha_2 < \cdots < \alpha_n = b$. Moreover, if*

$$F(x) = P_n^* \int_a^x f(t)\, dt,$$

https://doi.org/10.4153/CMB-1978-026-1 Published online by Cambridge University Press

then, for $\alpha_s \le x < \alpha_{s+1}$, *we have*

$$(3.6)\qquad (-1)^s \int_{(\alpha_i)}^{x} f(t)\, d_n t = F(x) - \sum_{i=1}^{n} \lambda(x; \alpha_i) F(\alpha_i),$$

where

$$\lambda(x; \alpha_i) = \prod_{j \neq i} \frac{(x-\alpha_j)}{(\alpha_i - \alpha_j)}.$$

Proof. Let $Q(x)$, $q(x)$ be P_n^*-major and minor functions, respectively, of $f(x)$ on $[a, b]$. Then

$$(3.7)\qquad \bar{Q}(x) = Q(x) - \sum_{i=1}^{n} \lambda(x; \alpha_i) Q(\alpha_i)$$

$$(3.8)\qquad \bar{q}(x) = q(x) - \sum_{i=1}^{n} \lambda(x; \alpha_i) q(\alpha_i)$$

are P^n-major and minor functions, respectively, of $f(x)$ on $[a, b]$. Moreover given $\varepsilon > 0$, $Q(x)$ and $q(x)$ may be chosen so that $\bar{Q}(x) - \bar{q}(x) < \varepsilon$, $x \in [a, b]$ and then (3.6) follows from (3.7) and (3.8).

In [3] Bullen proves the equivalence of the $C_{n-1}P$-integral [4] and his unsymmetric P^n-integral:

THEOREM 3.7. (Theorem 16, [3]): *f is P^n-integrable on $[a, b]$ if and only if it is $C_{n-1}P$-integrable in $[a, b]$. If F is the P^n-integral of f then*

$$F_{(n-1)}(x) = C_{n-1}P \int_a^x f(t)\, dt,$$

and

$$F(x) = P \int_a^x C_1 P \int_a^{x_1} C_2 P \int_a^{x_2} \cdots C_{n-1} P \int_a^{x_{n-1}} f(t)\, dt\, dx_{n-1} \cdots dx_1.$$

The unsymmetric integral of [3] thus is an n-fold iterated integral while the symmetric integral of [7] differs from the P_n^*-integral by a polynomial of degree $(n-1)$. The relationship between the integrals in Theorem. 3.6 may be described in a manner which is more relevant to our investigation by rewriting (3.6) in the form

$$(3.9)\qquad (-1)^s \int_{(\alpha_i)}^{x} f(t) d_n t = V_n(F; \alpha_1, \alpha_2, \ldots, \alpha_n, x) \cdot \prod_{i=1}^{n} (x - \alpha_i),$$

where $V_n(F; \alpha_1, \alpha_2, \ldots, \alpha_n, x)$ is the divided difference of order n of F over the points $\alpha_1, \alpha_2, \ldots, \alpha_n, x$. Thus the definite symmetric P^n-integral is, except for a multiplicative constant, the nth divided difference of the P_n^*-integral which may be thought of as an n-fold integral.

https://doi.org/10.4153/CMB-1978-026-1 Published online by Cambridge University Press

This explains why our Theorem 4.3 gives the representation of the coefficients of a trigonometrical series in terms of a divided difference of the P_n^*-integral.

4. Trigonometric Series. Following the notation of James [11] we identify the following conditions which may be imposed on series (1.1):

(4.1) $$a_n = o(n^k), \qquad b_n = o(n^k),$$

(4.2) $$A_n^{k-1}(x_0) = o(n^k),$$

(4.3) $$a_0/2 + \sum_{n=1}^{\infty} a_n(x_0) = f(x_0), \qquad (C, k).$$

We integrate series (1.1) formally term-by-term to obtain:

(4.4) $$\frac{a_0 x}{2} + \sum_{n=1}^{\infty} \frac{a_n \sin nx - b_n \cos nx}{n} \equiv \tfrac{1}{2}a_0 x - \sum_{n=1}^{\infty} \frac{b_n(x)}{n} \equiv \tfrac{1}{2}a_0 x - \sum_{n=1}^{\infty} c_n(x).$$

We shall make use of the following theorem:

THEOREM 4.1. (Theorem 3.1, [11]). *If condition* (4.1) *is satisfied, then the series obtained by integrating* (1.1) *formally term-by-term* $k+2$ *times converges to a continuous function* $F(x)$. *If conditions* (4.1) *and* (4.2) *are both satisfied, then* $D^{k+2-2r}F(x_0)$ *exists for* $1 \le r \le (k+1)/2$ *and* F *is* $(k+2)$*-smooth at* x_0. *If conditions* (4.1) *and* (4.3) *both hold, then* F *is* $(k+2)$*-smooth at* x_0 *and*

(4.5) $$\frac{a_0 x_0^{2r}}{2(2r)!} + (-1)^r \sum_{n=1}^{\infty} \frac{a_n(x_0)}{n^{2r}} = D^{k+2-2r}F(x_0), \qquad (C, k-2r),$$

for $0 \le r \le (k+1)/2$.

THEOREM 4.2. *Supose the series* (1.1) *is summable* (C, k) *to a finite function* $f(x)$ *for all* $x \in [0, 2\pi) - E$, *where* E *is at most countable, and let* $f(x) = 0$, $x \in E$. *If* $A_n^{(k-1)}(x) = o(n^k)$ *for* $x \in E$ *and* $B_n^{k-1}(x) = o(n^k)$ *for* $x \in [0, 2\pi]$ *then there exists a set* $F \subset [0, 2\pi]$, $|F| = 2\pi$, *such that* $f(x)$, $f(x)\cos px$, $f(x)\sin px$ *are each* P_n^**-integrable on* $[\alpha, \alpha + 2\pi]$, $\alpha \in F$.

Proof. The series obtained by integrating (1.1) formally $(k+2)$ times converges uniformly to a continuous function $F(x)$. It follows from Theorem 4.1 and the proof of Theorem 3.2 [11] that $F_{(r)}(x)$, $0 \le r \le k$, exists in $[0, 2\pi]$, $D^{(k+2)}F(x)$ exists and equals $f(x)$ in $[0, 2\pi] - E$, and $F(x)$ is n-smooth at each point of $(0, 2\pi)$. Moreover the set of points where either $\underline{D}^n F(x) = -\infty$ or $\bar{D}^n F(x) = +\infty$ is a scattered set ([11], Theorem 5.1).

It is well known that the series (4.4) is summable $(C, k-1)$ almost everywhere in $[0, 2\pi]$. Let α be a point of the set A_0 of summability of (4.4). Since the function $F(x)$ is also the function obtained by integrating (4.4) formally $k+1$ times, it follows from Theorem 4.1 that $D^{(k+1)}F(\alpha)$ exists. We

https://doi.org/10.4153/CMB-1978-026-1 Published online by Cambridge University Press

have, for k even, for $\alpha \in A_0 \cap([0, 2\pi]-E)$,

$$\frac{F(\alpha+h)-F(\alpha-h)}{2}=\sum_{r=0}^{k/2} D_{2r+1}F(\alpha)\frac{h^{2r+1}}{(2r+1)!}+o(h^{k+1}) \tag{4.6}$$

and, since $D^{k+2}F(\alpha)$ exists,

$$\frac{F(\alpha+h)+F(\alpha-h)}{2}\sum_{r=0}^{k+2/2} D_{2r}F(\alpha)\frac{h^{2r}}{(2r)!}+o(h^{k+2}), \tag{4.7}$$

and similar equalities hold when k is odd. Together, (4.6) and (4.7) show that $F_{(k+1)}(\alpha)$ exists which, of course, equals $D^{k+1}F(\alpha)$. Now it is clear that the function defined by

$$Q(x)=F(x)-\sum_{r=1}^{k+1} F_{(r)}(\alpha)\frac{(x-\alpha)^r}{r!}$$

is both a P^*_{k+2}-major and minor function for $f(x)$ on $[\alpha, \alpha+2\pi]$. Moreover fo $x \in [\alpha, \alpha+2\pi]$,

$$P^*_{k+2}\int_\alpha^x f(t)\,dt=F(x)-\sum_{r=1}^{k+1} F_{(r)}(\alpha)\frac{(x-\alpha)^r}{r!}\equiv G_0(x).$$

As in [11] we can write for $x \in [0, 2\pi]-E$

$$\sum_{n=0}^{\infty} u_n(x)=f(x)\cos px, \qquad (C, k), \tag{4.8}$$

where $u_n=o(n^k)$, $U_n^{k-1}(x)=o(n^k)$ for all x, $u_n(x)$ is the nth term of the series which is the formal product of series (1.1) and $\cos px$, and $U_n^{k-1}(x)$ is the $(k-1)$st Cesàro mean of the same series.

An application of Theorem 4.1 shows that the series obtained by integrating (4.8) formally term-by-term $k+2$ times converges uniformly to a continuous function $G(x)$ such that

$$\lim_{h\to 0} h\theta_{k+2}(G; x, h)=0,$$

for all x, and,

$$\frac{u_0x^{2r}}{2(2r)!}+(-1)^r\sum_{n=1}^{\infty}\frac{u_n(x)}{n^{2r}}=D_{k+2-2r}G(x), \qquad (C, k-2r),$$

for $0 \le r \le (k+1)/2$ and $x \in [0, 2\pi]-E$.

Moreover it was shown in [7] that $G_{(k)}(x)$ exists everywhere in $[0, 2\pi]$, and it follows, as before, that $G_{(k+1)}(x)$ exists in a set A_P of full measure in $[0, 2\pi]$. We have then, since condition (3.4) of Definition 3.1 is obviously satisfied for $G(x)$,

$$P^*_{k+2}\int_{\beta_P}^x f(t)\cos pt\,dt=G(x)-\sum_{r=1}^{k+1} G_{(r)}(\beta_P)\frac{(x-\beta_P)^r}{r!}\equiv G_P(x), \tag{4.9}$$

https://doi.org/10.4153/CMB-1978-026-1 Published online by Cambridge University Press

for $x \in [\beta_P, \beta_P + 2\pi]$, $\beta_P \in A_P$. Similarly, if the series

$$\sum_{n=0}^{\infty} U_n(x)(=f(x)\sin px) \tag{4.10}$$

is the formal product of series (4.1) with $\sin px$ and $H(x)$ is the sum of the series obtained by integrating (4.10) formally $(k+2)$ times we have

$$P^*_{k+2} \int_{\gamma_P}^{x} f(t)\sin pt\,dt = H(x) - \sum_{r=1}^{k+1} H_{(r)}(\gamma_P)\frac{(x-\gamma_P)^r}{r!} \equiv H_P(x), \tag{4.11}$$

for $x \in [\gamma_P, \gamma_P + 2\pi]$, $\gamma_P \in B_P$, where B_P is a set full measure on $[0, 2\pi]$. The theorem follows by choosing $F = \bigcap_{P=0}^{\infty} (B_{P+1} \cap A_P)$.

THEOREM 4.3. *Under the hypothesis of Theorem* 4.2 *the coefficients of series* (1.1) *are given by*

$$a_P = 2(k+2)!\ V_{k+2}(G_P), \qquad P = 0, 1, 2, \ldots, \tag{4.12}$$

$$b_P = 2(k+2)!\ V_{k+2}(H_P), \qquad P = 1, 2, \ldots, \tag{4.13}$$

where $V_{k+2}(G_P) = V_{k+2}(G_P; x_1, x_2, \ldots, x_{k+2}, x_{k+3})$ *is the divided difference of* G_P *of order* $k+2$ *at the* $k+3$ *points*

$$\begin{aligned} B_1 &\equiv (x_1, x_2, \ldots, x_{k+2}, x_{k+3}), \\ &\equiv (\alpha-(k+2)\pi, \alpha-k\pi, \ldots, \alpha-2\pi, \alpha+2\pi, \ldots, \alpha+k\pi, \alpha+(k+2)\pi, \alpha), \end{aligned}$$

or

$$\begin{aligned} B_2 = (x_1, x_2, \ldots, x_{k+2}, x_{k+3}) \equiv (&\alpha-(k+1)\pi,\ \alpha-(k-1)\pi, \ldots, \\ &\alpha-2\pi, \alpha+2\pi, \ldots, \alpha+(k+1)\pi, \alpha+(k+3)\pi, \alpha) \end{aligned}$$

depending on whether k is even or odd.

Proof. In order to verify (4.12) for $P = 0$ we note first that

$$V_{k+2}(G_0) = V_{k+2}(F)$$

since any $(k+2)$ divided difference of a polynomial of degree $(k+1)$ is 0. Next we write

$$F(x) \equiv G_1(x) + \frac{a_0 x^{k+2}}{2(k+2)!}$$

where $G_1(x)$ is periodic of period 2π. The divided difference of order $(k+2)$ of the function $G_1(x)$ at the $(k+3)$ points of B is just the divided difference of the constant $G_1(\alpha)$ which is 0. Since the divided difference of the function x^{k+2} is equal to 1, we have

$$V_{k+2}(G_0) = \frac{a_0}{2(k+2)!}$$

Formula (4.12) for $P=1,2,\ldots$ may be verified in exactly the same way since the constant term in (4.8) is $a_P/2$. A similar remark applies to formula (4.13).

Because of Theorem 3.6, the formulae (4.12) and (4.13) may be written in terms of the P^n-integral. For example, (4.12) becomes

$$a_P = 2(k+2)!\ V_{k+2}\left(P^{k+2}\int_{(\alpha_i)}^{x} f(t)\cos pt\, dt\right), \qquad P=0,1,2,\ldots$$

where (α_i) is any basis in $[\alpha, \alpha+2\pi]$. This follows from (3.6) using the fact again that a divided difference of order $k+2$ of a polynomial of degree $k+1$ is 0.

Bibliography

1. P. S. Bullen, *Construction of Primitives of Generalized Derivatives with Applications to Trigonometric Series*, Can. J. Math. vol. **13** (1961) pp. 48–58.

2. ——, *A Criterion for n-Convexity*, Pacific J. Math. vol. **36** (1971) pp. 81–89.

3. ——, *The P^n-integral*, J. Australian Math. Soc., (2) vol. **14** (1972) pp. 219–236.

4. J. C. Burkill, *The Cesàra-Perron Scale of Integration*, Proc. London Math Soc. (2) vol **39** (1935) pp. 541–552.

5. ——, *Integrals and Trigonometric Series*, Proc London Math. Soc. (3) vol. **1** (1951) pp. 46–57.

6. George Cross, *The Expression of Trigonometrical Series in Fourier Form*, Can. J. Math. vol **12** (1960) pp. 694–698.

7. ——, *The P^n-integral*, Canad. Math. Bull. vol **18** (1975) pp. 493–497.

8. ——, *Additivity of the P^n-integral*, (to appear).

9. ——, *The $SC_{k+1}P$-integral and Trigonometric Series*, (to appear).

10. R. D. James, *Generalized nth Primitives*, Trans. Amer. Math. Soc. vol **76** (1954) pp. 149–176.

11. ——, *Summable Trigonometric Series*, Pacific J. Math. vol **6** (1956) pp. 99–110.

12. S. N. Mukhopadhyay, *On the Regularity of the P^n-integral*, Pacific Journal of Math., vol **55** (1974) pp. 233–247.

13. E. C. Titchmarsh, *Theory of Functions*, Oxford, 1939.

University of Waterloo
Waterloo, Ontario