INVERSE SEMIGROUPS WHOSE FULL INVERSE SUBSEMIGROUPS FORM A CHAIN

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The structure of semigroups whose subsemigroups form a chain under inclusion was determined by Tamura [9]. If we consider the analogous problem for inverse semigroups it is immediate that (since idempotents are singleton inverse subsemigroups) any inverse semigroup whose inverse subsemigroups form a chain is a group. We will therefore, continuing the approach of [5, 6], consider inverse semigroups whose full inverse subsemigroups form a chain: we call these inverse V-semigroups.

In §1 we show that the non-trivial $\mathcal{J}$-classes of an inverse V-semigroup form a chain, the associated principal factors being either cyclic or quasi-cyclic $p$-groups with zero ($p$ a prime) or isomorphic to $B_5$, the five-element combinatorial Brandt semigroup. Inverse V-semigroups are then characterized by these properties together with (C): for any non-idempotents $x$ and $y$ with $J_x < J_y$, $x = xx^{-1}y^n$ for some non-zero integer $n$.

In §2 the property (C) is used to further elucidate the properties of inverse V-semigroups. It is shown, for instance, that each element of such a semigroup $S$ has index at most 2. If $S$ has an infinite subgroup $G$, then $G$ contains all the non-idempotents of $S$; if $S$ has a non-trivial subgroup $G$ of prime-power, but not prime, order, then $x^2 \in G$ for every non-idempotent $x$ of $S$.

Meakin [8] described a very special class of inverse V-semigroups: those with no proper inverse subsemigroups whatsoever. In the present paper the characterization given by the author in [6] of those inverse semigroups whose lattice of full inverse subsemigroups is distributive is clearly relevant.

1. A characterization. We begin with some terminology and notation, and a summary of the results from [5] which we will use. For background on lattices of full inverse subsemigroups the reader is referred to [5, 6].

Denote by $\mathcal{L}(S)$, or just $\mathcal{L}$, the lattice of full inverse subsemigroups of the inverse semigroup $S$ (that is, the lattice of those inverse subsemigroups of $S$ containing the semilattice $E$ of idempotents of $S$). If $A \subseteq S$, denote by $\langle A \rangle$ the inverse subsemigroup of $S$ generated by $A$, and by $\langle E, A \rangle$ the full inverse subsemigroup generated by $A$, that is, $\langle E \cup A \rangle$. In general we use the notation of [4].

**Result 1.1 ([5, Corollary 1.2, Proposition 1.3]).** For each $\mathcal{J}$-class $J$ of $S$ the relation $\gamma_J$ on $\mathcal{L}$, defined by $A \gamma_J B$ if $A \cap J = B \cap J$, is a congruence. In the lattice of congruences on $\mathcal{L}$, $\langle \{ \gamma_J : J \in S/\mathcal{J} \} \rangle = 0$. Hence $\mathcal{L}$ is a subdirect product of the lattices $\mathcal{L}[\gamma_J]$, $J \in S/\mathcal{J}$. Moreover $\mathcal{L}/\gamma_J \cong \mathcal{L}(P(F(J))$, where $P(F(J)$ is the principal factor associated with $J$, under the isomorphism $\psi_J : \mathcal{L}/\gamma_J \to (A \cap J) \cup \{0\}$.

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COROLLARY 1.2. The lattice $\mathcal{L}_F$ is distributive if and only if $\mathcal{L}_F(\operatorname{PF}(J))$ is distributive for each $\mathcal{J}$-class $J$.

RESULT 1.3 ([5, Corollary 3.6]). If $S$ is a completely 0-simple inverse semigroup with $\mathcal{L}_F$ distributive then $S$ is either a group with zero or is isomorphic to $B_5$, the combinatorial Brandt semigroup with five elements.

(Note: as in [5] we define $\operatorname{PF}(J) = J \cup \{0\}$, the product of two elements of $J$ being their product in $S$ if they lie in $J$, and all other products being zero. Thus $\operatorname{PF}(J)$ is always 0-simple).

Finally, we require a characterization of $\sqcap$-groups, (that is groups whose subgroups form a chain under inclusion).

RESULT 1.4 ([10, Theorem 5]). A group is a $\sqcap$-group if and only if it is a cyclic or quasi-cyclic $p$-group, for some prime $p$.

(The notation $\mathbb{Z}(p^\infty)$ is often used for quasi-cyclic $p$-groups. The reader is referred to [3] for their properties).

Since $\mathcal{L}_F(B_5)$ is a two-element chain (by Theorem 3.2 of [5]), it is immediate from Result 1.3 that the completely 0-simple $\sqcap$-semigroups are just $B_5$ and the $\sqcap$-groups with zero adjoined. We now show that these are the only 0-simple inverse $\sqcap$-semigroups; thus by Result 1.1, every inverse $\sqcap$-semigroup is completely semisimple.

Suppose $S$ is an inverse $\sqcap$-semigroup which is 0-simple but not completely 0-simple. Clearly $\mathcal{L}_F(S)$ is distributive. It was shown in [6] that such a semigroup is in fact a simple semigroup $S^* (= S \setminus \{0\})$ with zero adjoined and that $\mathcal{L}_F(S^*) = \mathcal{L}_F(S)$. Further $S^*$ is $E$-unitary (that is $ex = e$, $e \in E$, implies that $x \in E$) and if $\sigma$ denotes the least group congruence on $S^*$ then the morphism $\sigma^\sharp$ of $S^*$ upon $S^*/\sigma$ induces a lattice morphism of $\mathcal{L}_F(S^*)$ upon $\mathcal{L}_F(S^*/\sigma)$, the lattice of subgroups of $S^*/\sigma$. Since $S$ is a $\sqcap$-semigroup, $S^*/\sigma$ is a $\sqcap$-group, whence, by Result 1.4, a $p$-group for some prime $p$. But this is impossible, for (since $S$ is not completely 0-simple) $S^*$ contains an element of infinite order ([1, Theorem 2.54]), whose image in $S^*/\sigma$ again has infinite order.

We have proved the necessity of the property (B) in the following characterization of inverse $\sqcap$-semigroups.

THEOREM 1.5. An inverse semigroup $S$ is a $\sqcap$-semigroup if and only if

(A) the non-trivial $\mathcal{J}$-classes of $S$ form a chain,
(B) each non-trivial $\mathcal{J}$-class is either a cyclic or quasi-cyclic $p$-group for some prime $p$, or has principal factor isomorphic to $B_5$,
(C) for each $x, y \in S \setminus E$ with $J_x < J_y$ there is a non-zero integer $n$ such that $x = xx^{-1}y^n$.

Proof. Suppose $S$ is a $\sqcap$-semigroup, and put $E = E_S$. Let $x, y \in S \setminus E$. Either $\langle E, x \rangle \subseteq \langle E, y \rangle$ so that $x \in \langle E, y \rangle$ and $J_x \leq J_y$, or $\langle E, y \rangle \subseteq \langle E, x \rangle$ so that $y \in \langle E, x \rangle$ and $J_y \leq J_x$. This proves (A); (B) has already been shown.

Again let $x, y \in S \setminus E$, with $J_x < J_y$. Clearly $y \notin \langle E, x \rangle$, so $x \in \langle E, y \rangle$. By expressing $x$ as a product involving $E$ and $y$ and permuting idempotents if necessary (c.f. Lemma 2.1 of

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we may write \( x = ey^n \) for some \( e \in E \) and non-zero integer \( n \). Then \( xx^{-1} \leq e \), so
\[ x = ey^n = xx^{-1}ey^n = xx^{-1}y^n, \]
proving (C).

Conversely, suppose \( S \) is an inverse semigroup satisfying (A), (B) and (C), and let \( A, B \in \mathcal{L}(S) \), where \( A \neq E, B \neq E \). Suppose \( A \not\subseteq B \) and let \( a \in A \setminus B \). Put \( J = J_a \). From (B) and the comments following Result 1.4, \( PF(J) \) is an inverse \( \mathcal{V} \)-semigroup. Now from Result 1.1 the map \( C \rightarrow (C \cap J) \cup \{0\} \) is a lattice morphism of \( \mathcal{L}(S) \) upon \( \mathcal{L}(PF(J)) \).

Hence since \( a \in (A \cap J) \setminus (B \cap J) \), so that
\[ (A \cap J) \cup \{0\} \not\subseteq (B \cap J) \cup \{0\}, \]
we have
\[ (B \cap J) \cup \{0\} \subseteq (A \cap J) \cup \{0\}, \]
that is, \( B \cap J \subseteq A \cap J \).

On the other hand if \( b \in B \setminus E \) and \( b \notin J \), then by (A), either \( J_b > J \) or \( J_b < J \). But if \( J_b > J \) then, using (C), \( a = (aa^{-1})b^n \in B \), for some \( n \neq 0 \), a contradiction. Thus \( J_b < J \) and, using (C) again, \( b = (bb^{-1})a^n \in A \), for some \( n \neq 0 \). Therefore \( B \subseteq A \).

Hence \( S \) is a \( \mathcal{V} \)-semigroup.

2. Some consequences. Throughout this section \( S \) will be an inverse \( \mathcal{V} \)-semigroup, with \( E = E_S \). The properties (A), (B) and (C) will be those in Theorem 1.5.

We consider first the restrictions that (B) places on a non-idempotent \( x \) of \( S \). Clearly if \( x \) belongs to a subgroup, that is, \( x \) has index 1 (in the terminology of \([4, \S 1.2]\)), it has prime-power period. Suppose now that \( x \) does not belong to a subgroup. Then \( J_x \) has precisely 4 elements: \( J_x = \{x, x^{-1}, xx^{-1}, x^{-1}x\} \). Consider the monogenic inverse subsemigroup \( \langle x \rangle \) of \( S \): since \( S \) is completely semisimple so is \( \langle x \rangle \), and, further each non-group \( J \)-class of \( \langle x \rangle \) has at most four elements; from the description of all monogenic inverse semigroups (given in, for example, \([2]\)) it is apparent that \( x \) has index 2, that is, \( x^2 \) lies in the kernel \( K_x \) of \( \langle x \rangle \). From (B), again, \( K_x \) is a (cyclic) group of order \( p^k \), for some prime \( p \) and some \( k \geq 0 \). (If \( k = 0 \), \( K_x = \{x^2\} \). Thus
\[ x^2 = x^{2+p^k}, \]
that is, \( x \) has period \( p^k \). The identity \( f \) of \( K_x \) is \( x^{p^k} \) if \( k \geq 1 \), or \( x^2 \) if \( k = 0 \).

Let \( J \) be the \( J \)-class of \( S \) containing \( K_x \). Let \( z \in S \) be such that \( z \not\in f \). Then, by (C), there is a non-zero integer \( n \) such that \( z = fx^n \). Thus \( z \in K_x \). Thus \( J \) is a group and in fact \( J = K_x \). We have thus established

**Proposition 2.1.** In an inverse \( \mathcal{V} \)-semigroup \( S \) every element \( x \) which is not in a subgroup of \( S \) has index 2 and period \( p^k \) for some prime \( p \) and some \( k \geq 0 \). The kernel \( K_x \) of \( \langle x \rangle \) is an entire group \( J \)-class of \( S \), with identity \( x^{p^k} \) if \( k \geq 1 \), or \( x^2 \) if \( k = 0 \).

It is easily verified that any monogenic inverse semigroup generated by an element of index 2 and period \( p^k \), \( k \geq 0 \), satisfies (A), (B) and (C) and is therefore a \( \mathcal{V} \)-semigroup.
We now show that the property (C) imposes major restrictions on the permissible combinations of non-trivial $\mathcal{J}$-classes of $S$. First, however, a technical lemma, whose proof is routine, is needed.

**Lemma 2.2.** Let $T$ be any inverse semigroup, $G$ a group $\mathcal{J}$-class of $T$, with identity $e$, and $U$ an inverse subsemigroup of $T$ such that $e \leq f$ for all $f \in E(U)$. Then the map $u \mapsto eu$ is a morphism of $U$ into $G$.

Now let $G$ and $H$ be non-trivial group $\mathcal{J}$-classes of $S$ with identities $e$ and $f$, respectively, such that $e < f$ (so $G < H$ as $\mathcal{J}$-classes). From (C) it follows that the morphism $\phi_{f,e} : u \mapsto eu$ of $H$ into $G$, defined in the lemma, is surjective. In fact $K\phi_{f,e} = G$ for every non-trivial subgroup $K$ of $H$. In particular this implies $\ker \phi_{f,e} = \{f\}$, so $\phi_{f,e}$ is a bijection.

Furthermore $H$, being a cyclic or quasi-cyclic $p$-group, certainly contains a subgroup of order $p$. Thus $|G| = |H| = p$. Applying (A) we therefore have

**Proposition 2.3.** If $S$ is an inverse $\mathcal{J}$-semigroup with more than one non-trivial maximal subgroup then there is a prime $p$ such that every non-trivial subgroup of $S$ has order precisely $p$.

Now let $J$ be a non-trivial $\mathcal{J}$-class of $S$ containing an element $x$ of index 2, and let $G$ be a non-trivial group $\mathcal{J}$-class of $S$, with identity $e$, such that $G < J$. The case $K_x = G$ was covered in Proposition 2.1. By Exercise 3, §8.4 of [1], $e < xx^{-1}$, $e < x^{-1}x$ and so $e \leq (xx^{-1})(x^{-1}x) = f$, the identity of $K_x$. Hence $K_x \supseteq G$ and there is a morphism of $\langle x \rangle$ upon $G$ (using (C)), as defined in Lemma 2.2, whose restriction to $K_x$ is the identity if $K_x = G$, and is the bijection $\phi_{f,e}$ defined above if $K_x > G$. (Note that the $\mathcal{J}$-class $K_x$ cannot be trivial, for if so, we have $x^2 = x^3$, from which, using (C), it follows that $z^2 = z^3$ for any $z \in G \setminus \{e\}$, a contradiction). Summing up, we have

**Proposition 2.4.** Let $x$ be an element of $S$ of index 2 and let $G$ be a non-trivial group $\mathcal{J}$-class of $S$, with identity $e$, such that $G < J_x$. Then $G \leq K_x$ and the map $u \mapsto eu$ ($u \in \langle x \rangle$) is a morphism of $\langle x \rangle$ upon $G$ whose restriction to $K_x$ is a bijection upon $G$. Thus if $K_x$ is trivial so is every group $\mathcal{J}$-class $G < J_x$.

When the order of a group and a non-group $\mathcal{J}$-class is reversed the situation is rather different.

**Proposition 2.5.** Let $x$ be an element of $S$ of index 2 and let $G$ be a non-trivial group $\mathcal{J}$-class of $S$, with identity $e$, such that $G > J_x$. Then $|G| = 2$ and $x = xx^{-1}z$, where $z$ is the involution of $G$. In that case $x$ has period at most 2.

**Proof.** Since $J_x < G$, we have $x = (xx^{-1})z^n$ for some $n \neq 0$, for any $z \in G \setminus \{e\}$. Thus $xx^{-1} < e$ and for any such $z$,

$$xx^{-1}zRxx^{-1}e = xx^{-1}.$$  

But $R_{xx^{-1}} = \{xx^{-1}, x\}$ and if $xx^{-1}z = xx^{-1}$ then $xx^{-1}z^n = xx^{-1}$ for every non-zero integer $n$, contradicting (C). Hence $xx^{-1}z = x$ for every $z$ in $G \setminus \{e\}$.
Suppose some element \( z \) of \( G \) has order \( l > 2 \). Then
\[
x = xx^{-1}z = xx^{-1}z^2,
\]
so \( xz = x \) and \( xz^n = x \) for all \( n \geq 1 \). But then
\[
x = xz^{l-1} = (xx^{-1}z)z^{l-1} = xx^{-1}z = xx^{-1}e = xx^{-1},
\]
a contradiction. So every non-identity element of \( G \) has order 2. Since \( G \) is cyclic or quasi-cyclic, \(|G| = 2\).

The last statement is an application of Propositions 2.3 and 2.1.

An interesting application of these results is the following.

**Corollary 2.6.** If an inverse \( \mathcal{V} \)-semigroup \( S \) contains a quasi-cyclic maximal subgroup \( G \), then \( G \) constitutes the only non-trivial \( \mathcal{V} \)-class of \( S \).

If \( S \) contains a maximal subgroup \( G \) of prime-power, but not prime order then \( G \) is the only non-trivial maximal subgroup of \( S \) and \( G = K_x \) for every element \( x \) of \( S \) of index 2.

**Proof.** First let \( G \) be any non-trivial maximal subgroup of \( S \) not of prime order. By Proposition 2.3, \( G \) is the only non-trivial maximal subgroup of \( S \). Suppose \( x \in S \) has index 2. By Proposition 2.5, \( G \not\supset J_x \). Thus \( G < J_x \), so that by Proposition 2.4, \( G \leq K_x \) and \(|G| = |K_x| \). Since \( G \) is non-trivial, so is \( K_x \). By Proposition 2.1, \( K_x \) is an entire \( \mathcal{J} \)-class of \( S \) and hence \( G = K_x \). Again by Proposition 2.1, \( K_x \) is finite so \( G \) cannot be quasi-cyclic, proving the first statement.

We consider, finally, the relationship between two non-group \( \mathcal{J} \)-classes \( J < J' \) of \( S \).

Let \( x \in J \setminus E \) and \( y \in J' \setminus E \). From Proposition 2.1 we have that \( K_y \) is a group \( \mathcal{J} \)-class of \( S \), so \( K_y \not\supset J \). Consider the case \( K_y > J \) and suppose \( K_y \) is trivial, that is, \( K_y = \{y^2\} \). From Exercise 3, §8.4 of [1] again, either \( y^2 > xx^{-1} \) or \( y^2 > x^{-1}x \). In the former case
\[
xx^{-1}y^n = xx^{-1}(y^2y^n) = xx^{-1}
\]
for any non-zero integer \( n \), contradicting (C). Since the latter case similarly contradicts the obvious dual of (C), \( K_y \) is therefore non-trivial. In fact, by Proposition 2.5, \(|K_y| = 2 \) and \( x = xx^{-1}y^3 \) (since \( y^3 \) is the involution in \( K_y \)), so that \( xx^{-1} < y^2 \) and
\[
x = (xx^{-1}y^2)y = xx^{-1}y.
\]

Suppose next that \( K_y < J \). Then by Proposition 2.4, \( K_y \leq K_x \). From (C),
\[
x = xx^{-1}y^{2n+1}
\]
(since \( J_y < J_x \) for \(|n| \geq 2 \)). When \( x = xx^{-1}y \), we obtain
\[
x^2 = (xx^{-1}yxx^{-1}y^{-1})y^2,
\]
so \( K_y = K_x \). We obtain a similar result when \( x = xx^{-1}y^{-1} \).
Finally suppose $K_y$ and $J$ are incomparable. Then, by (A), we have that $K_y$ is trivial, so that, by Proposition 2.4, $K_x$ is also. Again we have $x = xx^{-1}y^{-1}$. Summarizing,

**Proposition 2.7.** If $x$ and $y$ in $S$ have index 2 and $J_x < J_y$ then either

(i) $K_y > J_x$, in which case $|K_y| = 2$ and $|K_x| = 2$, or
(ii) $K_y < J_x$, in which case $K_y = K_x$, or
(iii) $K_y$ and $J_x$ are incomparable, in which case $K_x$ and $K_y$ are trivial.

In each case $x = xx^{-1}y^{-1}$.

Before continuing, we provide examples to show that each of these cases may occur.

First, let $E$ be the semilattice in Fig. 1(a), and let $G$ be a group of order 2, with $g$ its involution and 1 its identity. Let $G$ act on $E$ on the left by order automorphisms so that $g$ acts by "reflection". Let $U$ be the semidirect product of $E$ and $G$: that is $U = E \times G$, with product

$$(x, h)(x', h') = (x \cdot hx', hh').$$

(In the terminology of [7], $U = P(G, E, E)$). Then $U$ has 2 non-trivial group $J$-classes, $J_{(f,1)}$ and $J_{(e,1)}$, of order 2, and two non-group $J$-classes, $J_{(a,g)}$ and $J_{(c,g)}$, each with principal factor isomorphic with $B_3$. Clearly (A) and (B) are satisfied and (C) is easily verified. Here $K_{(a,g)} > J_{(c,g)}$ and $|K_{(a,g)}| = |K_{(c,g)}| = 2$. By taking the Rees quotient modulo $J_{(c,g)}$, we obtain a similar example with $|K_{(a,g)}| = 2$ and $|K_{(c,g)}| = 1$.

Now let $E$ be the semilattice in Fig. 1(b), let $G$ be as above, $g$ again acting by "reflection", and form the semidirect product $V$ of $E$ and $G$. In this case

$K_{(a,g)} = K_{(c,g)} = J_{(e,1)}$.

(Here $|K_{(a,g)}| = 2$ but examples may be similarly constructed where $K_{(a, g)}$ has arbitrary prime-power order).

Finally let $E$ be the semilattice in Fig. 1(c) and let $W$ be the full inverse subsemigroup of $T_E$ (see [4, Chapter V]) generated by the isomorphism $y$ taking $ae$ to $be$ and fixing $f$. Then $W$ is an inverse $\nabla$-semigroup. If $x$ is the isomorphism taking $ce$ to $de$ then it is easily verified that, if we take $S = W$, then $S$ has the properties described in Proposition 2.7.(iii).

We now continue the theme of Corollary 2.6.
**COROLLARY 2.8.** If an inverse $\nabla$-semigroup $S$ contains a maximal subgroup of prime order $p \neq 2$, then every non-trivial subgroup has order $p$. Each element of $S$ of index 2 has the same kernel, $K$ say, and the same period $p$. Further $G \leq K$ for each non-trivial subgroup $G$ of $S$.

**Proof.** That every non-trivial subgroup has order $p$ follows from Proposition 2.3. Let $G$ be such a subgroup and let $x$ be any element of $S$ of index 2. By Proposition 2.5, $G \not\leq J_x$. Hence $G < J_x$, so by Proposition 2.4 we have that $G \leq K_x$ and $|G| = |K_x|$. So $x$ has period $p$.

If $y$ is another element of index 2 then $|K_y| = p$ also and we have $K_y \leq K_x \leq K_y$, that is, $K_x = K_y = K$, say, and $G \leq K$ for every non-trivial subgroup $G$ of $S$.

When $S$ has a maximal subgroup of order 2 then we can similarly show that every element of index 2 has period at most 2. However the examples above show that not every such element need have period 2, and the kernels may be disjoint.

**REFERENCES**


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