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True-and-error models violate independence and yet they are testable

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Abstract

Birnbaum (2011) criticized tests of transitivity that are based entirely on binary choice proportions. When assumptions of independence and stationarity (iid) of choice responses are violated, choice proportions could lead to wrong conclusions. Birnbaum (2012a) proposed two statistics (correlation and variance of preference reversals) to test iid, using random permutations to simulate *p*-values. Cha, Choi, Guo, Regenwetter, and Zwilling (2013) defended methods based on marginal proportions but conceded that such methods wrongly diagnose hypothetical examples of Birnbaum (2012a). However, they also claimed that "true and error" models also satisfy independence and also fail in such cases unless they become untestable. This article presents correct true-and-error models; it shows how these models violate iid, how they might correctly identify cases that would be misdiagnosed by marginal proportions, and how they can be tested and rejected. This note also refutes other arguments of Cha et al. (2013), including contentions that other tests failed to violate iid "with flying colors", that violations of iid "do not replicate", that type I errors are not appropriately estimated by the permutation method, and that independence assumptions are not critical to interpretation of marginal choice proportions.

Keywords:

1 Introduction

Although there is much to admire in the approach of Regenwetter, Dana, and Davis-Stober (2011) for testing transitivity in choice experiments, Birnbaum (2011) criticized its focus on marginal choice proportions rather than response patterns. Birnbaum pointed out that when choice responses are not independent and identically distributed (iid), any method based strictly on marginal choice proportions could easily reach wrong substantive conclusions.

The main problem is that data generated from mixtures that include intransitive preference patterns can appear to satisfy transitivity by such methods. Birnbaum (2011) therefore suggested that one should analyze response patterns or at least, that iid should be tested before drawing any conclusions about transitivity from marginal choice proportions. Regenwetter, Dana, Davis-Stober, and Guo (2011) replied that, when response patterns are to be analyzed, one must collect more extensive data, and they stated that they were unaware of statistical tests of iid for small samples, such as the little study by Tversky (1969), or the small replication by Regenwetter et al. (2011).

Birnbaum (2012a) proposed two statistics using Monte Carlo simulation methods (based on Smith and Batchelder, 2008) to test iid with small samples. One statistic is the correlation coefficient between preference reversals and the gap in time between two presentations of the same choice problems. The other is the variance of preference reversals between responses to the same problems in different blocks of trials. These two statistics were designed to detect violations of iid that would occur if people systematically changed their true preferences during the study. Reanalysis via these two tests indicated that the data of Regenwetter et al. (2011) violate the assumptions of iid. Violations of iid might arise from many different sources, including the possibility that a true-and-error (TE) model describes responses in choice tasks.

Cha, Choi, Guo, Regenwetter, and Zwilling (2013) defended methods analyzing only marginal proportions but they conceded that those methods wrongly diagnose hypothetical examples presented by Birnbaum (2012a). However, they falsely claimed that TE models also assume iid and would therefore also fail to correctly diagnose such cases. They next claimed if a TE model allowed a person to change "true" preferences between blocks (to violate iid), the model would become untestable and therefore useless. These statements and

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others in that paper need to be corrected, because TE models violate iid, they correctly diagnose such hypothetical examples, and yet they are testable.

2 True-and-error models

Cha et al. (2013, p. 67) claimed that the "standard true-and-error model" satisfies independence. However, I specifically argued that violations of iid are produced in TE models within an individual's data when the same person changes "true" preferences from block to block (Birnbaum, 2011, p. 680) and that iid could also be violated between-persons when different people have different "true" preferences (Birnbaum, 2011, p. 679). The so-called "standard" true-and-error model presented by Cha et al. is merely a special case of the TE model in which there is only a single true preference pattern and not a mixture. A brief history of these models might be useful.

The models now called "true and error" trace their development to a paper by Lichtenstein and Slovic (1971), who wished to state a clear null hypothesis in which preference reversals between two ways of evaluating lotteries could be analyzed. Sarah Lichtenstein developed the basic concepts from "common sense" (Slovic, 2013 and Lichtenstein, 2013, April 3, personal communications).

A paper by Conlisk (1989, Appendix I) presented a very clear statement of a simpler form of their model, which was used to justify the statistical test of correlated proportions, which has been the standard test whether or not choice proportions in two choice problems are significantly different. In his version of the model, it was assumed that all choice problems had the same rate of error (whereas the earlier model of Lichtenstein & Slovic [1971] allowed that choice and bidding tasks might have different rates of error).

Harless and Camerer (1994) applied the assumption of a single error rate and stated that a more elaborate theory had not yet been developed. The theory of homogeneous error rates has sometimes been called the "tremble" theory because it seemed to say that response errors arise between intention and response, as if the only reason for pushing the wrong response key was the result of a "trembling hand". However, that interpretation is not necessary and might be misleading, because errors more likely arise earlier in processing (Birnbaum, 2011). A person might misread the problem, misremember the information, misaggregate the information, misremember the decision, or push the wrong key, any of which could produce an error. Some of the rival models of error have been reviewed and analyzed by Wilcox (2008), Carbone and Hey (2000), and Loomes and Sugden (1998).

When testing transitivity, the constant-error-rate version implies that inequality of different types of intransitive preference patterns could be taken as evidence for a violation of transitivity (Loomes, Starmer, & Sugden, 1991). Sopher and Gigliotti (1993) disputed this interpretation, however, noting that, if error rates for different choice problems are not equal, then asymmetry of different types of cycles would not qualify as evidence of systematic intransitivity. Their approach could, in turn, be criticized because the model had (in principle) more parameters than degrees of freedom in the data to which it would be applied. Perhaps this limitation is why someone might claim that these models become untestable.

However, as noted in several recent papers (Birnbaum, 2011, p. 678; Birnbaum & Bahra, 2012a; Birnbaum & Schmidt, 2008; Birnbaum & LaCroix, 2008), the use of replications, as proposed by Birnbaum (2004) and improved upon in subsequent papers, provides a way to estimate error rates that may differ for different choice problems and still leave degrees of freedom to test the model. In particular, it is assumed that repetitions of the same choice problem by the same person in the same block of trials are governed by the same true preferences and differ only because of error.

In tandem with these theoretical developments, recent empirical results forced consideration of models that can violate iid (in contrast to earlier models, which did not allow violation of iid). Birnbaum and Bahra (2007b) reported cases where individuals perfectly reversed preferences on twenty out of twenty choice problems between blocks of trials; such cases are extremely unlikely given the assumption of iid within a person. Instead, it seems more plausible that individuals changed true preferences from block to block. Birnbaum and Bahra (2012b) repeated the experiment with different people and varied procedures and continued to find perfect reversals and other strong evidence against iid. Birnbaum (2011, 2012a) noted that, if a person changed "true" preferences during a long study, it could create violations of iid. When iid is violated, marginal choice proportions might be misleading.

That is the crux of Birnbaum's (2011) criticism of the approach of Regenwetter et al. (2011), which was focused on binary choice proportions. Cha et al. (2013) did not accurately describe appropriate TE models. I describe here how one can apply TE models to the investigation of transitivity, if one is willing to do a more up-to-date study than that done by Tversky (1969). I will describe how such a study can be used to test iid, the TE models, and transitivity.

I dispute other assertions by Cha et al. (2013) concerning algorithms for testing iid via Monte Carlo simulation in Appendix A; but first, let us consider the simplest TE model for the simplest case to show that it violates response independence, even though the errors are independent.

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2.1 One choice problem presented twice per block

Imagine that one person is asked to respond to many different choice problems, and embedded in each block of, say, 50 trials (which include many different choice problems), a given choice problem (between A and B) is presented twice, separated by many other intervening choice problems that are termed *fillers* within each block. Each block is separated by another task that requires, for example, 50 trials of other choice problems, called *separators*. This study might be done with a different block of trials on each of several different days.

The two versions of the same choice problem are denoted Choices AB and $A'B'$. They might differ only in which button should be pressed in order to respond that A is preferred to B. The use of repetitions within blocks adds constraints that make TE models highly testable. This experimental paradigm is similar to that used by Birnbaum and Bahra (2012b, Exp. 3). [The design of Tversky (1969) and the replication by Regenwetter et al. (2011) did not include repetitions within blocks.]

Table 1 shows a hypothetical matrix of responses, where "*0*" indicates that the person chose alternative A or A' and "*1*" indicates a preference for B or B'. Each row represents a different block of trials, and the two entries within each row represent the responses to the two, separated presentations of the same choice problem in the same block. The marginal choice proportions are the column sums, divided by the number of blocks (20). The two marginal choice proportions are each 0.6.

Table 1: A hypothetical table of results to responses by the same person to the same choice problem presented twice each in 20 blocks of trials.

Blocks	Choice AB	Choice $A'B'$
1	*0*	*0*
2	*0*	*0*
3	*0*	*0*
4	*0*	*0*
5	*0*	*0*
6	*0*	*0*
7	*0*	*0*
8	*0*	*0*
9	*1*	*1*
10	*1*	*1*
11	*1*	*1*
12	*1*	*1*
13	*1*	*1*
14	*1*	*1*
15	*1*	*1*
16	*1*	*1*
17	*1*	*1*
18	*1*	*1*
19	*1*	*1*
20	*1*	*1*
Marginal choice proportion	0.6	0.6

Let each response combination in a block (row of Table 1) be called a *response pattern*. There are four possible response patterns: *00*, *01*, *10*, and *11*. In order to be consistent within a block, the person had to push opposite buttons on separate trials to indicate preference for A and A' (*00*), or for B and B' (*11*).

If *response independence* held, the probability of each response pattern would be given by the product of the marginal probabilities. For these data, the predicted proportion for *11* would be (.6)(.6) = .36, and the predicted proportion of reversals *01* or *10* would be (.4)(.6) + (.6)(.4) = 0.48. However, Table 2 shows that the response patterns do not satisfy independence. Instead, the proportion of *11* was 0.6 (instead of 0.36), and this person was perfectly consistent within blocks (0% reversals, instead of 48%), thereby violating response independence.

By either the Fisher exact test on Table 2, or by means of Birnbaum's (2012a) test on the variance of preference reversals between blocks (on Table 1), one can reject response independence with $p < .0001$. In every case where this person chose A in a block, that same person chose A' in the same block, and in every block where the person chose B, that person chose B'. This person was perfectly consistent within blocks but changed preferences between blocks, resulting in violations of response independence.

A second, sequential type of violation of iid is also apparent in the Table 1. In particular, this person chose A and A' on every choice for the first 8 blocks and then switched to choosing B and B'. Something changed over trials, resulting in a systematic violation of stationarity within each column. This type of sequential violation is detected by Birnbaum's (2012a) correlation test, which evaluates the correlation coefficient between the mean number of preference reversals between rows and the gap (in blocks) between rows. In this case, the correlation, r = 0.94, and $p < .0001$.

These two statistics show that the data in Table 1 systematically violate the assumptions of independence and stationarity (a.k.a., iid), but they do not say how or why. One model that can describe certain violations of independence (but not all) is the true-and-error (TE) model.

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Table 2: Data from Table 1 are organized in a cross-tabulation to evaluate independence. These data violate independence, because products of marginal proportions fail to reproduce joint proportions.

	Response to Choice $A'B'$	
Response to Choice *AB*	*0*	*1*
0	8	0
1	0	12

2.2 True-and-error model for One Choice Problem Repeated in Each Block

A TE model can be expressed for this situation as follows: Suppose that two responses by the same person to the same choice problem within a block are governed by the same "true" preferences, except for random error, and suppose that responses in different blocks might be governed by different "true" preferences (Birnbaum, 2011; Birnbaum & Bahra, 2012a).

Suppose that if the person is in the "true state" of preferring *A* the error probability is *e*, which is the probability of choosing *B* when the true preference is *A*. Assume that if the person is in the "true state" of preferring *B*, that the probability of making an error is also *e*. Suppose that the probability of being in the state of truly preferring *B* is *p*. Assume that *p* and the error rate, *e*, are stationary (remain constant) throughout the study, that errors are mutually independent, and that $e < ½$.[1]

Do these iid assumptions concerning *p* and *e* mean that choice responses are independent, as in the so-called "standard true-and-error model" by Cha et al. (2013)? No, absolutely not. This TE model violates response independence, as shown next.

The predicted probabilities of the four response patterns are as follows:

$$P(00) = p(e)(e) + (1–p)(1–e)(1–e) \quad (1)$$

$$P(01) = P(10) = e(1–e) \quad (2)$$

$$P(11) = p(1–e)(1–e) + (1–p)(e)(e) \quad (3)$$

$P(11)$ is the predicted probability in the TE model for showing the response pattern *11* in a block, *p* is the probability of "truly" preferring *B* and B', and *e* is the probability of a random error. The marginal choice probability of choosing *B* in a single *AB* choice problem is given as follows: $P(1*) = P(10) + P(11) = p(1–e) + (1–p)e$, where $P(1*)$ is the marginal, binary probability of choosing *B* over *A*.

Do Equations 1–3 satisfy response independence? That is, can we write $P(11) = P(1*)P(*1)$? No. If p = .6 and $e = 0$, for example, this model is perfectly consistent with the data of Table 2 that systematically violate response independence.

This violation of response independence by the TE model does not require the error rate to be zero; for example, if $p = 0.63$ and $e = 0.11$, then $P(1*) = P(*1) = 0.6$, so $P(1*)P(*1) = 0.36$, whereas $P(11) = 0.50$.

So even though errors are independent of each other, responses are not predicted to be independent, except in special cases, such as when $p = 1$. Put another way: even though probability of the conjunction of two errors is represented by the product of their probabilities, the probability of a conjunction of two responses is not given by the product of response probabilities, but instead by Equations 1–3.

Cha et al. (2013, Equation 6) presented a model that satisfies response independence and called it the "standard true-and-error" (STE) model. Independence can hold in special cases of TE, such as when $p = 1$, but Expressions 1-3 do not satisfy independence in general (Birnbaum, 2011). The Cha et al. STE model is not a standard TE model; instead, it is only a special case in which there is only one true preference pattern; that model is not relevant to this debate, as noted by Birnbaum (2011, p. 680).

Cha et al. (2013, p. 70) next claimed that if a TE model allowed that people changed true preference between blocks (to account for violations of iid), the TE model would become un-testable. That claim is also false, even in this simplest case of a single choice problem, as shown next.

Table 3 displays four different hypothetical cross-tabulations of a repeated choice to show that response independence and error independence can separately fly or fail; that is, failure or satisfaction of one neither guarantees nor refutes the other. (Entries in Table 3 sum to 100, so they can be viewed as percentages, or divided by 100 to facilitate calculations on proportions). Each of these models (independence and TE) can be tested by a Chi-Square on the same 2×2 array with 1 df, since each model uses two parameters. Response independence implies that each cell entry can be reconstructed from the row and column marginal proportions, and TE says that each entry can be reconstructed from *p* and *e*, using Equations 1-3.

In the two examples in Table 4 that perfectly satisfy response independence, each entry can be perfectly reproduced as the products of their marginal choice proportions.

[1]More complex models can also be tested (Birnbaum, 2012b; Birnbaum & Schmidt, 2012). For example, it is possible to test models in which error rates depend on a person's "true" preference state.

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Table 3: Hypothetical cross-tabulations illustrating that response independence and TE independence can be separately satisfied or violated by repeated responses to a single choice problem. Both models are satisfied in the example in the upper left and both are violated in the case in the lower right.

	Independence satisfied		Independence violated	
TE satisfied	A'	B'	A'	B'
A	16	24	30	10
B	24	36	10	50
TE violated	A'	B'	A'	B'
A	8	32	20	25
B	12	48	5	50

In the two examples of Table 4 that violate response independence, people are more consistent than expected (i.e., the entries on the major diagonal are greater than expected from products of marginal proportions). In the two cases violating TE, the matrices are not symmetric (i.e., one type of preference reversal between replications is more probable than the other).

Treating the hypothetical entries in the tables as observed frequencies, the $\chi^2(1)$ for cases violating independence are 34.0 and 16.5 in the first and second rows, respectively. The $\chi^2(1)$ for examples violating TE are 9.66 and 22.14, respectively. The critical value of $\chi^2(1)$ with $\alpha = .01$ is 6.63, so each of these would be considered "significant". In the two cases satisfying TE, the parameters are $p = 1$ and $e = .4$ in the case that also satisfies response independence and $p = .63$ and $e = 0.11$ in the case that violates response independence. Chi-Squares are zero for models that fit perfectly.[2]

These four examples refute the claims by Cha et al. (2013) that the standard TE model implies independence, and they refute the claim that if TE models violated independence, they would be rendered un-testable. See also Birnbaum and Bahra (2012a, pp. 407–408), including their example in which the TE model would be rejected.

[2]In order to justify comparing the calculated Chi-Squares with the Chi-Square distribution to test either response independence or TE independence, higher order independence assumptions would be made; namely, each datum in the table entered its cell independently of the other entries. These higher-order assumptions to justify the significance test do not assume response independence. It is reasonable to question in empirical applications whether or not these higher-order assumptions are satisfied.

2.3 True-and-error model with multiple subjects and one block each

Now suppose that the data in Table 1 instead represented results from 20 *different* participants, each of whom participated in only one block (instead of 20 blocks by the same person). That is, suppose each row of Table 1 represents responses by a different person, tested separately. What is the "standard" TE model for that situation? In that case, Equations 1–3 are the same, but the interpretations of parameters differ. It is again assumed that the same person in the same block of trials is governed by the same "true" preferences, but in this case, it is assumed that different people might have different "true" preferences. In this case, p represents the proportion of people who "truly preferred B" in their first (and only) block. In this case, violation of independence arises because different people have different "true" preferences.

To emphasize the distinction between these two cases, Birnbaum and Bahra (2012a) used the terms *i*TET (*individual* True and Error Theory) and *g*TET (*group* True and Error Theory) to denote cases where violations of iid arise from an individual changing preferences from block to block (*i*TET) and where violations of iid arise from individual differences in a group of people (*g*TET), respectively. Both of these cases were discussed in Birnbaum (2011, p. 679) and Birnbaum (2011, p. 680), respectively.

In the simplest version of *g*TET, e represents an error rate that is assumed to be the same for all persons. When there are different people, however, one might hypothesize that different people might have different amounts of noise in their data, so violations of this assumption might show up as violations of this model. Indeed, Birn-

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Table 4: Hypothetical data in a test of transitivity for a single person who receives three choice problems twice in each of 20 blocks of choice problems, where each of the six choice trials was separated by many filler trials and each block of trials was also separated by multiple separator trials.

	Choice Problems					
Blocks	*AB*	*BC*	*CA*	$A'B'$	$B'C'$	$C'A'$
1	*0*	*0*	*0*	*0*	*0*	*0*
2	*0*	*0*	*0*	*0*	*0*	*0*
3	*0*	*0*	*0*	*0*	*0*	*0*
4	*0*	*0*	*0*	*0*	*0*	*0*
5	*0*	*0*	*0*	*0*	*0*	*0*
6	*0*	*0*	*0*	*0*	*0*	*0*
7	*0*	*0*	*0*	*0*	*0*	*0*
8	*0*	*0*	*0*	*0*	*0*	*0*
9	*1*	*1*	*1*	*1*	*1*	*1*
10	*1*	*1*	*1*	*1*	*1*	*1*
11	*1*	*1*	*1*	*1*	*1*	*1*
12	*1*	*1*	*1*	*1*	*1*	*1*
13	*1*	*1*	*1*	*1*	*1*	*1*
14	*1*	*1*	*1*	*1*	*1*	*1*
15	*1*	*1*	*1*	*1*	*1*	*1*
16	*1*	*1*	*1*	*1*	*1*	*1*
17	*1*	*1*	*1*	*1*	*1*	*1*
18	*1*	*1*	*1*	*1*	*1*	*1*
19	*1*	*1*	*1*	*1*	*1*	*1*
20	*1*	*1*	*1*	*1*	*1*	*1*
Marginal choice proportion	0.6	0.6	0.6	0.6	0.6	0.6

baum and Gutierrez (2007) found evidence that this simple model could be rejected in favor of a more complex model in which different people had different error multipliers.[3]

In *i*TET, stability of *e* means that the individual maintains the same error rate throughout the study, which would be violated in cases where a person becomes better with practice or where she might become fatigued over trials. It is important to realize that these theorized violations of the model can be tested against more general models that allow the violations of these assumptions.

It is also important to keep in mind that violations of independence in these two cases, as interpreted by the *i*TET and *g*TET models come from different origins. So even though the equations are the same, the violations of independence have different empirical interpretations.

For example, the correlation test proposed by Birnbaum (2011, 2012a) is really anticipated to be violated only in the case of *i*TET, because that statistic is sensitive to violations of iid that arise from sequential effects; that is, preference reversals between rows that are related to the temporal separation between blocks. If that test were to show significant violations in the *g*TET paradigm, it would mean that the order in which people were tested somehow affected the results, for example, that participants communicated via some type of ESP that depended on the sequence in which they were tested. Therefore, the data of Table 1 are not realistic for the *g*TET situation, if the rows represented the order in which different participants were separately tested. A more realistic example for *g*TET could be created from Table 1 by randomly switching rows, which would create a random sequence within each column; however, that resorting of rows would preserve the same cross-tabulation as in Table 2.

At this point, it is also worthy of note that, whereas the results in Table 2 are perfectly consistent with *i*TET, that model does not allow one to predict nor to fit the sequential information in the data of Table 1. In order to describe the obvious sequential effects in Table 1, one would need additional theory, such as that proposed by Birnbaum (2011, p. 680), and described more fully here in Appendix B. For example, one might theorize that the parameters of a decision making model (such as the TAX model of Birnbaum, 2008) might drift from block to block as in a random walk.

Although one might propose a TE model in which an individual randomly and independently samples a preference pattern in each block of trials, I doubt that such a model would be an accurate descriptive model, based on the findings with the correlation tests on data of Birnbaum and Bahra (2012b) and of Regenwetter et al. (2011) as analyzed in Birnbaum (2012a).

Thus, while Table 2 would be consistent with a TE model and would refute iid, the sequential effects in Table 1 would require some additional theory to be described. In the case of *i*TET, that theory might involve a model of risky decision-making in which parameters of the model change (Birnbaum, 2011); and in the case of *g*TET, that extra theory might involve communication among participants via ESP or some form of "cheating."

[3] Birnbaum and Gutierrez (2007) also hypothesized that each person might have only a single pattern of "true" preferences; however, their study was not designed to test that conjecture, as this issue is moot in a *g*TET study. When it was tested by Birnbaum and Bahra (2007b, 2012a, 2012b), this hypothesis was rejected.

Cha et al. (2013, p. 68) made a peculiar argument about the relation between these two cases as follows: First, they claimed that *i*TET models imply independence. (They do not.) Second, they argued that, if Birnbaum and Bahra (2012b) found violations of independence within a person (which they did), it would invalidate the between-subjects model of Birnbaum and Bahra (2007a), which it would not. The model in the 2007a paper was a *g*TET model that violates iid because of individual differences. In essence, Cha et al. argued that if iid is violated in the case of *i*TET it means that *g*TET is rendered "untenable". That is neither logical nor reasonable.

One might plausibly argue just the opposite; namely, if individuals change true preferences from block to block, it seems likely that people would also differ from each other. Therefore, evidence of violations of iid in the *i*TET case (Birnbaum & Bahra, 2012b) would seem from common sense to suggest that one *should* expect to find violations of iid in the *g*TET case (Birnbaum & Bahra, 2007a).

But keep in mind that neither *i*TET nor *g*TET satisfy iid and there is no *a priori* connection between the two. That is, violations of iid in one neither guarantees nor rules out violations of iid in the other case. For example, it is logically possible (if intuitively implausible) that, although each person might change true preferences from block to block, all humans might go through such changes in the same exact sequence.

Empirical results show that neither between-subjects data of Birnbaum and Bahra (2007a) nor the within subjects data of Birnbaum and Bahra (2007b, 2012a, 2012b) satisfied independence. Neither the *i*TET nor *g*TET models used in those studies implies independence. Further, empirical intuition leads one to anticipate that if iid is violated in the *i*TET case, one should expect to find violations in the *g*TET case. So, the claims by Cha et al. (2013, p. 68) that violations of iid in Birnbaum and Bahra (2012b) render the *g*TET model of Birnbaum and Bahra (2007a) "not tenable" is not correct.

The violation of response independence is the main issue in this debate, which is that such violations could lead to wrong conclusions concerning transitivity, if a person analyzed only marginal choice proportions. The assumption of iid is not merely some statistical nicety that justifies significance tests; violations mean that the interpretation of marginal proportions can be misleading, as shown in the next section.

3 Testing independence, TE models, and transitivity

Transitivity of preference asserts that, if *A* is preferred to *B* and *B* is preferred to *C*, then *A* should be preferred to *C*. To test this principle, we need at least three choices. To test TE models for this case, each of these three choices can be repeated within each block. So, this new experimental setup is like the previous one, except that within each block of choice problems, there are three choice problems that are each repeated within each block; these six problems are spaced out by multiple fillers, and blocks are separated as before.

3.1 Three choices repeated twice in each block

Suppose there are three choice problems: *AB*, *BC*, and *CA*. Let Choice $A'B'$ represent a second version of the *AB* choice that might require the person to switch buttons in order to indicate the same preference response. Choices $B'C'$ and $C'A'$ are similarly constructed.

Again, let us start with the paradigm of a single participant who serves in 20 blocks of trials that include six separated choice problems: 3 basic choice problems repeated within blocks, with all choices separated by intervening fillers, and blocks separated by numerous separators. Hypothetical data are shown in Table 4.

Data are coded so that *000* is the intransitive response pattern of choosing *A* over *B*, *B* over *C*, and *C* over *A*. The pattern *111* represents the intransitive cycle of choosing *B* over *A*, *C* over *B*, and *A* over *C*. The other six response patterns are transitive.

In Table 4 we see that the participant had perfectly intransitive response patterns within every single block of the study. The person began with the intransitive cycle *000* and switched to the opposite intransitive cycle, *111*. Weak stochastic transitivity is violated in the binary choice proportions, because $P(B \succ A) > ½$, $P(C \succ B) > ½$ and $P(A \succ C) > ½$. Yet the marginal choice proportions (0.6, 0.6, 0.6) are perfectly consistent with a mixture of linear orders, satisfying the triangle inequality, $P(AB) + P(BC) + P(CA) \leq 2$. If an investigator analyzed only marginal choice proportions, the conclusion from the triangle inequality would be that transitivity can be retained. By examining response patterns, however, it is easy to see that every individual response pattern was intransitive.

Cha et al. (2013, pp. 66–68) conceded this point, but they claimed that the TE models would also fail to detect intransitivity in such cases. However, that claim depends on their assumption that the TE model satisfies independence, which it does not.

The response patterns from Table 4 are cross-tabulated in Table 5. Table 5 is perfectly consistent with *i*TET in this case, which can violate response independence.

Table 5 also shows the response frequencies predicted from the iid model. The hypothetical data do not satisfy the predictions of iid at all. Among the predictions of iid, note that if the marginal choice proportions are each

Table 5: Analysis of response patterns from hypothetical data of Table 4.

Response pattern	Observed ABC	Observed $A'B'C'$	Observed both	Predicted ABC (iid)	Predicted both (iid)
000	8	8	8	1.28	.08
001	0	0	0	1.92	.18
010	0	0	0	1.92	.18
011	0	0	0	2.88	.41
100	0	0	0	1.92	.18
101	0	0	0	2.88	.41
110	0	0	0	2.88	.41
111	12	12	12	4.32	.93
Sum	20	20	20	20	2.81

0.6, the total probability that a response pattern will be repeated within blocks is only 0.14; so out of 20 trials, the person is expected to agree in choice pattern only 2.81 times within blocks. In these hypothetical data, however, the participants were perfectly consistent (20 agreements = 100%). Do real data show higher self-consistency than predicted by independence? They do (Birnbaum & Bahra, 2012a, Footnote 4; Birnbaum & Bahra, 2012b, Appendix H).

The examples in Birnbaum (2012a), like that in Table 5, are cases where the cross-tabulations are perfectly consistent with TE models and error rates are zero. Perhaps these perfect features of the examples led Cha et al. (2013, p. 70) to state that, if TE models are allowed to violate iid, they always fit perfectly and are therefore not testable. The next sections show the appropriate TE models for testing transitivity in the presence of error, and illustrate cases where the TE model leads to different conclusions from those reached by the methods used by Regenwetter et al. (2011). Examples where TE models can be rejected are also presented.

3.2 True-and-error model for Test of Transitivity

There are 8 possible response patterns for each test with three choice problems testing transitivity: *000*, *001*, *010*, *011*, *100*, *101*, *110*, and *111*. In the TE model, the predicted probability of showing the intransitive pattern, *111*, is given as follows:

$$\begin{aligned} P(111) &= p_{000}(e_1)(e_2)(e_3) + p_{001}(e_1)(e_2)(1-e_3) \\ &+ p_{010}(e_1)(1-e_2)(e_3) + p_{011}(e_1)(1-e_2)(1-e_3) \\ &+ p_{100}(1-e_1)(e_2)(e_3) + p_{101}(1-e_1)(e_2)(1-e_3) \\ &+ p_{110}(1-e_1)(1-e_2)(e_3) \\ &+ p_{111}(1-e_1)(1-e_2)(1-e_3). \end{aligned} \tag{4}$$

$P(111)$ is the theoretical probability of observing the intransitive response cycle of *111*; p_{000}, p_{001}, p_{010}, p_{011}, p_{100}, p_{101}, p_{110}, *and* p_{111} are the probabilities that the person has these "true" preference patterns, respectively (these 8 terms sum to 1); e_1, e_2, and e_3 are the probabilities of error on the AB, BC, and CA choices, respectively. These error rates are assumed to be mutually independent, and each is less than ½.

There are seven other equations like Equation 4 for the probabilities of the other seven possible response patterns.

Because each choice is presented twice in each block, there are 64 possible response patterns for all six responses within each block. If error rates are assumed to be the same for choice problems $A'B'$, $B'C'$, and $C'A'$ as for choice problems AB, BC, and CA, respectively, the probability of showing the same pattern, *111*, on both versions in a block is the same as in Equation 4, except each of the error terms, e or $(1 - e)$ in Equation 4 are squared. In this way, one can write out 64 expressions for all 64 possible response patterns that can occur in one block.

A hypothetical set of data is shown in Table 6. The rows show the 8 possible response combinations for the ABC choices and the columns show the 8 possible response patterns for the $A'B'C'$ choices. Each entry is the frequency with which each response pattern occurred. For example, the 17 in the first row and column shows that on 17 out of 200 blocks, the person showed the intransitive pattern, *000*, in both versions of the three choice problems.

The *general* TE model for this case has 8 parameters for the 8 "true" probabilities and 3 error rates (one for each choice problem).[4] Because these 8 "true" probabil-

[4]More general models can also be tested; this version is called the "general" model here to distinguish it from special cases that assume

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Table 6: Hypothetical data containing error that illustrate testing independence, TE model, and transitivity. Analyses described in the text show that these data violate independence, they satisfy TE model, and they violate transitivity.

	Response Pattern in $A'B'C'$ choices								
ABC	*000*	*001*	*010*	*011*	*100*	*101*	*110*	*111*	Sum
000	17	3	4	1	6	1	2	2	36
001	3	1	1	1	1	1	0	1	9
010	5	1	0	1	2	1	1	2	13
011	1	1	1	3	1	3	2	11	23
100	6	1	2	1	2	0	1	2	15
101	1	1	1	3	1	2	2	8	19
110	2	0	1	2	1	2	2	6	16
111	1	1	3	11	2	9	6	36	69
Sum	36	9	13	23	16	19	16	68	200

ities sum to 1, they use 7 df, so the model uses 10 df to account for 64 frequencies of response patterns; the data have 63 df because they sum to the total number of blocks. It should be clear that there are many ways for 64 frequencies to occur that are not compatible with a model with 11 parameters. Two examples will be presented later.

The *transitive* TE model is a special case of this general TE model in which the two probabilities of intransitive patterns are set to zero, $p_{000} = p_{111} = 0$. If the error rates are not zero, a set of data can (and typically would) still show some instances of intransitive response patterns, even though the "true" probabilities of these patterns are each zero.

One can conduct at least three types of statistical tests. First, one can test independence. Second, one can test this general TE model. Third, if the general TE model provides a reasonable approximation, one can test the special case of transitivity within that model.[5]

According to response independence, it should be possible reproduce the entries in Table 6 from just three numbers: the marginal choice proportions, $P(AB)$, $P(BC)$, and $P(CA)$. These are all 0.6, so the predicted entry for the upper left cell of Table 6 (*000*, *000*) would be $[1 - P(AB)]^2[1 - P(BC)]^2[1 - P(CA)]^2 = (.4)^6 = 0.004$. Multiplying by the total frequency (200), the predicted frequency is 0.82, far less than the observed value of 17.

For an overall index of fit, one can compute $\chi^2 = \Sigma(f_i - F_i)^2/F_i$. Where f_i are the observed frequencies and F_i are the corresponding predicted frequencies, based on independence (or below, predicted from the TE model). There are 63 df in the data, and we used 3 df to estimate the three parameters (the marginal choice proportions), so this test of independence has 60 df. In this case, $\chi^2(60) = 505.4$, so the conclusion would be that these data do not satisfy response independence.[6]

Next, one can use a function minimizer, such as the solver in Excel, to estimate best-fit parameters for the general TE model. Those estimates are $p_{000} = .333$, $p_{111} = .667$, $p_{001} = p_{010} = p_{011} = p_{100} = p_{101} = p_{110} = 0$; $e_1 = 0.25$, $e_2 = 0.20$, and $e_3 = 0.15$. In this case, 10 df were used to estimate the parameters, and $\chi^2(53) = 11.5$, showing that the general TE model fits these hypothetical data well.

However, when we fix $p_{000} = p_{111} = 0$, in order to test the transitive special case, and solve for the best-fit parameters, we find that the transitive TE model does not fit these data, $\chi^2(55) = 108.3$. The difference, $\chi^2(2) = 108.3 - 11.5 = 96.8$, indicates that transitivity is not satisfactory as a description of these same data.

These calculations show that, in principle, one can estimate the models and assess their fit in the 8 × 8 matrix as in Table 6. In practice, however, it might be difficult or impractical to obtain sufficient data for such a full analysis. When the data are thinner, one might partition the data in various ways and still test independence, TE model, and transitivity, as described next.

3.3 Partitions of the data

In order to test independence to compare the iid models with TE models, one might partition the data from the 8 × 8 matrix (as in Table 6) into three, 2 × 2 matrices, in order to test the models of repetitions, as was done in Tables 2 and 3. For example, one can tabulate the *AB* choice by the $A'B'$ choice. From Table 6, the four frequencies are 44, 37, 37, and 82, for 00, 01, 10, and 11, respectively. These violate independence by a standard chi-square test, $\chi^2(1) = 10.8$. However, the same values fit the TE model, $\chi^2(1) = 0.2$, with $e_1 = 0.25$. Similarly, the tabulations for the other two choice problems also violate response independence, $\chi^2(1) = 20.7$ and 40.3, and also satisfy TE independence, $\chi^2(1) = 0.1$, and 0.5, with $e_2 = 0.2$, and $e_3 = 0.15$, respectively. Because these error estimates do not assume or imply the property of transitivity, they might be used to constrain solutions to other partitions of the data that can be used to test transitivity.

transitivity.

[5]An Excel spreadsheet that implements these analyses is available with this article.

[6]Another index of fit is the *G*-squared statistic, which arises as a maximum likelihood, $G^2 = -2\Sigma\ln(f_i / F_i)$, which is considered a better asymptotic approximation to the theoretical distribution. One could also simulate either of these statistics via Monte Carlo for cases with small samples.

Table 7: Hypothetical examples testing transitivity; these examples illustrate use of partitioned data to compensate for small sample sizes. Marginal choice proportions are the same in all examples. Examples 1-3 violate iid. Example 1 satisfies transitivity, which is violated in Examples 2 or 3. Frequencies under "*ABC*" represent response patterns to Choice *AB*, *BC*, and *CA*, so *000* and *111* are intransitive; frequencies under "Both" indicate the same response pattern repeated within blocks. Example 4 satisfies iid model of Regenwetter et al. (2011), which wrongly concludes that all four of these examples satisfy transitivity.

Pattern	Example 1		Example 2		Example 3		Example 4	
	ABC	Both	*ABC*	Both	*ABC*	Both	*ABC*	Both
000	2	0	27	20	11	7	6	0
001	5	1	4	0	4	0	10	1
010	5	0	4	0	4	0	10	1
011	28	20	5	1	21	14	14	2
100	13	7	4	0	20	13	10	1
101	20	13	5	1	5	0	14	2
110	20	13	5	0	5	1	14	2
111	7	1	46	33	30	20	22	5
Total	100	55	100	55	100	55	100	14
χ^2 Indep	335.18		1139.25		408.95		0.52	

A useful partition for testing transitivity is to count the frequencies of the 8 possible response patterns in the *AB*, *BC*, and *CA* choices and the frequencies of repeating the same patterns on both *ABC* and $A'B'C'$ choices within blocks. These values can be found in the row sums of Table 6 and on the major diagonal, respectively. But these frequencies contain cases in common, so they are not mutually exclusive. One can construct a mutually exclusive, exhaustive partition by counting the frequency of showing each pattern on both repetitions of the same choice problems and the frequency of showing each of 8 response patterns in the *ABC* choices and not in both cases. For example, in Table 6, the frequency of showing the 000 pattern in the *ABC* choice problems and not in both forms is 36–17 = 19.

This partition of the data reduces the 64 cells as in Table 6 to 16 cells. This partition has the effect of increasing the frequencies in each cell, but reducing the degrees of freedom in the test. In this partition, we can also test independence, TE model, and transitivity. The purpose of the partition is to increase the frequencies within each cell, in order to meet the assumptions of the Chi-Square or *G*-Square statistical tests.

Four examples of hypothetical data are shown in Table 7 to illustrate different cases that might be observed with this type of partition. The numbers have been chosen to sum to 100 so that they could be easily converted to proportions to facilitate calculations for the models.

All four examples in Table 7 have identical marginal choice proportions, so any method of analysis that focused strictly on marginal choice proportions treats these four examples as identical, but they are quite different from each other. The marginal choice proportions, $P(AB)$, $P(BC)$, and $P(CA)$ are all 0.6, so these examples all violate weak stochastic transitivity, and all satisfy the triangle inequality. However, they have different interpretations, as shown below.

These response patterns are listed in terms of the *ABC* choice pattern and repeated patterns. To convert to a mutually exclusive and exhaustive partition, subtract the "both" frequencies from the *ABC* frequencies, as described above. The Chi-Squares are then computed in the conventional way comparing observed frequencies with those predicted by the models.

First, we can test independence, which is the assumption that products of marginal choice proportions correctly reproduce all 16 cells in this partition of the data. Three parameters (three marginal choice proportions) are calculated from the data, leaving 15 – 3 = 12 df for the test of independence. The critical value of $\chi^2(12)$ with α = .01 is 26.22. The last row of Table 6 shows these χ^2 tests; only Example 4 satisfies independence.

Second, we can test the general TE model. The TE model can also be tested by this partition because there are 15 df in the data and the model uses 7 df for the 8 "true" probabilities and 3 df for the three error rates (e_1, e_2, and e_3 for Choices *AB*, *BC*, and *CA*, respectively).

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Table 8: Best-fit solutions of TE models to Example 2 of Table 7. These hypothetical data satisfy the triangle inequality yet are perfectly intransitive, according to the fit of the TE models. Fixed values are shown in parentheses and constrained values are shown in brackets. Constrained errors are estimated strictly from preference reversals to the same choice problem within blocks, using the three, 2 × 2 partitions as in Table 3.

	Unconstrained errors			Constrained errors		
Parameter	General	Transitive	Intransitive	General	Transitive	Intransitive
p_{000}	0.378	(0)	0.378	0.378	(0)	0.375
p_{001}	0.000	0.342	(0)	0.000	0.141	(0)
p_{010}	0.000	0.045	(0)	0.000	0.116	(0)
p_{011}	0.011	0.000	(0)	0.011	0.184	(0)
p_{100}	0.000	0.030	(0)	0.000	0.116	(0)
p_{101}	0.011	0.015	(0)	0.011	0.184	(0)
p_{110}	0.000	0.568	(0)	0.000	0.259	(0)
p_{111}	0.600	(0)	0.622	0.601	(0)	0.625
e_1	0.095	0.024	0.112	[0.1]	[0.1]	[0.1]
e_2	0.095	0.024	0.112	[0.1]	[0.1]	[0.1]
e_3	0.103	0.500	0.091	[0.1]	[0.1]	[0.1]
χ^2	1.98	88.48	3.29	2.02	4194.97	3.67

That leaves 5 df to test the model in this partition.[7]

Third, if the TE model fits, we can test the transitive model by fixing the values of $p_{000} = p_{111} = 0$, which means that the solution is restricted to be purely transitive. The difference in fit between the general case where probabilities of all "true" patterns are free and the transitive special case provides a test of transitivity on 2 df.

Example 1 of Table 7 violates independence [$\chi^2(12) = 335.18$]; however, it satisfies both the TE model and transitivity. The TE model fit these data with error rates constrained to match the preference reversals data only, where $e_1 = e_2 = e_3 = 0.1$, where $p_{000} = p_{111} = 0$ were fixed, and where the best-fit solution yielded $p_{001} = 0$, $p_{010} = 0$, $p_{011} = 0.325$, $p_{100} = 0.125$, $p_{101} = 0.25$, and $p_{110} = 0.25$. This model has $\chi^2 = 2.04$, so it should be clear that there is no room for a significant improvement by making the model more complex. So this case violates iid, but satisfies the TE model and transitivity.

However, Example 2 is a very different case from Example 1, as shown in Table 8. Six models have been fit to those data, including the general TE model (all 8 response patterns allowed), the transitive special case (both intransitive patterns are fixed to zero), and a purely intransitive model (only intransitive patterns are allowed). Parameters shown in parentheses are fixed, and those shown in square brackets are constrained.

[7] We also had opportunities to reject the TE model via the three, 2 × 2 cross-tabulations of repeated choices.

When the general TE model fits the data, one might *constrain* the error rates in this analysis to agree with values estimated strictly from replications data (from the three, 2 × 2 cross-tabs). The constrained version provides greater power for the test of transitivity.

The fact that the TE general model fits either with or without constrained errors shows that we can retain the general TE model. The differences in Chi-Squares between the general model and the transitive special case are large enough to reject the transitive model either with or without constrained errors ($\chi^2(2) = 4192.95$ and 86.50, respectively). The purely intransitive special case also fits these data acceptably because the difference in Chi-Squares between the general model and purely intransitive model is not significant in either constrained or free cases.

Table 9 shows the corresponding analyses for Example 3. The general TE model is again acceptable with or without constrained errors. In this case, however, both the purely transitive model and the purely intransitive model can be rejected. Therefore, one would conclude that these data are best represented as a mixture of transitive and intransitive response patterns.

Example 4 satisfies independence. In that case, one could say that the data might be compatible with a mixture of strictly transitive patterns, but one might also say that the data could have arisen from a mixture that included intransitive patterns. In the analyses of Regenwetter et al. (2011), this case would be declared consistent

Table 9: Fit of TE models to Example 3 of Table 7. These hypothetical data satisfy the triangle inequality but they contain a mixture of transitive and intransitive response patterns. Neither the purely transitive nor purely intransitive solutions yields an acceptable fit.

	Unconstrained errors			Constrained errors		
Parameter	General	Transitive	Intransitive	General	Transitive	Intransitive
p_{000}	0.128	(0)	0.406	0.128	(0)	0.476
p_{001}	0.000	0.000	(0)	0.000	0.000	(0)
p_{010}	0.000	0.001	(0)	0.000	0.000	(0)
p_{011}	0.255	0.597	(0)	0.256	0.520	(0)
p_{100}	0.242	0.360	(0)	0.243	0.257	(0)
p_{101}	0.000	0.000	(0)	0.000	0.130	(0)
p_{110}	0.011	0.042	(0)	0.011	0.093	(0)
p_{111}	0.363	(0)	0.594	0.364	(0)	0.524
e_1	0.101	0.500	0.493	[0.1]	[0.1]	[0.1]
e_2	0.104	0.107	0.000	[0.1]	[0.1]	[0.1]
e_3	0.090	0.000	0.142	[0.1]	[0.1]	[0.1]
χ^2	1.66	23.32	27.22	1.69	1067.77	1069.16

with transitivity, as would all of these examples. In the approach of Regenwetter et al. (2011), no statistical test would be conducted because the model "fits perfectly" in all of these cases.

Table 10 provides two hypothetical examples showing that the TE model need not always fit. Marginal choice proportions in Examples 5 and 6 are the same as in Examples 1–4 of Table 7. In both cases iid is violated, but in both cases the general TE model fails. In Example 5, the response patterns observed are mostly intransitive and in Example 6 most of the response patterns observed are transitive.

To understand what went wrong for the TE models in these examples, recall that errors are assumed to be mutually independent. Although TE models violate independence of responses, they satisfy independence of errors, and the errors in these examples violate that assumption. In these examples, the participant was not completely consistent so errors are not zero; we know that there are substantial errors because people did not repeat the same response patterns in both versions very often. But if errors are not zero and are mutually independent, we should have observed more instances of response patterns 001, 010, 100, 011, 100, 101, and 110 in Example 5, and yet too few such cases are observed. Instead, whenever a person made an error on one choice problem, they too often made an error on other choice problems. Example 6 also violates TE because data violate independence of errors. Examples 5 and 6 violate iid and violate TE model.

In summary, one can separately test independence, TE, and transitivity. These examples illustrate how the TE model can be applied and they refute the claims by Cha et al. (2013) that TE models must satisfy response independence or become vacuous.

Difficulties in the approach of Regenwetter et al. (2011) are illustrated by these examples. Based on marginal choice proportions, all of these examples are the same and all are perfectly consistent with transitivity. When we examine the data as in Tables 6-10, we see that some cases systematically violate iid and among those, some cases systematically violate transitivity and others satisfy it. When iid assumptions are satisfied, then marginal choice proportions contain all of the useable information in the data, but when iid is violated, we need to examine response patterns to correctly diagnose the substantive issue of transitivity.

These hypothetical examples illustrate why it is important to know whether iid assumptions, especially response independence, are empirically satisfied in choice experiments. Birnbaum and Bahra (2007b, 2012b) found that iid was violated in a series of experiments testing transitivity. Birnbaum's (2012a) reanalysis of Regenwetter et al. (2011) also concluded that iid was not satisfied for those data.

However, Cha et al. (2013) claimed that the findings from Birnbaum (2012a) were "not replicated within subjects" when other data sets from Regenwetter et al. (2011) were examined, that the tests proposed have "unknown" p-values that are significantly different from those obtained by another method of simulation, and that certain

Table 10: Hypothetical examples violating both response independence and the TE model with all parameters free. As in Table 7, these examples have the same marginal choice proportions (all 0.6).

	Example 5		Example 6	
Pattern	*ABC*	Both	*ABC*	Both
000	28	4	1	0
001	4	0	1	0
010	4	0	1	1
011	4	0	37	4
100	4	0	19	4
101	4	0	19	4
110	4	0	19	4
111	48	6	3	1
Total	100	10	100	18
$\chi 2$ Indep	156.5		93.1	
$\chi 2$ TE	84.8		69.8	

other tests of "iid" were satisfied "with flying colors" for the Regenwetter et al. (2011) data. Each of these claims is refuted in Appendix A, where it is shown that the evidence against iid is significant in all three sets of data reviewed by Cha et al., that the *p*-values estimated by Birnbaum's methods are conservative relative to the method used by Cha et al., and that the tests of "iid" that were not significant "with flying colors" in Cha et al. do not test response independence.

3.4 Birnbaum and Bahra data violate iid

Birnbaum and Bahra (2007b, 2012b) also used three designs for each participant in each study; they used 136 participants (in three studies) compared to 18 in Regenwetter et al. (2011) and they asked each person to respond twice to each choice problem in each block (compared to once per block in Regenwetter et al., 2011). They also used a greater variety of choice problems that might be expected to create more interference in memory, and blocks were properly separated by numerous intervening tasks. Therefore, this 2012b paper with 136 participants must be accorded corresponding greater weight in relation to a study with only 18 participants. As shown in Birnbaum and Bahra (2012b), evidence against iid in those studies was extremely strong.

Birnbaum and Bahra (2007b, 2012b) found that a number of participants completely reversed preferences for 20 out of 20 choice problems between blocks; this provides a clear refutation of the theory of iid. Because each block of each design in that study contained 20 experimental choice problems (excluding fillers and separators), a complete reversal has a probability of ½ to the 20^{th} power, assuming iid, which is less than one in a million. There were 18 people out of 136 who showed at least one such perfect reversal of 20 out of 20 responses between blocks, and these 18 produced hundreds of instances of such perfect reversals. In fact, one person reversed preferences perfectly between 60 choice problems (all three designs) between blocks (see Table 2 of Birnbaum & Bahra, 2012b). These and other analyses of those data show that iid can be rejected.

4 Discussion

In my opinion, the empirical results obtained so far tell us that any viable approach to analyzing formal properties in choice data should be able to handle the possibility that the assumptions of response independence is violated. It should allow for the possibility that people behave more consistently than allowed by the simplifying assumptions of iid. The TE models can handle certain violations of response independence. These models do not satisfy response independence and yet they are testable because they cannot handle all such violations.

As shown in the examples presented here, TE model can distinguish and diagnose cases that look identical to tests defined on marginal choice proportions (such as weak stochastic transitivity and the triangle inequality). All of the examples in Tables 6, 7, and 10 have the same binary choice proportions. However, I think it proper to conclude that Example 1 of Table 7 satisfies transitivity and that Examples 2 and 3 in Table 7 violate it. Example 4 satisfies iid and is therefore open to debate, because a person might have a mixture that is purely transitive or might have a mixture including intransitive patterns and still produce such data.

These different conclusions for these different examples could not be reached by examination of the marginal choice proportions alone, because they all have the same marginal proportions. My advice to those testing transitivity or other properties is that they should analyze data at the level of response patterns rather than at the level of marginal choice proportions.

4.1 Are criticisms of using marginal proportions dependent on the TE model?

No, these criticisms apply whenever iid is violated, whatever the cause. The TE models provide one approach, but this family is not the only way that violations of iid might be represented. The criticism of using only marginal choice proportions applies to any case in which iid is violated, whether those violations satisfy a TE model or not. As shown in Examples 5 and 6 of Table 10, data might violate iid and also violate the TE model. These examples, including those in Table 3, show that the assumption of independence of errors is a testable property of these models, but it is not the same as response independence.

One could argue that TE models are only approximate because they allow that a person can change "true" preferences between blocks but not within a block. A more accurate or more general model might allow that a person might change "true" preferences at any point during the study. Such a model would include the iid model used by Regenwetter et al. (2011) and TE models as special cases. According to the model used by Regenwetter et al. (2011), independence is supposed to hold on every experimental trial, as long as there are three filler trials separating experimental trials.

4.2 What if TE models are wrong or incomplete?

The TE Models are testable and they might be rejected when appropriate studies have been done. A test of *i*TET requires a larger quantity of data from each participant to conduct a proper analysis, whereas tests of *g*TET require a large numbers of participants, each of which might contribute a smaller amount of data. Whereas a number of experiments in the *g*TET paradigm have been published, we do not yet have experimental results comparable to the hypothetical Table 6 for the *i*TET case, and one might reasonably wish for even more data than described in that example.

Although TE models can allow different error rates in different choice problems, and although more general versions can be tested in which different people might have different amounts of noise in their data, even these more general TE models do not provide any fundamental explanation for the sources of the errors.

Nor do TE models proposed so far provide an explanation for the kinds of sequential effects that might arise from a process such as described by Birnbaum (2011), in which the parameters of a model of risky decision making change systematically from trial to trial, as elaborated in Appendix B. Therefore, although TE models provide a testable framework within which issues of independence and transitivity can be explored, they do not provide specific or satisfying answers to important deeper questions. Appendix B shows how sequential models might account for violations of iid including response independence as well as violations detected by the correlation test of Birnbaum (2011, 2012a). These models allow that parameters representing probability weighting or risk aversion might fluctuate from block to block, but they do not identify the causes of changing parameter values.

But it is important to realize that even if TE models are wrong, as in Examples 5 and 6 of Table 9, or if they are approximate, incomplete or even misguided, criticism of TE models does not mitigate the problems of assuming iid as a basis for testing transitivity. The key problem is that when iid is violated, analysis of marginal choice proportions can easily lead to wrong conclusions.

4.3 Are assumptions of iid only used to justify statistical tests?

It might be argued that because the assumptions of iid are used to justify statistical tests, that this is their only role in the approach of Regenwetter et al. (2011). That is not true: in fact, it is the assumption of iid that justifies analysis of marginal choice proportions. As shown in the examples of Table 6 and 7, when iid is violated, marginal choice proportions might satisfy the triangle inequality despite systematic violations of transitivity within blocks, revealed in the response patterns.

The statistical issue (that violations of iid affect the *p*-value of a significance test) is far less important, in my opinion, than the danger of drawing wrong descriptive, substantive conclusions concerning a theoretical property (such as transitivity) from marginal choice proportions. Indeed, when the triangle inequalities are satisfied, as they are in all of the examples analyzed here, the Regenwetter et al. (2010, 2011) approach conducts no statistical test at all, because the model is said in all of such cases to fit "perfectly". For example, in Table 7 the triangle inequality can be "perfectly" satisfied in a case in which a different, deeper analysis (Table 8) would refute any mixture of transitive patterns in favor of a mixture of purely intransitive patterns.

There is another distinction that might be helpful to eliminate some confusion in this dispute. The random preference model used by Regenwetter et al. (2011) allows any set of preference patterns to be in the "mind" of the participant. These hypothesized preference patterns can violate independence. Indeed, in the linear order, no intransitive patterns are allowed, so it might seem that this model violates a type of independence in the postulated mental set.

However, on each trial, the Regenwetter et al. (2011) model assumes that the choice response can be represented as the result of an independent, random sample from the collection of mental preference patterns. Because that "random preference" sampling is random, it means that overt responses will satisfy independence, even when the theoretical preference patterns in the hypothesized collection (in the mind) violate independence.

Regenwetter et al. (2011) have pointed out that it is not possible to recover the distribution of true preference patterns from the choice responses of a person, because the assumed independence of responses means that overt responses do not permit recovery of the distribution of preference patterns in the mind of the participant.

The assumption of response independence thus justifies analysis of marginal choice proportions, while it also makes it impossible to identify the distribution of theorized preference patterns. When iid is assumed, one might, in principle, reject transitivity in this approach, but one cannot recover the distribution of "true" response patterns in this model nor can one definitively rule out mixtures containing intransitive response patterns when the marginal means satisfy transitivity. As shown here, when iid is violated, satisfaction of the triangle inequality can co-exist with systematic violations of transitivity.

The statistical tests used here to illustrate analyses in the TE model also make independence assumptions, but these do not assume nor imply response independence. Obviously, these higher order assumptions might be empirically wrong. But if and when the model is appropriate, it can be used to estimate the distribution of "true" response patterns.

4.4 The details are in the data

The question of how much detail should be analyzed in data arises in all research problems. The analyses of the various partitions of the data here should clarify that whenever data are aggregated, information can be lost. This debate can be viewed as a debate of how much useful information is contained in data and how much detail should be represented by a model.

For example, there are $20 \times 6 = 120$ Values in Table 4. In a larger experiment, there might be a hundred rows. Such a table of data might be summarized by 3 binary choice proportions, by 6 column proportions, by 3, 2×2 cross-tabulations of repeated responses, by 8 proportions of showing each response pattern involving three choice problems, by 8×2 proportions showing each response pattern on the ABC choice (but not both) and in both versions, by the 8×8 cross-tabulation of the eight response patterns in the ABC $\times$ A′B′C′ repetitions, or at the level of the original data.

When response independence is violated, as appears to be the case for all three sets of data in Regenwetter et al. (2011) as well as the series of studies in Birnbaum and Bahra (2012b), it means that marginal choice proportions do not tell the whole story. As shown here, response independence and error independence are different properties, but both properties can be tested, and I think it would be a true error not to test them both.

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Appendix A: Additional data, simulations, and reanalyses

Other Data of Regenwetter et al. (2011)

Birnbaum (2012a) analyzed only that part of the study of Regenwetter et al. (2011) that replicated Tversky's (1969) study, called "Cash 1." Cha et al. (2013) claimed that when two other sets of data from that study, "Cash 2" and "Noncash", which used different stimuli, are examined, Birnbaum's tests were "not replicated within subjects" and when other statistics were computed, that iid was satisfied "with flying colors". However, when I reviewed all three sets of data, I do not concur with their conclusions.

According to iid, there should be no correlation between the number of preference reversals between two blocks and the separation between blocks. If observed correlation coefficients differ from zero, we expect half to be positive and half negative, if iid holds. Birnbaum (2012a) previously reported that 15 of 18 participants had positive correlations in Cash 1, that the median correlation was 0.58, and that the mean correlation was significantly greater than zero by a conventional t-test.

When these correlations are calculated for Cash 2 and Noncash conditions, 12 and 15 of the 18 correlation coefficients are greater than zero, respectively; the median correlation coefficients were 0.39 and 0.63, respectively. Mean correlations were significantly greater than zero by t-tests in both Cash 2 and Noncash ($t(17) = 3.21$ and 5.00, respectively). Across all three sets of data, 42 of 54 correlation coefficients (3 data sets by 18 participants) are greater than zero, which is significantly more than half (binomial $p = .00003$). The overall median correlation was 0.51, which seems substantial.

I conclude that adding Cash 2 and Noncash datasets makes the case against iid even stronger than it was without them, even when conventional statistics are applied. The evidence against iid was indeed "replicated" in a "within subjects design" if these terms are interpreted as finding a significant test statistic with new stimuli in tests with the same participants.

Simulated p-values

Cha et al. (2013, Table 5, left side) presented the results of 108 statistical tests of iid for the Regenwetter et al. (2011) data. They used Birnbaum's (2012a) R-program to simulate two statistical tests for each of 18 participants in three sets of data with 10,000 random simulations per data set.

Separating the tests by data sets, there were 9, 11, and 9 "significant" violations ($p < 0.05$) in Cash 1, Cash 2, and Noncash conditions, respectively. For the correlation test alone, 5 out of 18 were significant in each of the three conditions. The binomial probability to observe five or more significant results out of 18 with $p = .05$ is .0015, so each condition provides significant refutations of iid taken alone.

Similarly, there were 4, 6, and 4 who violated the variance test in the three conditions. The binomial probability to observe 4 or more with $p = .05$ is .011, so iid can be rejected separately in all three conditions for this test as well. Thus, all six tests are significant: 3 data sets by 2 tests per data set.

Given the lack of power—there are only 18 participants who responded to each choice only 20 times—these findings seem strong evidence against iid. Summing across participants, data sets, and tests, Cha et al. reported that 29 of 108 tests were "significant" with $p < .05$, using Birnbaum's algorithm.

Suppose we adopted the following extra-conservative standard: Suppose we required the simulated p-value to be .02 or less in order to conclude it is actually "significant" at .05. According to Table 5 of Cha et al, there are 25 such values. According to the binomial distribution with $n = 108$ and $p = .05$, the probability to observe 25 or more "significant" tests, if the null hypothesis is true, is less than one in a billion. So, despite using a doubly conservative procedure to evaluate the significance tests, these data say we should reject the hypothesis that these participants are a random sample of people who satisfy iid beyond all conventional significance levels.

The claim that the tests "Do not replicate"

Although the data show overwhelming evidence to refute iid in all three sets of data with the same participants, Cha et al. (2013, pp. 62–64) made a very unusual argument, claiming that the statistical tests "do not replicate within participant" (p. 55). They do not spell out their logic clearly because strong evidence against iid was obtained in the same group of people in all three subsets of data, which I would interpret to say that the significant violations of iid were indeed "replicated" when the same participants were tested with different stimuli.

It might be argued that if a person had *nonsignificant* violations, we should conclude that this same person actually *satisfies* iid that if a person has nonsignificant violations of iid with one set of stimuli that this same individual should never show a significant violation of iid with any other stimuli. But that requires us to infer that a nonsignificant result means that the null hypothesis is true and it requires us to generalize this null hypothesis from one set of stimuli to all other stimuli. Or it might be argued that if a test is *significant* for one person with one set of stimuli, then it should also be significant in any new test with new stimuli. But that argument would also be specious.

If either of these principles is intended to justify the statement that the results "did not replicate," I find these arguments unreasonable. Recall that Regenwetter et al. (2011b) argued that it would take very large samples to test iid, and Birnbaum's (2012a) tests were devised to address the need to test iid in small samples. With such small samples (low power), it seems best not to expect every test to be significant even if the null hypothesis were false for all persons and all sets of stimuli.

Only 5 of 18 participants of Regenwetter et al. (2011) did not show at least one "significant" violation of iid, according to Table 5 of Cha et al. (2013). Does that mean that those five nonsignificant cases actually satisfy iid? I do not think so. Nor do I think we should conclude that there are no individual differences with respect to violating iid. I suspect that iid is false for all participants, but I think we should withhold judgment until there are better data before deciding whether iid is violated for all persons or just for some of them.

Comparing methods for simulation of p-levels

To investigate Birnbaum's (2012a) suggested statistical tests of iid, Cha et al. (2013, pp. 59–62) conducted 3,000 simulations for three of their participants. They used a computer program to simulate data, based on the assumption that the estimated parameters for each person were the population values. Their simulations were created via a program that is expected (but not guaranteed) to produce the same marginal proportions and expected (but not guaranteed) to satisfy iid. For each of these 3,000 simulations, they then used Birnbaum's (2012a) program to create 10,000 simulated permutations of the simulated data. They reported that the percentage of Type I errors in their simulated data (which they called the "actual") were close to the values simulated by Birnbaum's (2012a) two tests, although they found some small differences, including some statistically significant ones, where Birnbaum's tests were conservative relative to theirs.

Examining their findings, I conclude that these two methods for simulating Type I errors are fairly close in agreement, despite some small differences. For example, Table 4 of Cha et al. (2013) shows that what they call "actual" (more accurately, simulated) percentages ranged from .7% to 1.4% for 3,000 samples at the "nominal" 1% level (as simulated by Birnbaum's methods). "Actual" (simulated from 3,000 samples) data showed 4.2% to 5.6% violations corresponding to Birnbaum's nominal (simulation) at 5%. Their results showed that the two methods for simulating p-levels are highly correlated and in fairly close agreement, despite small differences such as one might expect when comparing these two, slightly different methods for simulating data using pseudo random numbers generated by computer programs.

New simulations

Although they argued (implicitly) that they had a good method for simulating data, Cha et al. (2013) did not simulate Type I errors for all of the data. Instead, they used their method only to compare the estimates of p-values by two procedures for a few cases. Their method samples independently from binomials whereas Birnbaum's method used random permutations, as in Smith and Batchelder (2008).

In order to simulate data by the method used by Cha et al. (2013), find the line in Birnbaum's (2012a) code as follows:

```
for (jj in 1:nchoices) {xperm[,jj] <-
x[sample(nreps,nreps),jj]} }
```

and change it to the following:

```
for (jj in 1:nchoices) {xperm[,jj]
<- x[sample(nreps,nreps, replace =
TRUE),jj]} }
```

No other changes to the program are necessary. With this revision, each datum is randomly drawn from a binomial population with probability equal to the marginal choice proportion in the original data, but each new sample need not have the same marginal choice proportions as in the original data.

Out of 108 simulations (18 subjects × 3 data sets × 2 tests), each based on 10,000 computer generated samples, I found that 29 simulated Type I errors had $p < .05$ by either simulation method; two new cases became "significant" and two other cases that were "significant" by the permutation method dropped to non-significant. However, the estimated p-values were smaller in 78 cases for the sampling method and smaller in 22 cases for the permutation method, with 8 cases the same. Thus, the permutation method used by Smith and Batchelder (2008, p. 731) and implemented by Birnbaum (2012a) is *conservative* relative to the method that allows the marginal proportions to vary across samples.

A reason that Birnbaum's simulations look "conservative" compared to those of Cha et al. (2013), is apparently that the method used by those authors allows simulated proportions not to match the proportions in the original data, which means that it is possible to have samples for a "variable" with all *0*s or all *1*s in a column; in those samples, that so-called "variable" becomes a constant, which will be independent of all other variables and show no sequential effects. That allows iid to fit such simulated samples better than it would with the permutation algorithm, so the original data appear more rare (improbable) in comparison. Personally, I prefer the more conservative, permutation method (which constrains the marginal proportions to match those in the original data); however, for these data, these two methods do not produce any material difference to the conclusions.

Combining all three datasets

The tests applied separately to each design are designed to assess whether or not the assumption of iid is satisfied for trials that are separated by three intervening trials, as theorized by Regenwetter et al. (2011).

Table 11: Simulations of combined data from Regenwetter et al. (2011). Each data array is 20 × 30, Repetitions by Choice Problems. Mean = average number of preference reversals between blocks, var = variance of preference reversals; r = correlation between mean number of preference reversals and difference in trial blocks; p_V and p_r = estimated p-values for variance and correlation tests, respectively, based on 10,000 simulations.

Case	mean	var	p_V	r	p_r
1	9.70	10.03	0.39	0.88	0.00
2	4.75	8.70	0.00	0.91	0.00
3	2.04	1.44	0.62	0.67	0.18
4	5.44	6.53	0.00	0.33	0.50
5	3.48	10.36	0.00	− 0.24	0.72
6	7.24	9.77	0.00	0.91	0.00
7	4.96	8.14	0.00	0.94	0.00
8	3.87	5.04	0.00	0.93	0.00
9	10.71	11.56	0.40	0.92	0.00
10	2.78	4.27	0.01	0.50	0.46
11	1.43	1.16	0.42	0.02	0.98
12	8.89	9.17	0.22	0.48	0.21
13	11.78	14.22	0.03	0.76	0.00
14	0.47	0.34	0.52	0.61	0.40
15	6.79	8.62	0.00	0.62	0.08
16	4.55	7.02	0.00	0.89	0.00
17	8.64	8.07	0.62	0.70	0.03
18	9.20	8.86	0.55	0.55	0.13

Cha et al. (2013, p. 64) reasoned that, if iid assumptions were false, iid would be more likely to fail if all of the choice problems were analyzed together. To check this possibility, I combined all three designs, producing a 20 (Reps) × 30 (Choice problems) array for each person. When these are analyzed using the Monte Carlo method sampling procedure, 10 and 9 participants had significant violations of iid at the 0.05 level for the variance and correlation tests, respectively, and only 5 participants did not have at least one significant violation. The median correlation between preference reversals and distance between blocks increased to 0.68, and all correlation coefficients except one (out of 18) were positive. The results are shown in Table 11. Similar results were obtained with the more conservative, permutation method, with the same 5 individuals lacking a significant violation.

The tests that "passed with flying colors" do not test independence

Cha et al. (2013, p. 59) performed statistical tests on each column of data separately, which failed to show statistical significance "with flying colors". These tests do not assess response independence, and so these tests do not address the main issue of this debate, which is that the assumption of response independence could lead to wrong conclusions in a test of transitivity, as in Tables 6, 7, and 10. Violations of stationarity can produce violations of response independence (Birnbaum, 2012a, p. 104), however, which would be the main reason to test stationarity in this debate.

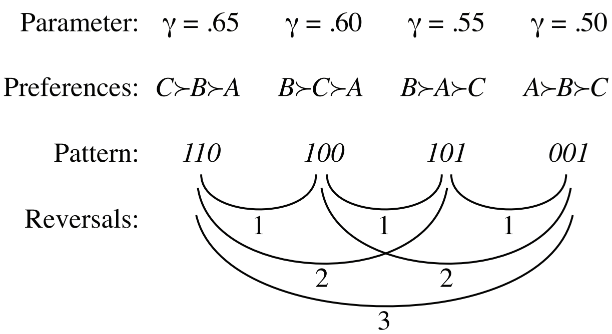
To understand why separate tests of each choice problem (each column) are not relevant to the issue of response independence, consider a debate between two researchers concerning whether X and Y are or are not positively correlated. Researcher 1 computes the correlation coefficient and finds that it is significant and positive; indeed, a scatterplot of data shows that as X increases, the conditional mean of Y given X also increases systematically. Researcher 2 then examines the distribution of X and the distribution of Y separately (but not the joint distribution), and declares that tests of iid were satisfied.

It should be fairly obvious that one cannot analyze independence of columns by analyzing each column separately. Clearly, Cha et al. (2013) must not have meant to say that iid was satisfied (based on finding a null hypothesis "with flying colors" for each column separately), because that would be like a person examining X and Y separately and saying they are "independent," without actually computing a correlation coefficient or examining the scatterplot of X and Y. Appendix C describes how the violation of response independence creates covariance that results in greater variance of preference reversals than allowed by iid in Birnbaum's (2012a) tests.

Appendix B: Stochastic process TE models

As noted in Birnbaum (2011, 2012a), results with the correlation test indicate that there are fewer preference reversals between two blocks of trials that occur closer together in time than between two blocks that are farther apart in time. This suggests that people are not randomly and independently choosing a true pattern of responses in each block of trials but instead that the true patterns in successive blocks are more similar. Birnbaum (2011, p. 680-681) suggested that such results might be compatible with a process model in which there are systematic changes of the parameters of a model of risky decision-making, such as the TAX model (Birnbaum, 2008). Such

Figure 1: Let A = (\$100, 0.5; \$0), B = (\$92, 0.58; \$0), and C = (\$84, 0.66; \$0). Four "true" preference patterns for Choices AB, BC, and CA can occur in the TAX model, as the parameter, γ, varies from 0.65 to 0.50, where the other parameters are fixed to conventional values: *110*, *100*, *101*, and *001*. In the absence of error, the number of preference reversals between these patterns varies from 0, when the person retains the same true preferences, to 3 out of 3, when this person's γ changes from 0.65 to 0.5.



a systematic drift in the value of a parameter might result from a deterministic process or from a stochastic process in which the value of a parameter at time t is likely to persist at time $t + 1$.

To illustrate one such process model, consider two-branch gambles of the form $G = (x, p; y)$, representing a gamble with a probability of p to win \$$x$ and otherwise win \$$y$, where $x > y \geq 0$. Suppose there are three gambles as follows: $A = (100, .50; 0)$, $B = (92, .58; 0)$ and $C = (84, .66; 0)$. Suppose that a person's choices are governed by the following TAX model for such gambles:

$$U(G) = \frac{au(x) + bu(y)}{a + b} \qquad (5)$$

where $a = p^{\gamma}(1\text{–}\delta/3)$ and $b = (1\text{–}p)^{\gamma} + p^{\gamma}\,\delta\,/3$, and $U(G)$ is the utility of the gamble. For American undergraduates with cash prizes ranging from \$0 to \$150, it has been found that one can approximate modal choices with $u(x) = x$, $\delta = 1$, and with γ between 0 and 1. Assume that a person chooses gamble G over gamble F if and only if $U(G) > U(F)$; models satisfying this assumption are transitive.

Figure 1 illustrates the four true preference patterns possible for plausible values of γ. In the case of the gTET model, one might assume that there are four different types of participants, with different values of γ. In the case of iTET, we assume that one individual might have different values of γ_t in different blocks of trials, where γ_t represents the parameter value in block t. The number of preference reversals, in the absence of error, between each pair of true preference patterns is shown in Figure 1; note that the number of reversals depends

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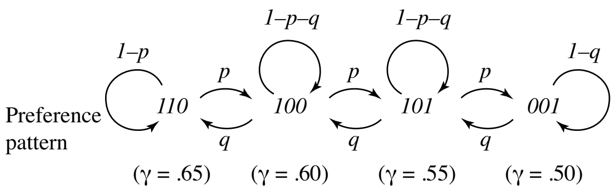
Figure 2: A random walk model on the four states of Figure 1. This model has two parameters: p and q are the probabilities to move to the state to the right or left between two blocks of trials, respectively. This model has fewer parameters than the general TE model that allows all eight possible true response patterns. Given the starting state, one can calculate the probabilities of being in each of the four states in a block, for a given number of trial blocks. This model also makes testable predictions for the probabilities of response patterns on one block, conditioned on responses in the previous block, as well as other testable implications.



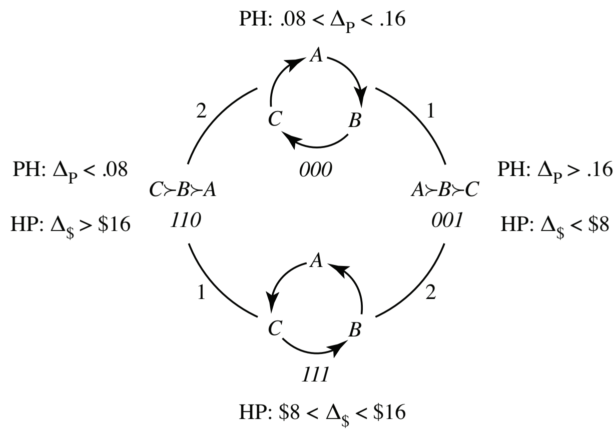
on the difference between the values of γ. So, it should be clear that there are many deterministic (or stochastic) processes in which γ_t would (likely) persist from block to block; that is, that γ_{t+1} would be equal to or similar to γ_t. Such models can imply a positive correlation between preference reversals and the gap between the blocks.[8]

For example, Figure 2 illustrates one such stochastic process model in which there are just four true preference patterns, and probabilities of transitions between states from block t to block $t + 1$ are the same for all values of t, governed by just two parameters. These probabilities represent increasing or decreasing the value of γ_t in the range from .65 to .50, by increments of .05. If p and q are both small, it means that the same true response pattern would likely persist for one or more blocks, since the probability to stay at the same value of γ_t is $1–p–q$ for the intermediate values of γ_t, and is even greater for the extreme values of γ_t.

In the model of Figure 2, it is not possible to switch from *001* to *100* without crossing via intermediate patterns. A rival stochastic process would represent the values of γ_{t+1} as a diffusion process in which the probability distribution of γ_{t+1} is specified as a function of the value of γ_t. A beta distribution with two parameters might be used for this purpose, with the mode fixed as the value of γ_t, which again means that the most likely transition between two successive blocks is to remain in the same state, but the parameter might change by variable

[8]The same four "true" response patterns in Figure 1 are also compatible with expected utility theory, which is a special case of the TAX model in which $\gamma = 1$ and $\delta = 0$, if $u(x) = x^{\beta_t}$, where β_t = the exponent of the utility function in block t. This model, however, can not account for systematic violations of coalescing and dominance (Birnbaum, 2008), so it could be tested by means of other choices in the same block besides those among A, B, and C.

Figure 3: If a person followed a lexicographic semiorder PH model, in which probabilities are first compared, and if their absolute difference exceeds Δ_P, decides based on probability and if not, decides based on the prizes, then that person might have true preference patterns 110, 000, or 001, depending on the value of Δ_P. If the person switched to the HP model, a lexicographic semiorder in which the highest consequences are compared first, then the intransitive cycle, 111, is possible as well as the same two transitive patterns. It is possible to define a stochastic process model that describes transitions among these states, analogous to Figure 2.



amounts in the range from 0 to 1. Such a model could allow transitions from any state to any other state.

In Figures 1 and 2, the only true states are transitive. Figure 3 illustrates a similar analysis of true response patterns for two lexicographic semiorder (LS) models that can handle intransitive true response patterns. In the PH LS model, a person compares two gambles of the form, $G = (x, g; 0)$ and $F = (y, f; 0)$ by first comparing their probabilities (P) to win the higher prize (H); if the absolute difference, $|g - f| \geq \Delta_P$, where Δ_P is the threshold parameter of probability, then the gamble with the higher probability to win is chosen; if not, choose the gamble with the better H. This model can produce transitive preferences for a given set of stimuli, $A \succ B \succ C$ or $C \succ B \succ A$, or it could produce the "clockwise" intransitive cycle illustrated in Figure 3 for A, B, and C, when $0.08 < \Delta_P < 0.16$.

However, if a person switched to the HP LS model, that person could show the "counterclockwise" intransitive cycle illustrated in Figure 3. In HP, the highest cash prizes are examined first and the difference, $|x - y|$, compared to a cash difference threshold, $\Delta_\$$.

The number of preference reversals between the true response patterns (in the absence of error) are shown along the curves; the number of reversals is greatest when comparing *110* versus *001* or *000* versus *111*, which re-

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quire larger changes in parameter values or a switch between HP and PH. Thus, by assuming that parameter values likely persist for several blocks, or that switching from HP to PH is less likely than a change in parameter value, such an intransitive model could imply violations of iid revealed by a positive correlation between preference reversals and the gap between blocks.

Appendix C: Violations of independence affect variance

Birnbaum's (2012a) approach detects violations of response independence. Let $x(i, j)$ = response to Choice Problem j in Block i. Suppose, as in Table 1, that $x(i, j) = 1$ if the participant reported preference for the second alternative and 0 otherwise. Each choice problem might have a different probability, p_j, that can be estimated from the column marginal proportion, $x(*, j)$. If iid were true, each column of data represents an iid sample from a binomial with probability p_j. The mean of Choice Problem (Column) j is expected to be p_j and the variance of Choice j is $p_j\ (1 - p_j)$. Note that the variance of a binomial is always less than its mean because $0 < p_j\ (1 - p_j) < p_j$.

Now, define a matrix of preference reversals between Blocks i and k, as follows:
$z(i, k, j) = 1$ if $x(i, j) \neq x(k, j)$ and $z(i, k, j) = 0$ otherwise. If iid is satisfied, the entries in z will have expected values of $\mu j = 2p_j\ (1 - p_j)$ with column variances $\sigma_j{}^2$. Note that this expression is independent of blocks, as long as the response in Block i is assumed independent of that in Block k; therefore, it should be the same for any choice of i and k, no matter how far apart or close together they are. This implies that there should be no correlation between the probability of a preference reversal and the gap between i and k.

Next, compute the sum of preference reversals between Blocks i and k as follows: $Z(i, k) = \Sigma z(i, k, j)$, where the summation is across Choices, j. If iid is satisfied in z, the mean of Z will be the sum of the column means, and the variance of Z will be the sum of the column variances, $\Sigma\sigma_j{}^2$. However, if independence of the columns of z is violated, then the variance of Z will be the sum of the variances plus the sum of all covariance terms, which implies that the variance of the sum might not equal the sum of the variances, as it would under the assumption of independence. For example, in Table 11, all of the significant cases in the variance test have variances greater than their means.

By combining across columns (choice problems), Birnbaum's (2012a) variance test will be significant when iid is violated, such that preference reversals for different choice problems have positive covariances with each other. This approach is, I think, appropriately conservative in that it uses random permutations of the data to simulate the distribution of variances of preference reversals under the null hypothesis of iid, rather than using asymptotic results for the sampling distribution of a variance to small samples. Nor does it use Monte Carlo simulations that allow marginal means to vary across samples, which is less conservative, but does not make any material difference in this case. However, aggregation of preference reversals across choice problems (columns) provides a statistic that is more diagnostic of response independence than any possible analysis done on each choice problem (column) separately. In particular, the variance of a sum will be relatively large when the terms aggregated are positively correlated compared to the variance of a sum when the variables are independent.