Oscillation theorems for semilinear hyperbolic and ultrahyperbolic operators

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The oscillation property of the semilinear hyperbolic or ultrahyperbolic operator $L$ defined by

$$L[u] \equiv \Delta u - \sum_{i,j=1}^{m} \frac{\partial}{\partial y_i} \left( a_{ij}(x, y) \frac{\partial u}{\partial y_j} \right) + f(x, y, u)$$

is studied. Sufficient conditions are provided for all solutions of $\omega L[u] \leq 0$ satisfying certain boundary conditions to be oscillatory. The basis of our results is the non-existence of positive solutions of the associated differential inequalities.

Oscillation criteria for linear hyperbolic differential equations have been obtained by Kahane [7], Kreith [2, 3], Pagan [7], Travis [8], and Young [9]. More recently, the author and Yoshida [5] established oscillation theorems for linear ultrahyperbolic operators. The purpose of this paper is to study the oscillation property of a class of nonlinear hyperbolic or ultrahyperbolic equations and inequalities. Use is made of some of the techniques and results developed by Naito and Yoshida [4] and Noussair and Swanson [6].

Let $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_m)$ denote points in $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively. Let $H$ be an unbounded domain in $\mathbb{R}^n$ defined by

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$H = \{ x = (x_1, \ldots, x_n) : 0 < x_i < \infty, i = 1, \ldots, n \}$,

and let $G$ be a bounded domain in $\mathbb{R}^m$ with piecewise smooth boundary.

The partial differential operator to be considered in this paper is

$$L[u] = \Delta_x u - \sum_{i,j=1}^{m} \frac{\partial}{\partial y_i} \left( a_{ij}(x, y) \frac{\partial u}{\partial y_j} \right) + f(x, y, u),$$

where $\Delta_x$ denotes the laplacian in $\mathbb{R}^n$; that is, $\Delta_x = \frac{\partial^2}{\partial x_1^2} + \ldots + \frac{\partial^2}{\partial x_n^2}$.

The coefficients $a_{ij}(x, y)$ are real-valued functions of class $C^1(\overline{H \times G})$, $(i, j = 1, \ldots, m)$, and $f(x, y, \xi)$ is a real-valued function of class $C^0(\overline{H \times G} \times \mathbb{R}^1)$. The matrix $(a_{ij})$ is assumed to be symmetric and positive definite in $H \times G$. The domain $D_L$ of $L$ is the set of all real-valued functions of class $C^2(\overline{H \times G}) \cap C^1(\overline{H \times G})$.

For each $u \in D_L$ we define the function $g(x)$ by

$$(1) \quad g(x) = \frac{1}{\kappa} \int_G u(x, y)dy, \quad \kappa = \int_G dy.$$

**Lemma 1.** Assume that:

(i) $f(x, y, \xi) \geq p(x)\phi(\xi)$ for all $(x, y) \in H \times G$ and for all $\xi > 0$, where $p$ is continuous and non-negative in $H$ and $\phi$ is continuous, non-negative, and convex in $(0, \infty)$;

(ii) $u(x, y) \in D_L$ is a positive solution of the inequality

$L[u] \leq 0$ in $H \times G$ and satisfies the boundary condition

$u = 0$ on $H \times \partial G$.

Then the function $g(x)$ given by (1) satisfies the differential inequality

$$(2) \quad \Delta_x g + p(x)\phi(g) \leq 0, \quad x \in H.$$

**Proof.** Since $\Delta_x g(x) = \frac{1}{\kappa} \int_G \Delta_x udy$, it follows from Green's formula that
\[ \Delta_x g(x) \leq \frac{1}{\kappa} \int_G \sum_{i,j=1}^m \frac{\partial}{\partial y_i} \left[ a_{ij}(x, y) \frac{\partial u}{\partial y_j} \right] dy - \frac{1}{\kappa} \int_G f(x, y, u) dy \]

\[ = \frac{1}{\kappa} \int_{\partial G} \frac{\partial u}{\partial \nu} d\tau - \frac{1}{\kappa} \int_G f(x, y, u) dy , \]

where \( \frac{\partial}{\partial \nu} = \sum_{i,j=1}^m a_{ij}(x, y) \nu_i \frac{\partial}{\partial y_j} \), \( \nu = (\nu_1, \ldots, \nu_m) \) being the unit exterior normal vector to \( \partial G \), and \( \tau \) denotes the measure on \( \partial G \). In view of the fact that \( u > 0 \) in \( H \times G \) and \( u = 0 \) on \( H \times \partial G \), \( \frac{\partial u}{\partial \nu} \) must be non-positive. Therefore, using hypothesis (i) and Jensen's inequality applied to \( \phi(u) \) over \( G \), we get

\[ \Delta_x g(x) \leq -\frac{p(x)}{\kappa} \int_G \phi(u) dy \]

\[ \leq -p(x) \phi \left[ \frac{1}{\kappa} \int_G u(x, y) dy \right] , \]

which is the desired inequality (2).

We shall use the notation

\[ H_r = H \cap \{ x \in \mathbb{R}^n : |x| > r \} , \ r > 0 . \]

**DEFINITION.** A function \( u(x, y) \in D_L \) which satisfies

(3) \[ wL[u] \leq 0 \text{ in } H \times G \text{ and } u = 0 \text{ on } H \times \partial G \]

is said to be oscillatory in \( H \times G \) if it has a zero in \( H_r \times G \) for every \( r > 0 \).

**PROPOSITION 1.** Every solution of (3) is oscillatory in \( H \times G \) if in addition to hypothesis (i) of Lemma 1 the following conditions are satisfied:

(i) \( f(x, y, -\xi) = -f(x, y, \xi) \) for all \( (x, y) \in H \times G \) and for all \( \xi > 0 \);

(ii) the differential inequality (2) has no solution which is positive in \( H_r \) for any \( r > 0 \).

**Proof.** Suppose to the contrary that there exists a solution \( u(x, y) \)
of (3) which has no zero in \( H^{r}, \times G \) for some \( r' > 0 \). If \( u > 0 \) in \( H^{r}, \times G \), then \( L[u] \leq 0 \) in \( H^{r}, \times G \), and by Lemma 1, the function \( g(x) \) defined by (1) is a positive solution of (2) in \( H^{r}, \times G \), contradicting the hypothesis (ii).

Likewise, \( u \) cannot be negative in \( H^{r}, \times G \), or else \( -u \) would be a positive solution of (3).

In the case when \( n = 1 \), the operator \( L \) reduces to a hyperbolic operator and the inequality (2) becomes the ordinary differential inequality

\[
\frac{d^2 g}{dx^2} + p(x)\phi(g) \leq 0, \ x > 0.
\]

Sufficient conditions for the non-existence of eventually positive solutions of (4) have recently been established by Naito and Yoshida [4] and Noussair and Swanson [6]. Here we present an oscillation criterion for the semilinear hyperbolic operator \( L \ (n = 1) \) which follows from Proposition 1 combined with a result of [4, Theorem 2.1].

**THEOREM 1.** Assume that the following conditions are satisfied:

(I) \( f(x, y, \xi) \geq p(x)\phi(\xi) \) for all \( (x, y) \in (0, \infty) \times G \) and for all \( \xi > 0 \), where \( p \) is continuous and non-negative in \((0, \infty)\) and \( \phi \) is continuous, non-negative, and convex in \((0, \infty)\);

(II) \( f(x, y, -\xi) = -f(x, y, \xi) \) for all \( (x, y) \in (0, \infty) \times G \) and for all \( \xi > 0 \);

(III) there exist positive continuous functions \( \phi_1 \) and \( \phi_2 \) in \((0, \infty)\) such that

(i) \( \phi(\xi) \geq \phi_1(\xi)\phi_2(\xi) \) for all \( \xi > 0 \),

(ii) \( \phi_1 \) is non-increasing and \( \phi_2 \) is non-decreasing for all \( \xi > 0 \),

(iii) \( \int_{\epsilon}^{\infty} \frac{d\xi}{\phi_2(\xi)} < \infty \) for some \( \epsilon > 0 \),

\[\phi \]
(iv) \[ \int_0^\infty \xi p(\xi) \phi_k(k\xi) d\xi = \infty \] for all \( k > 0 \).

Then every solution of (3) \((n = 1)\) is oscillatory in \((0, \infty) \times G\).

**COROLLARY 1.** Consider the semilinear hyperbolic equation

\[ \frac{\partial^2 u}{\partial x^2} - \sum_{i=1}^{m} \frac{\partial^2 u}{\partial y_i^2} + c(x)u^\gamma = 0, \]

where \( c(x) \) is a non-negative continuous function in \((0, \infty)\) and \( \gamma > 1 \) is the quotient of odd integers. Every solution \( u \) of (5) satisfying the boundary condition \( u = 0 \) on \((0, \infty) \times \partial G\) is oscillatory in \((0, \infty) \times G\) if

\[ \int_0^\infty xe(x) dx = \infty. \]

Next we consider the case \( n \geq 2 \). Letting \((r, \theta)\) denote hyperspherical coordinates for \( R^n\), \( H\) can be rewritten as

\[ H = \{(r, \theta) : 0 < r < \infty, \theta \in \Theta\}, \]

where \( \Theta \) is the domain defined by

\[ \Theta = \{\theta = (0, \ldots, 0) : 0 < \theta_i < \pi/2, i = 1, \ldots, n-1\}. \]

The following notation will be used:

\[ S_r = \{x \in R^n : |x| = r\}, \]

\[ H(r) = H \cap S_r, \]

\[ H(s, t) = \{x \in H : s < |x| < t\}. \]

The measure on \( S_r \) and \( S_1 \) will be denoted by \( \sigma \) and \( \omega \), respectively. The unit exterior normal vector to \( \partial H\) will be denoted by \( \eta \).

Associated with every function \( u \in D_L\), we define a function \( h(r) \) in \((0, \infty)\) by the equation

\[ h(r) = \frac{1}{\sigma_r} \int_{H(r)} g(x) d\sigma, \]

where \( g(x) \) is the function given by (1) and \( \sigma_r \) denotes the area of
By employing the technique of Noussair and Swanson [6], we obtain the following principal tool.

**Lemma 2.** Assume that the hypotheses (i) and (ii) of Lemma 1 hold and, moreover, that

(i) \( p(x) \geq q(|x|) \) in \( H_{r_0} \) for some \( r_0 > 0 \), where \( q \) is continuous and non-negative in \([r_0, \infty)\);

(ii) \( \frac{\partial g}{\partial n} \geq 0 \) on \( \partial H_{r_0} \), where \( g \) is given by (1).

Then the function \( h(r) \) defined by (6) satisfies the ordinary differential inequality

\[
\frac{d}{dr} \left( r^{n-1} \frac{dh}{dr} \right) + r^{n-1} q(r) \phi(h) \leq 0, \quad r \geq r_0.
\]

**Proof.** Green's formula yields the integral identity

\[
\int_{H_{r_0}} \Delta_x g dx = \int_{H(r)} \frac{\partial g}{\partial n} d\sigma - r^{n-1} \int_{H(1)} \frac{\partial g}{\partial r} d\omega
\]

for any \( r \geq r_0 \), where \( \omega \) denotes the measure on \( \partial \Omega \). Since the following identities hold,

\[
\frac{d}{dr} \left( \int_{H_{r_0}} \Delta_x g dx \right) = \int_{H(r)} \Delta_x g d\sigma,
\]

\[
\frac{d}{dr} \left( \int_{H} \frac{\partial g}{\partial r} d\sigma \right) = \omega_1 \frac{d}{dr} \left( r^{n-1} \frac{dh}{dr} \right),
\]

where \( \omega_1 \) denotes the area of \( H(1) \), differentiating (8) with respect to \( r \) and using condition (ii), we obtain

\[
\int_{H(r)} \Delta_x g d\sigma \geq \omega_1 \frac{d}{dr} \left( r^{n-1} \frac{dh}{dr} \right).
\]

On the other hand, applying Jensen's inequality to \( \phi(g) \) over \( H(r) \) and
using condition (i), we find

\[ (10) \quad \omega_1 \pi^{n-1} q(r) \phi(h) \leq \int_{H(r)} p\phi(g) dg . \]

The conclusion (7) now follows from Lemma 1, (9), and (10). This completes the proof.

**PROPOSITION 2.** Let the following conditions hold.

(i) \( f(x, y, \xi) \geq q(|x|) \phi(\xi) \) in \( H_{r_0} \times G \times (0, \infty) \) for some \( r_0 > 0 \),

where \( q \) is continuous and non-negative in \([r_0, \infty)\) and \( \phi \) is continuous, non-negative, and convex in \((0, \infty)\).

(ii) \( f(x, y, -\xi) = -f(x, y, \xi) \) for all \((x, y) \in H_{r_0} \times G\) and for all \( \xi > 0 \).

(iii) The ordinary differential inequality (7) has no positive solution in \([r, \infty)\) for any \( r \geq r_0 \).

Then every solution \( u \) of (3) which satisfies

\[ (11) \quad \frac{\partial u}{\partial \eta} - \lambda(x, y) u = 0 \]

on \( \partial H_{r_0} \times G \) is oscillatory in \( H \times G \), where \( \lambda(x, y) \) is a non-negative continuous function on \( \partial H_{r_0} \times G \).

Proof. If \( u \) is a solution of (3) which satisfies (11) and is positive in \( H_{r_1} \times G \) for some \( r_1 \geq r_0 \), then we find from (1) and (11) that

\[
\frac{\partial q}{\partial \eta} = \frac{1}{\kappa} \int_G \frac{\partial u}{\partial \eta} dy \\
= \frac{1}{\kappa} \int_G \lambda(x, y) u dy \geq 0 .
\]

Define the function \( h(r) \) by (6). Then, proceeding as in the proof of Proposition 1 and using Lemma 2, we can show that \( h(r) \) is a positive solution of (7) in \([r_1, \infty)\). But this is a contradiction.
The above proposition together with the results of Naito and Yoshida [4, Theorems 2.1 and 2.4] yields the following oscillation criteria for the semilinear ultrahyperbolic operator \( L \).

**THEOREM 2.** Let \( n = 2 \) and assume that:

(i) the hypotheses (i) and (ii) of Proposition 2 are satisfied;

(ii) there exist positive continuous functions \( \phi_1 \) and \( \phi_2 \) in \((0, \infty)\) such that (i), (ii), and (iii) of Theorem 1 (III) hold, and

\[
\int_0^\infty \xi (\log \xi) q(\xi) \phi_1(k \log \xi) d\xi = \infty \quad \text{for all } k > 0.
\]

Then every solution of (3) satisfying (11) is oscillatory in \( H \times G \).

**THEOREM 3.** Let \( n \geq 3 \) and suppose that:

(i) the hypotheses (i) and (ii) of Proposition 2 are satisfied;

(ii) there exist positive continuous functions \( \phi_1, \phi_2, \phi_3 \), and \( \phi_4 \) in \((0, \infty)\) such that

\[
\phi(\xi) \geq \phi_1(\xi) \phi_2(\xi) \quad \text{for all } \xi > 0,
\]

\( \phi_1 \) is non-increasing and \( \phi_2 \) is non-decreasing for all \( \xi > 0 \),

\[
\phi_2(\xi \xi) \geq \phi_3(\xi) \phi_4(\xi) \quad \text{for all } \xi, \xi \text{ such that } 0 < \xi < 1/\xi,
\]

\[
\int_0^\infty \frac{d\xi}{\phi_3(\xi)} < \infty \quad \text{for some } \varepsilon > 0,
\]

\[
\int_0^\infty \xi^{n-1} q(\xi) \phi_4 \left( \frac{\xi^{n-2}}{n-2} \right) d\xi = \infty.
\]

Then every solution of (3) satisfying (11) is oscillatory in \( H \times G \).

**COROLLARY 2.** Consider the semilinear ultrahyperbolic equation

\[
\sum_{i=1}^n \frac{\partial^2 u}{\partial x_i^2} - \sum_{j=1}^m \frac{\partial^2 u}{\partial y_j^2} + c(|x|) u^\gamma = 0,
\]

where \( c \) is a non-negative continuous function in \((0, \infty)\) and \( \gamma > 1 \) is the quotient of odd integers. Every solution \( u \) of (12) satisfying (11)
and \( u = 0 \) on \( H \times \partial G \) is oscillatory in \( H \times G \) if

\[
\int_0^\infty \psi_n(r)c(r)dr = \infty,
\]

where

\[
\psi_n(r) = \begin{cases} 
  r \log r & \text{if } n = 2, \\
  r^{n-1+\gamma(2-n)} & \text{if } n \geq 3.
\end{cases}
\]

References


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