

Rational points on nonlinear horocycles and pigeonhole statistics for the fractional parts of \sqrt{n}

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Abstract. In this paper, we investigate *pigeonhole statistics* for the fractional parts of the sequence \sqrt{n} . Namely, we partition the unit circle $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ into N intervals and show that the proportion of intervals containing exactly j points of the sequence $(\sqrt{n} + \mathbb{Z})_{n=1}^N$ converges in the limit as $N \rightarrow \infty$. More generally, we investigate how the limiting distribution of the first sN points of the sequence varies with the parameter $s \geq 0$. A natural way to examine this is via point processes—random measures on $[0, \infty)$ which represent the arrival times of the points of our sequence to a random interval from our partition. We show that the sequence of point processes we obtain converges in distribution and give an explicit description of the limiting process in terms of random affine unimodular lattices. Our work uses ergodic theory in the space of affine unimodular lattices, building upon work of Elkies and McMullen [Gaps in $\sqrt{n} \pmod{1}$ and ergodic theory. *Duke Math. J.* **123** (2004), 95–139]. We prove a generalisation of equidistribution of rational points on expanding horocycles in the modular surface, working instead on nonlinear horocycle sections.

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1. Introduction

Let $\mathbb{T} := \mathbb{R}/\mathbb{Z}$ denote the circle, $\mathbb{N} := \{1, 2, 3, \dots\}$ be the set of natural numbers, $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$ be the set of non-negative integers and \mathbb{R}^+ the set of non-negative real numbers. We investigate *pigeonhole statistics* for the sequence \sqrt{n} modulo 1. Specifically, we look at the limiting distribution of the numbers $(\sqrt{n} + \mathbb{Z})_{n=1}^N$ among partitions of \mathbb{T} into intervals of length $1/N$ as $N \rightarrow \infty$.

For $s \geq 0$, $x_0 \in [0, 1)$ and $N \in \mathbb{N}$ with $N \geq 1$, define

$$S_N(x_0, s) := \left| \left\{ 1 \leq n \leq sN : \sqrt{n} \in \left[x_0 - \frac{1}{2N}, x_0 + \frac{1}{2N} \right) + \mathbb{Z} \right\} \right|. \tag{1.1}$$

When x_0 ranges over the set $\Omega_N := \{k/N : 0 \leq k \leq N - 1\} \subset [0, 1)$, the N intervals $[x_0 - 1/2N, x_0 + 1/2N) + \mathbb{Z}$ will partition \mathbb{T} and so the average value of $S_N(x_0, s)$ as x_0 ranges over Ω_N will be $\lfloor sN \rfloor / N = s + O(1/N)$. As a result, it is natural to investigate the long term statistical properties of the sequences $\{S_N(x_0, s) : x_0 \in \Omega_N\}$ as $N \rightarrow \infty$ and, in particular, the proportion of terms equal to a given $j \in \mathbb{N}_0$ as $N \rightarrow \infty$. Indeed, for each $j \in \mathbb{N}_0$, we define

$$E_{j,N}(s) := \frac{1}{N} \left| \left\{ 0 \leq k \leq N - 1 : S_N\left(\frac{k}{N}, s\right) = j \right\} \right|. \tag{1.2}$$

This is the proportion of the intervals $\{[x_0 - 1/2N, x_0 + 1/2N) + \mathbb{Z} : x_0 \in \Omega_N\}$ containing exactly j of the points $\{\sqrt{n} : 1 \leq n \leq sN\}$. Here we show the following.

THEOREM 1.1. *For all $j \in \mathbb{N}_0$ and $s \geq 0$, $E_j(s) := \lim_{N \rightarrow \infty} E_{j,N}(s)$ exists. Moreover, the limiting distribution function $E_j(s)$ is C^2 with respect to s .*

Our proof of Theorem 1.1 builds upon the work of Elkies and McMullen in [7]. Here ergodic theory and, specifically, Ratner’s theorem are used to determine the gap distribution of the sequence $(\sqrt{n} + \mathbb{Z})_{n=1}^\infty$ via relating these properties to the equidistribution of a family of closed orbits of a certain unipotent flow in the homogeneous space

$$X = (\mathrm{SL}(2, \mathbb{Z}) \ltimes \mathbb{Z}^2) \backslash (\mathrm{SL}(2, \mathbb{R}) \ltimes \mathbb{R}^2). \tag{1.3}$$

We elaborate on this further in §1.1.

Remark 1.2. The limiting functions $E_j(s)$ are given more concretely by equation (5.7). They give the probability the lattice corresponding to a randomly chosen point $x \in X$ contains exactly j points in a fixed triangle of area s in the plane. The functions $E_j(s)$ agree with the limiting distribution for the probability of finding j of the points of the sequence $\{\sqrt{n} + \mathbb{Z} : 1 \leq n \leq sN\}$ in a randomly shifted interval of length $1/N$ in \mathbb{T} [7]. They also agree with the limiting functions found by Marklof and Strömbergsson for the probability of finding exactly j lattice points of a typical (two-dimensional) affine unimodular lattice in a ball of radius N whose directions all lie in a random open disc of radius proportional to s/N^2 on the unit circle [14, Theorem 2.1 and Remark 2.3]. As we will see in §5, the work of Marklof and Strömbergsson allows us to immediately infer the aforementioned differentiability of the limiting distribution functions.

Remark 1.3. We do not give exact formulae for the functions $E_j(s)$ in terms of explicit analytic functions in this paper. The analogous functions for rectangles were considered by Strömbergsson and Venkatesh in [17], who obtained explicit piecewise analytic formulae for small j . Based on their work, we would expect the functions $E_j(s)$ to be piecewise analytic with the functions becoming increasingly complex as j increases.

Remark 1.4. As is discussed in, for example, [18], the sequence of fractional parts of the sequence \sqrt{n} is of interest from the point of view of *fine-scale statistics*. The gap

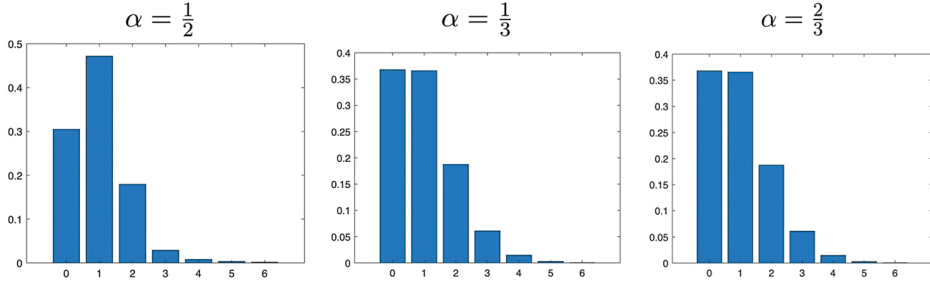


FIGURE 1. The proportion of $N = 10\,000\,000$ intervals in partition of \mathbb{T} containing $0 \leq j \leq 6$ points of $n^\alpha + \mathbb{Z}$ for $n \leq N$ when α is equal to $\frac{1}{2}$, $\frac{1}{3}$ and $\frac{2}{3}$. For $\alpha = 1/2$, these proportions approximate $E_j(s)$ for $s = 1$.

distribution of this sequence in not Poissonian (see also Remark 1.6), which contrasts with the conjectured gap distribution of the fractional parts of n^α for any other $\alpha \in (0, 1) \setminus \{\frac{1}{2}\}$. In our case, if we instead considered the fractional parts of n^α for $\alpha \in (0, 1) \setminus \{\frac{1}{2}\}$, we would expect Poissonian pigeonhole statistics in the sense that the corresponding limiting distribution functions $E_j(s)$ would equal $s^j e^{-j} / j!$. This contrasts with the case $\alpha = \frac{1}{2}$, as shown in Figure 1.

We can also recast our problem in a probabilistic setting. Indeed, for $N \in \mathbb{N}$, let W_N be a random variable which is distributed uniformly on the set Ω_N . Then, we define a sequence of stochastic processes Y_s^N for $N \in \mathbb{N}$ and $s \geq 0$ by setting

$$Y_s^N := S_N(W_N, s). \tag{1.4}$$

With this notation, Theorem 1.1 states that the sequence $\mathbb{P}(Y_s^N = j)$ converges as $N \rightarrow \infty$.

For each N , we can also think about each point $x_0 \in \Omega_N$ as giving us a locally finite Borel measure on \mathbb{R}^+ of the form

$$\eta_N(x_0) := \sum_{r=1}^{\infty} \delta_{s_r(x_0)},$$

where $s_1(x_0) < s_2(x_0) < s_3(x_0), \dots$, are the complete sequence of points $s \in (1/N)\mathbb{N}$ such that $\sqrt{sN} \in [x_0 - 1/2N, x_0 + 1/2N) + \mathbb{Z}$. Namely, these are points of discontinuity of the map $s \mapsto S_N(x_0, s)$. In this case, we have the relation

$$\eta_N(x_0)([0, s]) = S_N(x_0, s). \tag{1.5}$$

Again, recasting this in a probabilistic setting, we define the corresponding sequence of random measures/point processes ξ_N by setting

$$\xi_N := \eta_N(W_N). \tag{1.6}$$

Equation (1.5) above tells us that for an interval $(a, b) \subset \mathbb{R}^+$, we have that the point process and stochastic process are related via

$$\xi_N((a, b)) = Y_b^N - Y_a^N. \tag{1.7}$$

In this setting, we establish the following convergence result which helps us to understand how the limiting distribution of the points of our sequence varies with s .

THEOREM 1.5. *The point process $(\xi_N)_{N=1}^\infty$ converges in distribution to a point process ξ .*

The process ξ is defined similarly to the processes ξ_N as the sum of Dirac delta measures associated to the jump points of a stochastic process $Y_s : X \rightarrow \mathbb{R}$. Here the space X can be thought of as the homogeneous space of all two-dimensional affine unimodular lattices (which we show explicitly in §2) and $Y_s(x)$ gives the number of points of the lattice associated to $x \in X$ within a certain triangle of areas s in the plane. More concretely, if the lattice associated to $x \in X$ is $L \subset \mathbb{R}^2$ and

$$\tau(\infty) := \{(u, v) \in \mathbb{R}^2 : u \geq 0, -u \leq v \leq u\}, \tag{1.8}$$

then

$$\xi(x) = \sum_{(u,v) \in L \cup \tau(\infty)} \delta_{\sqrt{u}}. \tag{1.9}$$

As we illustrate in §6, ξ is a simple, intensity-1 process which does not have independent increments.

Remark 1.6. The pigeonhole statistics we consider were previously studied by Weiss and Peres for the fractional parts of the sequence $2^n \alpha$ (as well as higher dimensional generalisations). In this case, the analogous processes converge to a Poisson point process [19]. A Poisson point process is also (almost surely) the limiting process we would obtain if, instead of generating our point processes via considering how the points of the sequence \sqrt{n} distribute among shrinking partitions of \mathbb{T} , we instead consider the analogous processes defined for a sequence of points in \mathbb{T} generated by a sequence of independent and identically distributed random variables which are uniformly distributed on \mathbb{T} [8, §VI.6].

Similarly to what is observed in [6], even though our limiting processes is not Poissonian, its second moment is nearly Poissonian with an error resulting from the fact that, asymptotically, \sqrt{N} of the points $\{\sqrt{n} + \mathbb{Z} : 1 \leq n \leq N\}$ are 0.

COROLLARY 1.7.

$$\mathbb{E}[|Y_s^N|^2] \rightarrow \sum_{j=0}^\infty j^2 E_j(s)^2 + s = \int_X |Y_s|^2 dm_X + s = s^2 + 2s$$

as $N \rightarrow \infty$. In particular,

$$\text{Var}[Y_s^N] \rightarrow 2s$$

as $N \rightarrow \infty$.

Remark 1.8. If we desire the (more satisfactory) convergence of the variance of the random variables Y_s^N to those of Y_s , one has to avoid the escape of mass resulting from the term 0 appearing regularly in the sequence of fractional parts of \sqrt{n} . This can be done via removing the terms \sqrt{n} when n is a square and, in this case, we would have $\text{Var}[Y_s^N] \rightarrow s$ which is the variance we would obtain if the limiting point process were Poissonian. We will also use this approach in the proof of Corollary 1.7.

1.1. *Ergodic theory.* Let $G = \text{ASL}(2, \mathbb{R}) = \text{SL}(2, \mathbb{R}) \ltimes \mathbb{R}^2$ be the affine special linear group of \mathbb{R}^2 with multiplication law defined by

$$(M, x)(M', x') = (MM', xM' + x'),$$

where elements of \mathbb{R}^2 are viewed as row vectors. Let $\Gamma = \text{SL}(2, \mathbb{Z}) \ltimes \mathbb{Z}^2$ be the discrete subgroup of G consisting of elements with integer entries. As is discussed in §2, Γ is a lattice in G , meaning we have a fundamental domain \tilde{F} with finite volume (and hence, up to normalization, volume 1) under the Haar measure m_G on G . By restricting m_G to \tilde{F} and projecting to X , we have a right-invariant probability measure m_X on the space $X := \Gamma \backslash G$ which we call the Haar measure on X [4, Proposition 9.20]. Let

$$\Phi(t) := \left(\begin{pmatrix} e^{-t/2} & 0 \\ 0 & e^{t/2} \end{pmatrix}, (0, 0) \right)$$

and

$$a(N) := \Phi(\log(N)).$$

As in [7], we shall be concerned with the equidistribution of points on certain *horocycle sections* in the space X . Here, a horocycle section is a function $\sigma : \mathbb{R} \rightarrow G$ of the form

$$\sigma(t) := \left(\begin{pmatrix} 1 & 2t \\ 0 & 1 \end{pmatrix}, (x(t), y(t)) \right),$$

where $x(t)$ and $y(t)$ are smooth functions. We call $\sigma(t)$ a horocycle section of period $p \in \mathbb{N}$ if there exists some $\gamma_0 \in \Gamma$ such that $\gamma_0\sigma(t + p) = \gamma_0\sigma(t)$ for all $t \in \mathbb{R}$. Moreover, such a horocycle section is *nonlinear* if there exists some $\alpha, \beta \in \mathbb{Q}$ such that the set $\{t \in [0, p] : y(t) = \alpha t + \beta\}$ has zero Lebesgue measure. For such horocycle sections, the following equidistribution result is known.

THEOREM 1.9. [7, Theorem 2.2], [12, Theorem 4.2] *Let σ be a nonlinear horocycle section with period p . Then, for any bounded continuous function $f : X \rightarrow \mathbb{R}$,*

$$\frac{1}{p} \int_0^p f(\Gamma\sigma(x_0)\Phi(t)) dx_0 \rightarrow \int_X f dm_X$$

as $t \rightarrow \infty$.

Applying this to the nonlinear period 1 horocycle section

$$n(t) := \left(\begin{pmatrix} 1 & 2t \\ 0 & 1 \end{pmatrix}, (t, t^2) \right),$$

one can determine the distribution of $\{\sqrt{n}\}_{n=1}^N$ among the intervals $[x_0 - 1/2N, x_0 + 1/2N) + \mathbb{Z}$ when x_0 is uniformly distributed on $[0, 1)$. In our setting, we restrict x_0 to lying in the set Ω_N for each N and the corresponding equidistribution we desire is that of rational points on such a horocycle section. We therefore prove the following result which, like Theorem 1.9, applies more generally to functions $f : X \rightarrow \mathbb{R}$ which are piecewise continuous: functions $f : X \rightarrow \mathbb{R}$ whose points of discontinuity are contained in a set of measure zero with respect to m_X .

THEOREM 1.10. *Let σ be a nonlinear horocycle section with period p . Then, for any bounded piecewise continuous function $f : X \rightarrow \mathbb{R}$ and $C \geq 1$,*

$$\frac{1}{pN} \sum_{k=0}^{pN-1} f\left(\Gamma\sigma\left(\frac{k}{N}\right)a(M)\right) \rightarrow \int_X f \, dm_X \tag{1.10}$$

as $N \rightarrow \infty$ and $1/CN \leq M \leq CN$.

As we show concretely in §5, for an appropriate $f : X \rightarrow \mathbb{R}$, we can approximate $\mathbb{P}(\xi_N((a, b]) = 0)$ (or more generally $\mathbb{P}(\xi_N(B) = 0)$ for B as in Lemma 1.13(ii)) by a sum of the above form in equation (1.10) and, using the above equidistribution result, show Theorem 1.5. The same principle applies in the case of Theorem 1.1.

Remark 1.11. Although Theorems 1.1, 1.5 and 1.10 are stated for the points/interval centres k/N for $0 \leq k \leq N - 1$, one can see the methods presented in this paper also give the analogous results when considering the points/interval centres $(k + \alpha)/N$ for any $\alpha \in \mathbb{R}$. The choice $\alpha = \frac{1}{2}$ in particular results in considering the points of the sequence \sqrt{n} in the intervals formed via partitioning by cutting \mathbb{T} at the points k/N for $0 \leq k \leq N - 1$.

Remark 1.12. There are many known results related to Theorem 1.10 when considering the equidistribution of discrete collections of points on expanding horocycle orbits. An effective equidistribution theorem for rational horocycle points $\{k/N + iy\}_{k=0}^{N-1}$ in the modular surface is proved by Burrin, Shapira and Yu in [1, Theorem 1.1]. Using spectral methods, [1] shows such points equidistribute when the number of such rational points N being considered at height y satisfies $N \gg y^{-(39/64+\epsilon)}$ for some $\epsilon > 0$. This contrasts with Theorem 1.10, which corresponds to the case when $N \asymp y^{-1}$. Using dynamical methods, Einsiedler, Luethi and Shah prove effective equidistribution results for the rational points

$$\left\{ \left(\text{SL}(2, \mathbb{Z}) \begin{pmatrix} 1 & k/N \\ 0 & 1 \end{pmatrix} \begin{pmatrix} N^{-1/2} & 0 \\ 0 & N^{1/2} \end{pmatrix}, \frac{k}{N} + \mathbb{Z} \right) : 0 \leq K \leq N - 1 \right\}$$

in the more general space $\text{SL}(2, \mathbb{Z}) \backslash \text{SL}(2, \mathbb{R}) \times \mathbb{T}$ [3]. The equidistribution of such points when projected $\text{SL}(2, \mathbb{Z}) \backslash \text{SL}(2, \mathbb{R})$ is implied by Theorem 1.10. Finally, in [13], Marklof and Strömbergsson prove for fixed $\delta > 0$, there is full measure set of $\alpha \in [0, 1)$ such that the points $\{m\alpha + iy\}_{m=1}^N$ equidistribute in the modular surface as $y \rightarrow 0$ whenever $y \asymp N^{-\delta}$.

1.2. Outline of proof. Recall the following conditions which are sufficient to give the convergence in distribution of a sequence of point processes [11, Theorem A2.2].

LEMMA 1.13. [11, Theorem A2.2] *Let $(\xi_n)_{n=1}^\infty$ and ξ be point processes defined on \mathbb{R}^+ with ξ being simple. Suppose the following:*

- (i) $\mathbb{E}[\xi_N((a, b])] \rightarrow \mathbb{E}[\xi((a, b))]$ as $N \rightarrow \infty$ for all $0 \leq a < b < \infty$;
- (ii) $\mathbb{P}[\xi_N(V) = 0] \rightarrow \mathbb{P}[\xi(V) = 0]$ for all V of the form $\bigcup_{j=1}^k (a_j, b_j]$ with $0 \leq a_1 < b_1 \leq a_2 < b_2 \leq \dots \leq a_k < b_k$.

Then $\xi_N \xrightarrow{d} \xi$, where \xrightarrow{d} denotes convergence in distribution.

For our processes ξ_N defined by equation (1.6), we will see that condition (i) merely amounts to the fact the average number of points of an affine unimodular lattice in a triangle of area s is s . We prove this more generally in Lemma 5.1.

Turning to (ii), we define the measures $(\nu_N)_{n=1}^\infty$ on X by

$$\nu_N(f) = \int_X f \, d\nu_N := \frac{1}{N} \sum_{k=0}^{N-1} f\left(\Gamma n\left(\frac{k}{N}\right)a(N)\right). \tag{1.11}$$

In §5, for a given set V as in Lemma 1.13(ii), we show how to choose the function $f : X \rightarrow \mathbb{R}$ such that $\nu_N(f)$ approximates $\mathbb{P}(\xi_N(B) = 0)$. The same is true in proving Theorem 1.1, where we choose a function $f : X \rightarrow \mathbb{R}$ such that $\nu_N(f)$ approximates $\mathbb{P}(Y_s^N = j)$. By taking $N \rightarrow \infty$, we can then show the required limiting values are attained using Theorem 1.10. For the remainder of this section, we thus focus on the proof of Theorem 1.10.

Proof outline of Theorem 1.10 for $\sigma(t) = n(t)$. By a standard approximation argument, it suffices to show we have $\nu_N(f) \rightarrow \int_X f \, dm_X$ for all $f \in C_c(X)$. This allows us to reduce to understanding weak-star limit points of the sequence of measure (ν_N) . In particular it suffices, by the Banach–Alaoglu theorem, to show any accumulation point ν of the measures (ν_N) is m_X .

As is shown in Proposition 3.1, moving from $n(k/N)a(N)$ to $n((k + 1)/N)a(N)$ corresponds, up to some negligible error, to right multiplication by the unipotent element $u(1)$, where

$$u(t) := \left(\begin{pmatrix} 1 & 2t \\ 0 & 1 \end{pmatrix}, (0, 0) \right) \tag{1.12}$$

for $t \in \mathbb{R}$. It will follow that any such ν is invariant under the action of the subgroup $\{u(k)\}_{k \in \mathbb{Z}}$.

The right-action of this subgroup on X is mixing, as is shown in Lemma 3.4. A consequence of this is that the system (X, U_t, m_X) , where $U_t(x) = xu(t)$, is disjoint from the linear rotation flow on $[0, 1)$ in the sense introduced by Furstenberg in [9] (as is shown in Lemma 3.3). To be precise, the linear rotation flow $R_t : [0, 1) \rightarrow [0, 1)$ is given by $R_t(s) = \{s + t\}$, where $\{\cdot\}$ gives the fractional part of a real number. This is used to extend to the flow U_t to $\tilde{U}_t : X \times [0, 1) \rightarrow X \times [0, 1)$ given by $\tilde{U}_t(\Gamma g, s) = (\Gamma gu(t), \{s + t\})$. Disjointness then tells us that the only \tilde{U}_t -invariant measure on $X \times [0, 1)$ whose marginals (projections to X and $[0, 1)$) are m_X , and the Lebesgue measure ds on $[0, 1)$ is the product measure $m_X \times ds$.

To use this fact, we consider the corresponding *special flow under the ceiling function 1*: namely, the flow $T_t : X \times [0, 1) \rightarrow X \times [0, 1)$ given by

$$T_t(\Gamma g, s) = (\Gamma gu(\lfloor s + t \rfloor), \{s + t\}).$$

This flow has $\nu \times ds$ as an invariant measure and is also conjugate to the flow \tilde{U}_t . Keeping track of the measure $\nu \times ds$ under this conjugation map (described explicitly in the proof of Proposition 3.6) and using Theorem 1.9, we see the resulting measure on $X \times [0, 1)$ indeed has marginals m_X and ds , and so is the product measure $m_X \times ds$. This in turn

gives us that $\nu \times ds = m_X \times ds$ by applying the inverse of the conjugation map and so $\nu = m_X$, as required. □

2. *The space X*

Here we overview, for completeness, some of the basic properties of the space $X = \Gamma \backslash G$ which we will be using. More details can be found in [12, §3.1] and [16, §1].

- X is a $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$ bundle over the base space $B := \text{SL}(2, \mathbb{Z}) \backslash \text{SL}(2, \mathbb{R})$. If \mathcal{F} is a fundamental domain for the left-action of $\text{SL}(2, \mathbb{Z})$ on $\text{SL}(2, \mathbb{R})$, then a fundamental domain for the left-action of Γ on G is

$$\begin{aligned} \tilde{\mathcal{F}} &= \{(I_2, x)(M, 0) : x \in [0, 1), M \in \mathcal{F}\} \\ &= \{(M, x) \in G : M \in \mathcal{F} \text{ and } x \in [0, 1)^2 M\}. \end{aligned}$$

We fix such \mathcal{F} and $\tilde{\mathcal{F}}$ for the remainder of paper.

- Let $m_{\text{SL}(2, \mathbb{R})}$ be the Haar measure on the unimodular group $\text{SL}(2, \mathbb{R})$, normalised so that $m_{\text{SL}(2, \mathbb{R})}(\mathcal{F}) = 1$. Using Fubini’s theorem and the translation invariance, it is easy to see $m_G = m_{\text{SL}(2, \mathbb{R})} \times dx$ is a (left) Haar measure on X , where dx represents the Lebesgue measure on \mathbb{R}^2 . The right-invariant measure m_X on X is obtained by then restricting this measure m_G to $\tilde{\mathcal{F}}$.
- There exists a left-invariant Riemannian metric d_G on G inducing the same topology on G as the product topology on the space $\text{SL}(2, \mathbb{R}) \times \mathbb{R}^2$. Fixing one such metric d_G , we construct a metric d on X via defining

$$d(\Gamma g_1, \Gamma g_2) := \inf_{\gamma \in \Gamma} d_G(\gamma g_1, g_2).$$

For more explicit details on these constructions, see [4, §9.3]. Throughout the remaining sections, continuity of functions $f : X \rightarrow \mathbb{R}$ will mean continuity with respect to this metric.

- Any element $(M, x) \in G$ gives us an affine unimodular lattice in \mathbb{R}^2 —namely the lattice $\mathbb{Z}^2 M + x$. Moreover, for any other $(M', x') \in G$, the lattice associated to $(M', x')(M, x)$ is given by $\mathbb{Z}^2 M' M + x' M + x$. These two lattices are identical if and only if $(M', x') \in \Gamma$. Thus we have a natural identification between elements of $X = \Gamma \backslash G$ and such lattices. We will use this identification in §5 to construct the functions $f : X \rightarrow \mathbb{R}$ to which we will apply Theorem 1.10.

3. *The special flow under 1*

Throughout this section, whenever (X_1, μ_1) is a measure space, X_2 is a measurable space and $\mathcal{T} : X_1 \rightarrow X_2$ is a measurable map, we will define the measure $\mathcal{T}_* \mu_1$ on X_2 by

$$\mathcal{T}_* \mu_1(A) = \mu_1(\mathcal{T}^{-1}(A))$$

for any measurable $A \subset X_2$.

PROPOSITION 3.1. *Let σ be a nonlinear horocycle section of period p . Define the measures $(\nu_N)_{n=1}^\infty$ on X by setting*

$$\nu_N(f) = \int_X f d\nu_N := \frac{1}{pN} \sum_{k=0}^{pN-1} f\left(\Gamma\sigma\left(\frac{k}{N}\right)a(N)\right) \tag{3.1}$$

for any continuous bounded $f : X \rightarrow \mathbb{C}$. Then, any weak-star limit point of the measures defined in equation (3.1) is invariant under the map $T : X \rightarrow X$ given by

$$T(\Gamma g) = \Gamma g u(1), \tag{3.2}$$

where $u(1)$ is defined by equation (1.12).

To see this, we will need the following lemma.

LEMMA 3.2. Any $f \in C_c(X)$ is uniformly continuous in the \mathbb{T}^2 direction. More precisely, for any $\epsilon > 0$, we can find $\delta > 0$ such that for all $M \in SL(2, \mathbb{R})$ and $u, v \in \mathbb{T}^2$ with $d_{\mathbb{T}^2}(u, v) \leq \delta$, we have

$$|f((I_2, u)(M, 0)) - f((I_2, v)(M, 0))| < \epsilon.$$

Proof. Take $f \in C_c(X)$ and let K be the projection of the support of f to the base space B . Here, K is a compact set and so the map $K \times \mathbb{T}^2 \ni (M, x) \mapsto f((I_2, x)(M, 0)) \in \mathbb{R}$ is uniformly continuous. This means f is uniformly continuous in the fibre direction over K in the sense that for any $\epsilon > 0$, we can find $\delta > 0$ such that for any $M \in K$ and $u, v \in \mathbb{T}^2$ with $d_{\mathbb{T}^2}(u, v) < \delta$, we have

$$|f((I_2, u)(M, 0)) - f((I_2, v)(M, 0))| < \epsilon.$$

Hence, since f is identically zero on the fibre above all base points outside of K , f is in fact uniformly continuous in the fibre direction over all of B . □

Proof of Proposition 3.1. Suppose ν is a weak-star limit of the sequence of measures (ν_{N_j}) where $N_j \nearrow \infty$.

Now, for any $N \in \mathbb{N}$ and $0 \leq k \leq pN - 1$, we will see via equations (3.3) and (3.4) that the two points $\sigma((k + 1)/N)a(N)$ and $\sigma(k/N)a(N)u(1)$ are identical in their $SL(2, \mathbb{R})$ components and, as the functions x and y are smooth and so bounded and Lipschitz on $[0, p]$, differ by a distance $O(1/N)$ in the \mathbb{T}^2 direction. Using this, we will see the measures $\{T_*\nu_{N_j}\}_{j \in \mathbb{N}}$ given by

$$T_*\nu_{N_j}(f) = \frac{1}{pN_j} \sum_{k=0}^{pN_j-1} f\left(\Gamma\sigma\left(\frac{k}{N_j}\right)a(N_j)u(1)\right), \quad f \in C_c(X)$$

will also converge to ν as $j \rightarrow \infty$ in the weak-star topology, since, for compactness, $f \in C_c(X)$, $f(\sigma(k/N_j)a(N_j)u(1))$ and $f(\sigma((k + 1)/N)a(N))$ will be uniformly close across all $0 \leq k \leq pN_j - 1$ in equation (3.5). This will follow from the Lemma 3.2.

Indeed, we have

$$\begin{aligned} \sigma\left(\frac{k+1}{N}\right)a(N) &= \left(\left(\begin{matrix} N^{-1/2} & 2(k+1)N^{-1/2} \\ 0 & N^{1/2} \end{matrix}\right), \left(x\left(\frac{k+1}{N}\right)N^{-1/2}, y\left(\frac{k+1}{N}\right)N^{1/2}\right)\right) \\ &= \left(I_2, \left(x\left(\frac{k+1}{N}\right), y\left(\frac{k+1}{N}\right) - \frac{2(k+1)}{N}x\left(\frac{k+1}{N}\right)\right)\right) \\ &\quad \times \left(\left(\begin{matrix} N^{-1/2} & 2(k+1)N^{-1/2} \\ 0 & N^{1/2} \end{matrix}\right), (0, 0)\right) \end{aligned} \tag{3.3}$$

and

$$\sigma\left(\frac{k}{N}\right)a(N)u(1) = \left(I_2, \left(x\left(\frac{k}{N}\right), y\left(\frac{k}{N}\right) - \frac{2k}{N}x\left(\frac{k}{N}\right)\right)\right) \tag{3.4}$$

$$\times \left(\begin{pmatrix} N^{-1/2} & 2(k+1)N^{-1/2} \\ 0 & N^{1/2} \end{pmatrix}, (0, 0)\right).$$

Take $f \in C_c(X)$ and let $\epsilon > 0$. Choose $\delta > 0$ as given by Lemma 3.2 for such ϵ . By the fact x, y are bounded and Lipschitz on $[0, p]$, for any sufficiently large j sufficiently large, we have that

$$d_{\mathbb{T}^2}\left(\left(x\left(\frac{k+1}{N_j}\right), y\left(\frac{k+1}{N_j}\right) - \frac{k+1}{N_j}x\left(\frac{k+1}{N_j}\right)\right), \left(x\left(\frac{k}{N_j}\right), y\left(\frac{k}{N_j}\right) - \frac{k}{N_j}x\left(\frac{k}{N_j}\right)\right)\right) < \delta \tag{3.5}$$

for all $0 \leq k \leq N_j - 1$. Hence, for such j ,

$$\begin{aligned} &|T_*(v_{N_j})(f) - v_{N_j}(f)| \\ &\leq \frac{1}{pN_j} \left| \sum_{k=0}^{pN_j-1} f\left(\Gamma\sigma\left(\frac{k}{N_j}\right)a(N_j)u(1)\right) - f\left(\Gamma\sigma\left(\frac{k}{N_j}\right)a(N_j)\right) \right| \\ &\leq \frac{1}{pN_j} \left| \sum_{k=0}^{pN_j-1} f\left(\Gamma\sigma\left(\frac{k}{N_j}\right)a(N_j)u(1)\right) - f\left(\Gamma\sigma\left(\frac{k+1}{N_j}\right)a(N_j)\right) \right| + O\left(\frac{\|f\|_\infty}{N_j}\right) \\ &\leq \epsilon + O\left(\frac{\|f\|_\infty}{N_j}\right). \end{aligned}$$

So $\limsup_{j \rightarrow \infty} |T_*(v_{N_j})(f) - v_{N_j}(f)| \leq \epsilon$ for any $\epsilon > 0$ and so $T_*(v) = \lim_{j \rightarrow \infty} T_*(v_{N_j})(f) = \lim_{j \rightarrow \infty} v_{N_j}(f) = v(f)$, as required. □

Next, as mentioned in §1.2, we will use the special flow under the ceiling function 1 to show that any weak-star limit point ν of the measures (3.1) is the Lebesgue measure. Specifically, the special flow will give us a system with invariant measure $\nu \times ds$ conjugate to a joining of the systems (X, U_t, m_X) and $([0, 1), R_t, ds)$, where $U_t(\Gamma g) = \Gamma g u(t)$ and $R_t(s) = \{s + t\}$. This will imply $\nu = m_X$ due to the following.

LEMMA 3.3. *The flows (X, U_t, m_X) and $([0, 1), R_t, ds)$ are disjoint.*

To see this, we will use the following lemmas.

LEMMA 3.4. *The system (X, U_t, m_X) is mixing.*

Proof. This follows from applying the proposition from [10, §2.2] to the system (X, U_t, m_X) (instead of a diagonal flow) and using that the horocycle flow on B is ergodic. □

LEMMA 3.5. [2, Proposition 2.2] *Let $T : X \rightarrow X$ be an ergodic measure-preserving transformation with respect to the measure m_X . Then, (X, T, m_X) is disjoint from any measure-preserving system given by the identity map $I : Y \rightarrow Y$ on a probability space (Y, μ) .*

Proof of Lemma 3.3. Let μ be a joining of (X, U_t, m_X) and $([0, 1), R_t, ds)$ an invariant measure on $X \times [0, 1)$ for the map $\tilde{U}_t(\Gamma g, s) = (\Gamma gu(t), \{s + t\})$ whose marginals are m_X and ds . Here, μ will be invariant under the map \tilde{U}_1 , and so is a joining of the systems (X, U_1, m_X) and $([0, 1), R_1, ds)$. Additionally, R_1 is the identity and, by Lemma 3.4, U_1 is mixing and hence ergodic. Thus it follows from Lemma 3.5 that $\mu = m_X \times ds$, as required. \square

We are now in a position to prove the following.

PROPOSITION 3.6. *Any weak-star limit point ν of the measures (3.1) is the Haar measure m_X on X .*

As mentioned, the main construction we will use in this proof is the special flow under the ceiling function 1.

LEMMA 3.7. [4, Lemma 9.23] *Let ν be a finite measure on X which is invariant under $u(1)$. Then $\nu \times ds$ is an invariant measure for the map $T_t : X \times [0, 1) \rightarrow X \times [0, 1)$ given by $T_t(\Gamma g, s) = (\Gamma gu(\lfloor s + t \rfloor), \{s + t\})$.*

Proof. If $\nu(X) > 0$, the result is given by [4, Lemma 9.23] (which applies to probability measures and hence any non-zero finite measure via normalizing). Otherwise, the result is trivial as $\nu \times ds$ is the zero measure. \square

Proof of Proposition 3.6. Note that for a weak-star limit point ν of the probability measures in (3.1), we have $\nu(X) \in [0, 1]$. Thus, Lemma 3.7 implies $\nu \times ds$ is T_t invariant, where T_t is as in the statement of Lemma 3.7.

Now let $\psi : X \times [0, 1) \rightarrow X \times [0, 1)$ be given by $\psi(\Gamma g, s) = (\Gamma gu(s), s)$ and recall the extension of the flow U_t to $X \times [0, 1)$ is given by $\tilde{U}_t(x, s) = (xu(t), \{s + t\})$. Using that $s + t = \lfloor s + t \rfloor + \{s + t\}$, we see that $\psi \circ T_t = \tilde{U}_t \circ \psi$, meaning T_t and \tilde{U}_t are conjugate via ψ and $\mu := \psi_*(\nu \times ds)$ is an invariant measure for the flow \tilde{U}_t . Denote the projection maps from $X \times [0, 1)$ to X and $[0, 1)$ by P_X and $P_{[0,1)}$, respectively. Here $P_{[0,1)*}(\mu)$ is invariant under all $R_t : [0, 1) \rightarrow [0, 1)$ with $t \in \mathbb{R}$ and so, if it is a probability measure, it is the Lebesgue measure ds on $[0, 1)$.

We now show $(P_X)_*\mu$ is the Haar measure m_X on X , which in turn shows μ is a probability measure. To do this, take $f \in C_c(X)$ and let $N_j \nearrow \infty$ be a sequence of natural numbers such that ν_{N_j} converges weak-star to ν as $j \rightarrow \infty$. Then

$$\begin{aligned} \int_X f d(P_X)_*(\mu) &= \int f \circ P_X \circ \psi d(\nu \times ds) \\ &= \int_0^1 \int_X f(xu(s)) d\nu(x) ds \\ &= \int_0^1 \lim_{j \rightarrow \infty} \frac{1}{pN_j} \sum_{k=0}^{pN_j-1} f\left(\Gamma\sigma\left(\frac{k}{N_j}\right)a(N_j)u(s)\right) ds \\ &= \lim_{j \rightarrow \infty} \int_0^1 \frac{1}{pN_j} \sum_{k=0}^{pN_j-1} f\left(\Gamma\sigma\left(\frac{k}{N_j}\right)a(N_j)u(s)\right) ds, \end{aligned} \tag{3.6}$$

where the last equality follows from the dominated convergence theorem (as f is bounded).

Similarly to as in the proof of Proposition 3.1, $n((k + s)/N)a(N)$ and $n(k/N)a(N)u(s)$ have the same base point and are a distance at most $O(1/N)$ apart in the \mathbb{T}^2 fibre direction whenever $s \in [0, 1]$. Hence, by Lemma 3.2, given any $\epsilon > 0$, we can ensure

$$\int_0^1 \frac{1}{pN_j} \sum_{k=0}^{pN_j-1} f\left(\Gamma\sigma\left(\frac{k+s}{N_j}\right)a(N_j)\right) ds \tag{3.7}$$

and the integrals in equation (3.6) differ by at most ϵ provided j is sufficiently large. However, by making the substitution $t = (s + k)/N_j$, we see that

$$\begin{aligned} \int_0^1 \frac{1}{pN_j} \sum_{k=0}^{pN_j-1} f\left(\Gamma\sigma\left(\frac{k+s}{N_j}\right)a(N_j)\right) ds &= \frac{1}{pN_j} \sum_{k=0}^{pN_j-1} \int_0^1 f\left(\Gamma\sigma\left(\frac{k+s}{N_j}\right)a(N_j)\right) ds \\ &= \frac{1}{p} \sum_{k=0}^{pN_j-1} \int_{k/N_j}^{(k+1)/N_j} f(\Gamma\sigma(t)a(N_j)) dt \\ &= \frac{1}{p} \int_0^p f(\Gamma\sigma(t)a(N_j)) dt \end{aligned} \tag{3.8}$$

and, by Theorem 1.9, equation (3.8) converges to $\int f dm_X$ as $j \rightarrow \infty$. Hence, we have shown that for any $\epsilon > 0$,

$$\left| \int_X f d(P_X)_*(\mu) - \int_X f dm_X \right| \leq \epsilon.$$

Therefore, $(P_X)_*(\mu) = m_X$ and so μ is a joining of (X, U_t, m_X) and (\mathbb{T}, R_t, ds) . Since Lemma 3.3 shows these two systems are disjoint, we conclude $\mu = m_X \times ds$. To see finally that this implies $\nu = m_X$ and note, since m_X is invariant under the right-action of G , we have

$$\begin{aligned} \int g d(\nu \times ds) &= \int g d\psi_*^{-1}(\mu) = \int_0^1 \int_X g(xu(-s), s) dm_X(x) ds \\ &= \int_0^1 \int_X g(x, s) dm_X(x) ds = \int g d(m_X \times ds) \end{aligned}$$

for any $g \in C_c(X \times \mathbb{T})$. Thus, $\nu \times ds = m_X \times ds$ and so $\nu = m_X$. □

4. Completing the proof of Theorem 1.10

Proof of Theorem 1.10. By the Banach–Alaoglu theorem, any subsequence of the measures (ν_N) defined in equation (3.1) has a further subsequence which converges weak-star to some limiting measure ν . By Proposition 3.6, $\nu = m_X$. This shows the sequence of measures (ν_N) indeed converges weak-star to m_X .

Notice that for any constant function $f : X \rightarrow \mathbb{R}$, it is immediate that $\int_X f d\nu_N \rightarrow \int f dm_X$ as $N \rightarrow \infty$. This convergence therefore also holds for continuous functions which are constant outside of a compact set, being the sum of a constant function and a function in $C_c(X)$. Now, let $f : X \rightarrow \mathbb{R}$ be a bounded continuous function and let $\epsilon > 0$. Then we can find continuous functions $f_-, f_+ : X \rightarrow \mathbb{R}$, which are constant outside some

compact set, with $f_- \leq f \leq f_+$ and for which

$$\int_X f_+ - f_- \, dm_X < \epsilon.$$

Then we have

$$\begin{aligned} \int f \, dm_X - \epsilon &\leq \int f_- \, dm_X = \liminf_{n \rightarrow \infty} \nu_N(f_-) \leq \liminf_{n \rightarrow \infty} \nu_N(f) \\ &\leq \limsup_{n \rightarrow \infty} \nu_N(f) \leq \limsup_{n \rightarrow \infty} \nu_N(f_+) = \int f_+ \, dm_X \leq \int f \, dm_X + \epsilon \end{aligned}$$

meaning

$$\liminf_{n \rightarrow \infty} \nu_N(f) = \limsup_{n \rightarrow \infty} \nu_N(f) = \int f \, dm_X,$$

as our choice of $\epsilon > 0$ was general. Thus, (ν_N) converges weakly to m_X and so $\nu_N(f) \rightarrow \int f \, dm_X$ as $N \rightarrow \infty$ for all piecewise continuous $f : X \rightarrow \mathbb{R}$ by the continuous mapping theorem.

Finally, let $C \geq 1$ and $(M_N)_{N=1}^\infty$ be a sequence satisfying $(1/C)N \leq M_N \leq CN$ for all $N \in \mathbb{N}$. Take an arbitrary subsequence $(M_{N_j})_{j=1}^\infty$ of the sequence (M_N) . By compactness of the interval $[1/C, C]$, we can find a further subsequence of (M_{N_j}) , which we will still index by N_j , such that $(M_{N_j}/N_j) \rightarrow c \in [1/C, C]$ as $j \rightarrow \infty$. Let $f \in C_c(X)$ and define $h \in C_c(X)$ by setting

$$h(\Gamma g) := f(\Gamma ga(c)).$$

Note that

$$\nu_{N_j}(h) = \frac{1}{pN_j} \sum_{k=0}^{pN_j-1} f\left(\Gamma\sigma\left(\frac{k}{N_j}\right)a(cN_j)\right).$$

Using the metric d defined in §2, we see

$$d\left(\Gamma\sigma\left(\frac{k}{N_j}\right)a(cN_j), \Gamma\sigma\left(\frac{k}{N_j}\right)a(M_{N_j})\right) \leq d_G\left(e, a\left(\frac{M_{N_j}}{cN_j}\right)\right) \rightarrow 0$$

uniformly in k as $j \rightarrow \infty$. Using this and the fact that as f is continuous and compactly supported, f is uniformly continuous, we have

$$\left| \frac{1}{pN_j} \sum_{k=0}^{pN_j-1} f\left(\Gamma\sigma\left(\frac{k}{N_j}\right)a(cN_j)\right) - \frac{1}{pN_j} \sum_{k=0}^{pN_j-1} f\left(\Gamma\sigma\left(\frac{k}{N_j}\right)a(M_{N_j})\right) \right| \rightarrow 0$$

as $j \rightarrow \infty$. Thus, given $\nu_{N_j}(h) \rightarrow \int h \, dm_X$ and $\int_X h \, dm_X = \int_X f \, dm_X$ by the right-invariance of m_X , we have

$$\frac{1}{pN_j} \sum_{k=0}^{pN_j-1} f\left(\Gamma\sigma\left(\frac{k}{N_j}\right)a(M_{N_j})\right) \rightarrow \int f \, dm_X \tag{4.1}$$

as $j \rightarrow \infty$. Since our original subsequence was arbitrary, equation (4.1) holds in the case where $N_j = j$ as required. This can also be extended to any piecewise continuous function $f : X \rightarrow \mathbb{R}$ by the standard approximation argument above. \square

5. Pigeonhole statistics

As mentioned in §1.1, to prove Theorem 1.13, we are going to apply Theorem 1.10 to a family of functions $f : X \rightarrow \mathbb{R}$ such that $\nu_N(f)$ gives us, up to some error of $o(1)$ in N , $\mathbb{P}(\xi_N((a, b]) = 0)$.

For a non-negative measurable function $f : \mathbb{R}^2 \rightarrow \mathbb{R}$, we define $\widehat{f} : X \rightarrow \mathbb{R}$ by setting $\widehat{f}(x)$ to be the sum of all the function values at the lattice points corresponding to $x \in X$. Explicitly,

$$\widehat{f}(\Gamma(M, x)) = \sum_{m \in \mathbb{Z}^2} f(mM + x).$$

For such functions, the following simple version of Siegel’s formula holds [15].

LEMMA 5.1. *Let $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ be a non-negative measurable function. Then*

$$\int_X \widehat{f} \, dm_X = \int_{\mathbb{R}^2} f \, dx.$$

Proof. Using the non-negativity of f , the fact $\text{SL}(2, \mathbb{R})$ consists of matrices on determinant 1 and the form of the Haar measure m_X described in §2, we see

$$\begin{aligned} \int_X \widehat{f} \, dm_X &= \int_{\mathcal{F}} \int_{[0,1]^2 M} \widehat{f} \, dx \, dm_{\text{SL}(2, \mathbb{R})} \\ &= \int_{\mathcal{F}} \int_{[0,1]^2} \sum_{m \in \mathbb{Z}^2} f((m + x)M) \, dx \, dm_{\text{SL}(2, \mathbb{R})}. \end{aligned}$$

The result then follows from the fact that

$$\int_{[0,1]^2} \sum_{m \in \mathbb{Z}^2} f((m + x)M) \, dx = \int_{\mathbb{R}^2} f(x) \, dx. \quad \square$$

For a set $A \subset \mathbb{R}^2$, we denote by $f_A : X \rightarrow \mathbb{R}$ the function $\widehat{\chi_A}$, where χ_A denotes the indicator function of the set A . Following [12, §4], we see how such functions can be used to approximate the values of the functions S_N . This will allow us to show the random variables Y_s^N defined by equation (1.4) converge to the same limit of a sequence of random variables \widetilde{Y}_s^N which will be defined by evaluating such a function f_A at the points $n(k/N)a(N)$ uniformly at random.

Indeed, fixing some $s > 0$ and setting $N' = \lfloor sN \rfloor$, the counting function $S_N(x_0, s)$ defined in equation (1.1) is given by

$$S_N(x_0, s) = \sum_{n=1}^{N'} \sum_{m \in \mathbb{Z}} \chi_{[-1/2, 1/2)}(N(\sqrt{n} - x_0 + m)). \tag{5.1}$$

It turns out $S_N(x_0, s)$ can be well approximated by $f_\tau(n(x_0)a(N))$, where

$$\tau = \tau(s) := \{(x, y) \in \mathbb{R}^2 : x \in [0, \sqrt{s}], y \in [-x, x]\} \tag{5.2}$$

is a triangle of area s in the plane (see Figure 2).

To see this, it is first useful to rewrite $S_N(x_0, s)$ using the constraint imposed on the summation over m in equation (5.1) by the inner indicator function. Indeed, the constraint

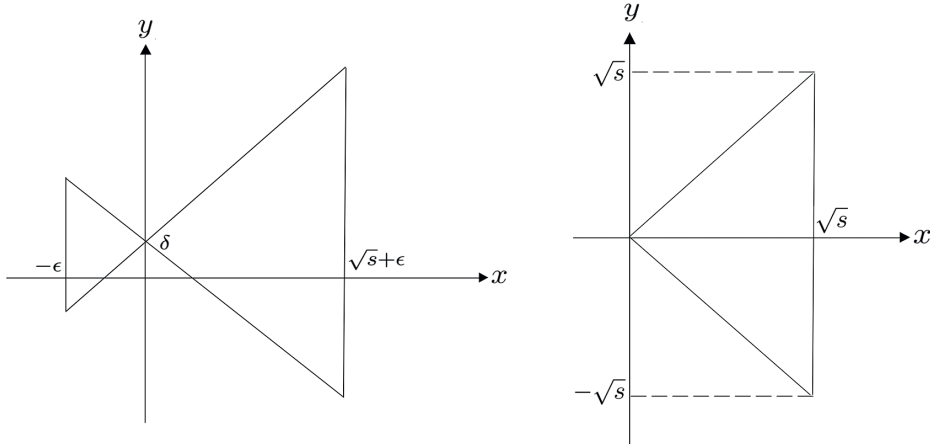


FIGURE 2. The boundaries of $A_{\epsilon,\delta}$ (left) and τ (right).

imposed on the inner sum is equivalent to

$$\left(x_0 - m - \frac{1}{2N}\right)^2 \leq n < \left(x_0 - m + \frac{1}{2N}\right)^2,$$

which amounts to

$$-\frac{1}{N}(x_0 - m) \leq n - (x_0 - m)^2 - \left(\frac{1}{2N}\right)^2 < \frac{1}{N}(x_0 - m),$$

giving us that

$$\chi_{[-1/2,1/2)}(N(\sqrt{n} - x_0 + m)) = \chi_{[-1,1)}\left(\frac{N^{1/2}(n - (x_0 - m)^2 - (1/2N)^2)}{N^{-1/2}(x_0 - m)}\right).$$

Note also, $|\sqrt{n} - x_0 + m| \leq 1/2N$, whenever (m, n) contributes to the sum in equation (5.1). So, the summation bound $1 \leq n \leq N'$ can be replaced by

$$\chi_{(0,1)}\left(\frac{x_0 - m + O(1/2N)}{\sqrt{N'}}\right) \tag{5.3}$$

giving us

$$S_N(x_0, s) = \sum_{(m,n) \in \mathbb{Z}^2} \chi_{(0,1)}\left(\frac{x_0 - m + O(1/2N)}{\sqrt{N'}}\right) \chi_{[-1,1)}\left(\frac{N^{1/2}(n - (x_0 - m)^2 - (1/2N)^2)}{N^{-1/2}(x_0 - m)}\right)$$

whenever $x_0 \neq 0$. The case $x_0 = 0$ can largely be ignored as the random variable W_N , which is uniformly distributed on the set $\Omega_N = \{k/N : 0 \leq k \leq N - 1\}$, has probability $1/N$ of taking this value and we are interested in the limit as $N \rightarrow \infty$.

Therefore, the counting function can be bounded above and below using the following family of functions depending on parameters ϵ and δ , which can be realised as

functions on X :

$$S_{N,\epsilon,\delta}(x_0, s) := \begin{cases} \sum_{(m,n) \in \mathbb{Z}^2} \chi_{(-\epsilon, \sqrt{s} + \epsilon]} \left(\frac{x_0 - m}{N^{1/2}} \right) \chi_{[-1,1)} \left(\frac{N^{1/2}(n - (x_0 - m)^2) + \delta}{N^{-1/2}(x_0 - m)} \right) & \text{if } x_0 \neq 0, \\ 0 & \text{if } x_0 = 0. \end{cases} \tag{5.4}$$

Note that equation (5.3), together with the fact that $N' = \lfloor sN \rfloor$, implies we have

$$S_{N,-\epsilon,\delta}(x_0, s) \leq S_N(x_0, s) \leq S_{N,\epsilon,\delta}(x_0, s) \tag{5.5}$$

for $\epsilon = \epsilon_N := 1/2N(N')^{1/2} + |(N')^{1/2}/N^{1/2} - \sqrt{s}|$, $\delta = \delta_N := -1/4N^{3/2}$ and $x_0 \neq 0$. As $N \rightarrow \infty$, the difference between the upper and lower bounds on $S_N(x_0, s)$ given by equation (5.5) converges to zero in probability as x_0 runs over Ω_N according to W_N , as is shown in Proposition 5.3.

The utility of introducing the functions $S_{N,\epsilon,\delta}$ is that they can be interpreted as functions of the form $f_A : X \rightarrow \mathbb{R}$ for suitable sets $A \subset \mathbb{R}^2$.

PROPOSITION 5.2.

$$S_{N,\epsilon,\delta}(x_0, s) = f_{A_{\epsilon,\delta}}(\Gamma n(x_0)a(N)), \tag{5.6}$$

where $A_{\epsilon,\delta} = A_{\epsilon,\delta}(s) := \{(x, y) \in \mathbb{R}^2 : x \in (-\epsilon, \sqrt{s} + \epsilon], ((y + \delta)/x) \in (-1, 1]\}$.

As one would expect, as ϵ and δ converge to zero, the domains $A_{\epsilon,\delta} = A_{\epsilon,\delta}(s)$ increasingly better approximate the triangle $\tau = \tau(s)$. This is shown in Figure 2.

Proof of Proposition 5.2. From equation (5.4), if we make the substitutions $(m, n) \mapsto (-m, -n)$ and then $n \mapsto n + m^2$ in the sum over n , we get

$$S_{N,\epsilon,\delta}(x_0, s) = \sum_{(m,n) \in \mathbb{Z}^2} \chi_{(-\epsilon, \sqrt{s} + \epsilon]} \left(\frac{x_0 + m}{N^{1/2}} \right) \chi_{[-1,1)} \left(\frac{N^{1/2}(n - x_0^2 + 2mx_0) + \delta}{N^{-1/2}(x_0 + m)} \right).$$

To realise this as the value of a function of the space X , note for

$$(M, x) := n(x_0)a(N) = \left(\begin{pmatrix} N^{-1/2} & 2x_0N^{1/2} \\ 0 & N^{1/2} \end{pmatrix}, \begin{pmatrix} x_0 \\ N^{1/2}, x_0^2N^{1/2} \end{pmatrix} \right),$$

we have that

$$(m, n)M + x = \left(\frac{x_0 + m}{N^{1/2}}, (2mx_0 + n + x_0^2)N^{1/2} \right).$$

Thus,

$$S_{N,\epsilon,\delta}(x_0, s) = f_{A_{\epsilon,\delta}}(\Gamma n(x_0)a(N)),$$

where $A_{\epsilon,\delta} = A_{\epsilon,\delta}(s) := \{(x, y) \in \mathbb{R}^2 : x \in (-\epsilon, \sqrt{s} + \epsilon], ((y + \delta)/x) \in (-1, 1]\}$, as required. □

To relate the random variables Y_s^N to those defined on the space X , we set $\tilde{S}_N(x_0, s) := f_{\tau(s)}(\Gamma n(x_0)a(N))$, $\tilde{Y}_s^N := \tilde{S}_N(W_N, s)$ and, more generally, $Y^{N,\epsilon_N,\delta_N} := S_{N,\epsilon,\delta}(W_N, s)$.

As we will now see, the limiting distribution of the variables Y_s^N is identical to that of \tilde{Y}_s^N . To see this, we first show the following.

PROPOSITION 5.3. $\mathbb{P}(Y_s^N \neq Y_s^{N,\epsilon_N,\delta_N}) \rightarrow 0$ as $N \rightarrow \infty$.

Proof. In light of equation (5.5), it is sufficient to prove

$$\mathbb{P}(Y_s^{N,-\epsilon_N,\delta_N} < Y_s^{N,\epsilon_N,\delta_N}) \rightarrow 0$$

as $N \rightarrow \infty$. Note

$$\begin{aligned} S_{N,\epsilon_N,\delta_N}(x_0, s) - S_{N,-\epsilon_N,\delta_N}(x_0, s) &= (f_{A_{\epsilon_N,\delta_N}} - f_{A_{-\epsilon_N,\delta_N}})(\Gamma n(x_0)a(N)) \\ &= f_{A_N}(\Gamma n(x_0)a(N)), \end{aligned}$$

where $A_N := A_{\epsilon_N,\delta_N} \setminus A_{-\epsilon_N,\delta_N}$. Now, we let the set A'_N be the union of the two rectangles $[-\epsilon_N, \epsilon_N] \times [-\eta, \eta]$ and $[1 - \epsilon_N, 1 + \epsilon_N] \times [-\eta, \eta]$, where $\eta := \sqrt{s} + 1$. Then, for N sufficiently large, $A_N \subset A'_N$ and $A'_N \searrow A_\infty := \{0, 1\} \times [-\eta, \eta]$ which has (Lebesgue) measure zero. Therefore, whenever $n \leq N$ are sufficiently large,

$$\mathbb{P}(Y_s^{N,-\epsilon_N,\delta_N} < Y_s^{N,\epsilon_N,\delta_N}) \leq \nu_N(f_{A'_n}).$$

Each of the functions $f_{A'_n}$ is piecewise continuous as discontinuities of $f_{A'_n}$ correspond to lattices with points in the boundary of A'_n , which has (Lebesgue) measure zero. So, taking $\limsup_{N \rightarrow \infty}$ and using Theorem 1.10, we get

$$\limsup_{n \rightarrow \infty} \mathbb{P}(Y_s^{N,-\epsilon_N,\delta_N} < Y_s^{N,\epsilon_N,\delta_N}) \leq \int_X f_{A'_n} dm_X$$

for all n sufficiently large. Taking $n \rightarrow \infty$ and applying the dominated convergence theorem (as each set A'_n is uniformly bounded and hence $f_{A'_n}$ is uniformly bounded by an integrable function for n sufficiently large), we have $\int_X f_{A'_n} dm_X \rightarrow \int_X f_{A_\infty} dm_X$. By Lemma 5.1, $\int_X f_{A_\infty} dm_X = 0$, which shows the required result. \square

The above then allows us to prove the following.

PROPOSITION 5.4. $\mathbb{P}(Y_s^N \neq \tilde{Y}_s^N) \rightarrow 0$ as $N \rightarrow \infty$.

Proof. This goes along similar lines to the proof of Proposition 5.3. First note that, by this result, it is sufficient to show

$$\mathbb{P}(Y_s^{N,\epsilon_N,\delta_N} \neq \tilde{Y}_s^N) \rightarrow 0$$

as $N \rightarrow \infty$. Points in Ω_N where these random variables differ correspond to lattices with points in exactly one of the sets τ or A_{ϵ_N,δ_N} . Hence, $\mathbb{P}(Y_s^{N,\epsilon_N,\delta_N} \neq \tilde{Y}_s^N) \leq \nu_N(f_{\tau \Delta A_{\epsilon_N,\delta_N}})$. We can also find a sequence $\{\tau_N\}_{N=1}^\infty$ of regions in \mathbb{R}^2 , each consisting of a triangular region with a smaller triangular region removed from its interior, such that $\tau \Delta A_{\epsilon_N,\delta_N,L} \subset \tau_N$ for all N and $\tau_N \searrow W$, where W has (Lebesgue) measure 0. By taking limsups and using Theorem 1.10, we get

$$\limsup_{N \rightarrow \infty} \mathbb{P}(Y_s^{N,\epsilon_N,\delta_N} \neq \tilde{Y}_s^N) \leq \int_X f_{\tau_n} dm_X$$

for all n . Taking $n \rightarrow \infty$, using the dominated convergence theorem and Lemma 5.1 again gives the result. □

Using these two propositions, we are now in a position to prove Theorems 1.1 and 1.5 using Theorem 1.10.

Proof of Theorem 1.1. Let $j \in \mathbb{N}_0$ and $s > 0$. By Proposition 5.4, it is enough to show

$$\lim_{N \rightarrow \infty} \mathbb{P}(\tilde{Y}_s^N = j)$$

exists. We have $\mathbb{P}(\tilde{Y}_s^N = j) = (1/N) \sum_{k=0}^{N-1} \chi_{\{x_0: \tilde{S}_N(x_0)=j\}}(k/N) = \nu_N(f_{j,s})$, where $f_{j,s} := \chi_{\{f_\tau(s)=j\}}$. Now, note that the points of discontinuity of $\chi_{\{f_\tau(s)=j\}}$ correspond to lattices with points in the boundary of the set $\tau(s)$, $\partial\tau(s)$. Namely, if $\Gamma(M, x)$ is a discontinuity point of $f_{j,s}$, then the lattice $\{mM + x\}_{m \in \mathbb{Z}^2}$ contains a point in $\partial\tau$. However, then $f_{\partial\tau(s)}(\Gamma(M, x)) \geq 1$. By Markov's inequality and Lemma 5.1, the set of all such discontinuity points is contained in a set of measure zero, namely the set $\{f_{\partial\tau(s)} \geq 1\}$. So we can apply Theorem 1.10, which gives us that

$$\lim_{n \rightarrow \infty} \mathbb{P}(\tilde{Y}_s^N = j) = \int f_{j,s} \, dm_X.$$

Hence, we see the limiting distribution $E_j(s)$ of the quantities $E_{N,j}(s)$ is given by

$$E_j(s) = m_X(\{\Gamma(M, x) \in X : |(\mathbb{Z}^2 M + x) \cap \tau(s)| = j\}). \tag{5.7}$$

Reference [14, Proposition 8.13] immediately tells us this function is C^2 . □

Proof of Theorem 1.5. Let $Y_s : X \rightarrow \mathbb{R}$ be given by $Y_s = f_{\tau(s)}$ and let ξ be the associated point process. Given a point $\Gamma(M, x) \in X$, this point process takes the form

$$\xi(\Gamma(M, x)) = \sum_{j=1}^{\infty} \delta_{s_j},$$

where $s_j = \inf\{s > 0 : Y_s(\Gamma(M, x)) \geq j\}$. In this setting we have, analogously to equation (1.7), that $\xi((a, b]) = Y_b - Y_a$. This agrees with equation (1.9) due to the definition of $\tau(s)$. If $s_j = s_{j+1}$ for some j , it must be the case that the lattice defined by (M, x) contains multiple points on the boundary of the triangle $\tau(s_j)$. Now, for any $s > 0$, the boundary of the triangle $\tau(s)$ is contained within the lines $y = x$, $y = -x$ and $x = \sqrt{s}$. So, if the lattice defined by $(M, x) \in X$ does intersect the boundary of the triangle $\tau(s)$ in a set of size at least two for some s , then either:

- the lattice contains a point in the line $y = x$;
- the lattice contains a point in the line $y = -x$;
- the lattice contains multiple points in the line $x = \sqrt{s}$ for some $s > 0$.

By Lemma 5.1, the measure of the set of all points $\Gamma(M, x) \in X$ whose corresponding lattice intersects the lines $y = x$ or $y = -x$ is zero. Moreover, in the case where we have multiple lattice points on the line $x = \sqrt{s}$ for some $s > 0$, we can find $(u, v) \in \mathbb{Z}^2$ such that $(u, v)M$ has first coordinate equal to 0. So, for $(u, v) \in \mathbb{Z}^2$, define

$$G_{u,v} := \{(M, x) \in G : (u, v)M = (0, y) \text{ for some } y \in \mathbb{R}\}.$$

For all (u, v) , $G_{u,v}$ is a codimension-one submanifold of G and so $m_X(G_{u,v}) = 0$. Therefore, the set of all lattices with multiple points on one of the vertical lines $x = \sqrt{s}$ has measure zero. Consequently, the points $(s_j)_{j=1}^\infty$ are almost surely distinct and so the process ξ is simple.

To verify condition (i) in Lemma 1.13 holds, take an interval $(a, b] \subset \mathbb{R}^+$. Then, $\mathbb{E}[\xi_N((a, b])] = \mathbb{E}[Y_b^N - Y_a^N] \rightarrow b - a$ as $N \rightarrow \infty$ since, for any $s > 0$, the interval $[x_0 - 1/2N, x_0 + 1/2N] + \mathbb{Z}$ contains, on average, $s + O(1/N)$ points from the sequence $(\sqrt{n})_{n=1}^{\lfloor sN \rfloor}$ as x_0 varies across Ω_N . By Lemma 5.1, $m_X(\xi((a, b])) = m_X(Y_b - Y_a) = b - a$.

To verify condition (ii) holds, let $k \in \mathbb{N}$ and $a_1 < b_1 \leq a_2 < b_2 \leq \dots \leq a_k < b_k$ be non-negative real numbers. Set $V = \bigcup_{j=1}^k (a_j, b_j]$.

$$\mathbb{P}(\xi_N(V) = 0) = \mathbb{P}\left(\bigcap_{j=1}^k \{Y_{b_j}^N = Y_{a_j}^N\}\right). \tag{5.8}$$

Now let $V_N := \bigcap_{j=1}^k (\{Y_{b_j}^N = \tilde{Y}_{b_j}^N\} \cap \{Y_{a_j}^N = \tilde{Y}_{a_j}^N\})$. By Proposition 5.4, $\mathbb{P}(V_N) \rightarrow 1$ as $N \rightarrow \infty$. Therefore,

$$\begin{aligned} & \limsup_{N \rightarrow \infty} \left| \mathbb{P}\left(\bigcap_{j=1}^k \{Y_{b_j}^N = Y_{a_j}^N\}\right) - \mathbb{P}\left(\bigcap_{j=1}^k \{\tilde{Y}_{b_j}^N = \tilde{Y}_{a_j}^N\}\right) \right| \\ & \leq \limsup_{N \rightarrow \infty} \mathbb{P}\left(\bigcap_{j=1}^k \{Y_{b_j}^N = Y_{a_j}^N\} \Delta \bigcap_{j=1}^k \{\tilde{Y}_{b_j}^N = \tilde{Y}_{a_j}^N\}\right) \\ & \leq \limsup_{N \rightarrow \infty} \mathbb{P}(B_N^c) = 0. \end{aligned} \tag{5.9}$$

Moreover,

$$\mathbb{P}\left(\bigcap_{j=1}^k \{\tilde{Y}_{b_j}^N = \tilde{Y}_{a_j}^N\}\right) = \nu_N(\chi_{\{f_D=0\}}), \tag{5.10}$$

where $D = D(a_1, b_1; a_2, b_2; \dots; a_k, b_k)$ is the set $\bigcup_{j=1}^k (\tau(b_j) \setminus \tau(a_j))$. Note that the function $\chi_{\{f_D=0\}}$ has discontinuities at points in X whose corresponding lattice contains a point in the boundary of D . Since the boundary of D is a union of the boundaries of the triangles $\tau(a_j)$ and $\tau(b_j)$, it has measure zero. Thus, we can apply Theorem 1.10 and deduce that

$$\nu_N(\chi_{\{f_D=0\}}) \rightarrow m_X(f_D = 0) \tag{5.11}$$

as $N \rightarrow \infty$. Finally, given

$$m_X(\xi(V) = 0) = m_X\left(\bigcap_{j=1}^k \{f_{\tau(a_j)} = f_{\tau(b_j)}\}\right) = m_X(f_D = 0), \tag{5.12}$$

we get

$$\lim_{N \rightarrow \infty} \mathbb{P}(\xi_N(V) = 0) = m_X(\xi(V) = 0)$$

by combining equations (5.8), (5.9), (5.10), (5.11) and (5.12), completing the proof. □

The proof of Corollary 1.7 relies on the following consequence of the Siegel integral formula.

LEMMA 5.5. [5, (3.7)] *Let $F_1, F_2 \in L^1(\mathbb{R}^2)$. Then*

$$\int_X \sum_{m_1 \neq m_2 \in \mathbb{Z}^2} F_1((m_1M + x))F_2((m_2M + x))dm_X(M, x) = \int_{\mathbb{R}^2} F_1 dx \int_{\mathbb{R}^2} F_2 dx.$$

We will also use non-escape of results proved by El-Baz, Marklof and Vinogradov in [6], which is the content of equation (5.13) below.

Proof of Corollary 1.7. Expanding the formula for $|Y_s(M, x)|^2$, we see that

$$\begin{aligned} \int_X |Y_s(M, x)|^2 dm_X(M, x) &= \int_X \sum_{m_1 \neq m_2 \in \mathbb{Z}^2} \chi_{\tau(s)}(m_1M + x)\chi_{\tau(s)}(m_2M + x)dm_X(M, x) \\ &\quad + \sum_{m \in \mathbb{Z}^2} \chi_{\tau(s)}(mM + x)dm_X(M, x). \end{aligned}$$

This equals $s^2 + s$ by Lemmas 5.5 and 5.1.

As in [6], we define

$$\mathcal{P}_N := \{\sqrt{n} + \mathbb{Z} : 1 \leq n \leq N \text{ and } n \text{ is not a square}\}.$$

Also, for an interval $I \subset \mathbb{R}$, we define the function $Z_N(I, \cdot) : [0, 1) \rightarrow \mathbb{R}$ by setting

$$Z_N(I, \alpha) := |(|\mathcal{P}_N|^{-1}I + \alpha + \mathbb{Z}) \cap \mathcal{P}_N|.$$

This gives the number of points of \mathcal{P}_N in the interval I when normalized and shifted by α . Now, equation (2.5) in [6] tells us

$$\lim_{R \rightarrow \infty} \limsup_{N \rightarrow \infty} \int_{\{Z_N(I, \cdot) > R\}} Z_N(I, \alpha) d\alpha = 0. \tag{5.13}$$

Fix $s > 0$. For any $N \in \mathbb{N}$, one can see from the inequality $\sqrt{t+1} - \sqrt{t} \geq 1/2\sqrt{t+1}$ that all points \mathcal{P}_{sN} lie a distance at least $1/2\sqrt{Ns} + 1$ away from $0 \in \mathbb{T}$. As a consequence, when N is sufficiently large, the only points of the sequence $\{\sqrt{n} + \mathbb{Z} : 1 \leq n \leq sN\}$ which lie in the interval of width $1/N$ centred at 0 are themselves 0 and correspond to squares less than sN . Therefore, when N is sufficiently large, we have $S_N(0, s) = \lfloor \sqrt{sN} \rfloor$ and, if we define $\widehat{E}_{j,N}(s)$ to be the proportion of the intervals $\{[x_0 - 1/2N, x_0 + 1/2N) + \mathbb{Z} : x_0 \in \Omega_N\}$ containing j points of \mathcal{P}_{sN}

$$|E_{j,N}(s) - \widehat{E}_{j,N}(s)| \leq \frac{1}{N}. \tag{5.14}$$

Now, for a large natural number R , we have that

$$\begin{aligned} & \left| \mathbb{E}[(Y_s^N)^2] - \int Y_s^2 dm_X - s \right| \\ & \leq \left| \frac{(S_N(0, s))^2}{N} - s \right| + \left| \frac{1}{N} \sum_{x_0 \in \Omega_N \setminus \{0\}} S_N(x_0, s)^2 - \int Y_s^2 dm_X \right| \\ & \leq \left| \frac{(S_N(0, s))^2}{N} - s \right| + \left| \sum_{j=0}^R j^2 \widehat{E}_{j,N}(s) - \sum_{j=0}^{\infty} j^2 E_j(s) \right| + \sum_{j=R+1}^{\infty} j^2 \widehat{E}_{j,N}(s). \end{aligned}$$

The first term above here clearly tends to 0 as $N \rightarrow \infty$, whilst the second tends to $\sum_{j=R+1}^{\infty} j^2 E_j(s)$ as a consequence of Theorem 1.1 and equation (5.14). So, to complete the proof, we need to show

$$\lim_{R \rightarrow \infty} \limsup_{N \rightarrow \infty} \sum_{j=R+1}^{\infty} j^2 \widehat{E}_{j,N}(s) = 0. \tag{5.15}$$

First, note

$$\begin{aligned} \sum_{j=R+1}^{\infty} j^2 \widehat{E}_{j,N}(s) &= \frac{1}{N} \sum_{x_0 \in \Omega_N \setminus \{0\}} S_N(x_0, s)^2 \chi_{\{S_N(\cdot, s) \geq R+1\}}(x_0) \\ &\leq \frac{1}{N} \sum_{x_0 \in \Omega_N} \left| Z_{sN} \left(\left[-\frac{s}{2}, \frac{s}{2} \right], x_0 \right) \right|^2 \chi_{\{Z_{sN}([-s/2, s/2], \cdot) \geq R+1\}} \left(\frac{k}{N} \right), \end{aligned}$$

since the shifted intervals $|\mathcal{P}_{sN}|^{-1}[-s/2, s/2) + x_0 + \mathbb{Z}$ contain $[x_0 - 1/2N, x_0 + 1/2N) + \mathbb{Z}$.

Second, for any $\alpha \in [x_0 - 1/2N, x_0 + 1/2N)$, as the interval $|\mathcal{P}_{sN}|^{-1}[-s, s) + \alpha + \mathbb{Z}$ contains $|\mathcal{P}_{sN}|^{-1}[-s/2, s/2) + x_0 + \mathbb{Z}$, we have $Z_{sN}([-s/2, s/2), x_0) \leq Z_{sN}([-s, s), \alpha)$ for any such α . Thus, we get

$$\begin{aligned} & \frac{1}{N} \sum_{x_0 \in \Omega_N} \left| Z_{sN} \left(\left[-\frac{s}{2}, \frac{s}{2} \right], x_0 \right) \right|^2 \chi_{\{Z_{sN}([-s/2, s/2], \cdot) \geq R+1\}}(x_0) \\ & \leq \frac{1}{N} \sum_{x_0 \in \Omega_N} N \int_{x_0 - 1/2N}^{x_0 + 1/2N} |Z_{sN}([-s, s), \alpha)|^2 \chi_{\{Z_{sN}([-s, s], \cdot) \geq R+1\}}(\alpha) d\alpha \\ & = \int_{\{Z_{sN}([-s, s], \cdot) > R\}} |Z_{sN}([-s, s), \alpha)|^2 d\alpha. \end{aligned}$$

Taking $N \rightarrow \infty$ and applying equation (5.13) gives us equation (5.15) and hence the result. □

6. Properties of the limiting process

As we noted in §1, ξ is a simple intensity-1 process which does not have independent increments. The simplicity of this process was shown in the proof of Theorem 1.5. The fact it has intensity 1 follows from Lemma 5.1 since for any interval $(a, b) \subset [0, \infty)$, we

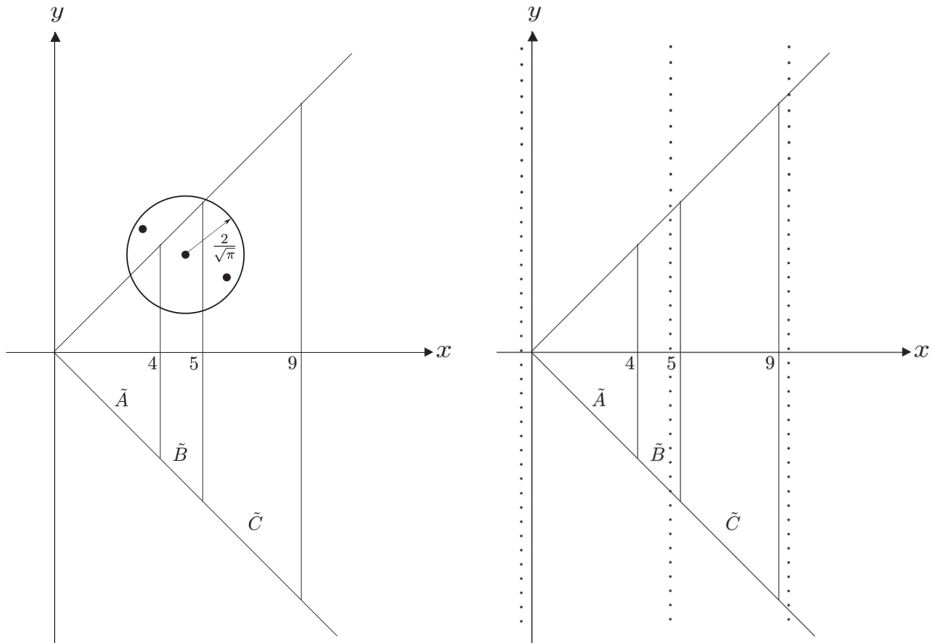


FIGURE 3. Any lattice containing a single point in \tilde{B} will contain one in either \tilde{A} or \tilde{C} (left). An example of a lattice with no points in $\tilde{A} \cup \tilde{C}$ (right).

have that

$$m_X[\xi((a, b))] = m_X(Y_b - Y_a) = b - a.$$

To see that ξ does not have independent increments, consider the intervals $A := [0, 2)$, $B := [2, \sqrt{5})$ and $C := [\sqrt{5}, 3)$. Then,

$$m_X(\xi(A \cup C) \geq 1 \mid \xi(B) = 1) = 1,$$

but

$$m_X(\xi(A \cup C)) < 1. \tag{6.1}$$

This follows from the fact that if there is exactly one point of the lattice given by $x \in X$ in the set $\tilde{B} = \{(u, v) \in \tau(\infty) \mid 4 \leq u < 5\}$ then, by Minkowski’s theorem, there is another lattice point in $\tau(\infty)$ which lies a distance at most $2/\sqrt{\pi}$ away. Given $\xi(x)(B) = 1$, this point must lie in either the set $\tilde{A} = \{(u, v) \in \tau(\infty) \mid 0 \leq u < 4\}$ or $\tilde{C} = \{(u, v) \in \tau(\infty) \mid 5 \leq u < 9\}$ giving us $\xi(x)(A \cup B) \geq 1$. Conversely, it can easily be seen via analysing the form of the Haar measure on X that a positive proportion of our affine unimodular lattices contain no points in $\tilde{A} \cup \tilde{B}$. An example of such a lattice is shown in Figure 3.

In terms of understanding the distribution of the points $\{\sqrt{n} + \mathbb{Z} : 1 \leq n \leq sN\}$ among our partition intervals, the lack of independent increments in the limiting point process tells us when $(a, b) \cap (c, d) = \emptyset$, the points $\{\sqrt{n} + \mathbb{Z} : aN < n \leq bN\}$ and $\{\sqrt{n} + \mathbb{Z} : cN < n \leq dN\}$ do not distribute among the partition intervals independently in the limit as

$N \rightarrow \infty$. For example, when $S_N(x_0, 2)$ is large, then, on average, $S_N(x_0, \sqrt{5}) - S_N(x_0, 2)$ will be also. This is made intuitively clear by the fact that if \tilde{A} contains many lattice points, then we would also expect \tilde{B} to do so also.

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