ON STRONG RIESZIAN SUMMABILITY

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1. Introduction. Recently H.-E. Richert [**10**] introduced a new method of summability, for which he completely solved the " summability problem " for Dirichlet series, and which led also to an extension of our knowledge of the relations between the abscissae of ordinary and absolute Rieszian summability. This non-linear method, which may best be characterized by the notion " strong Rieszian summability " †, depends on three parameters, on the order κ, the type λ, and the index p. While Richert's paper deals almost exclusively with the application of that method of summability in a specialized form (namely the case $p=2$, $\lambda_n=\log n$) to Dirichlet series, it is the object of the present paper, to consider the general theory of strong Rieszian summability.

I wish to thank Dr Richert for suggesting this problem to me and for his valuable comments. I am also indebted to the referee for valuable suggestions, according to which I have modified my paper.

Strong Rieszian summability, i.e. the $|R, \lambda, \kappa|^p$-method, is defined as follows :

Definition 1 [**10**, pp. 96, 98 and 109]. *A series* $\sum_{n=1}^{\infty} c_n$ *is summable* $|R, \lambda, \kappa|^p$ *of order* κ, *where* $\kappa > -\dfrac{1}{p}$ *and is real, of index* p, *where* $0<p<\infty$, *and of type* λ, *where* $0 \leqslant \lambda_1 < \lambda_2 < \ldots < \lambda_n \to \infty$, *to the sum* c, *if there exists a number* c *such that, as* ω *tends to infinity continuously, the relation*

$$\int_0^{\omega} | C_{\lambda}^{\kappa}(\tau) - c\tau^{\kappa} |^p \, d\tau = o(\omega^{\kappa p+1}) \quad \ldots\ldots (1)$$

holds, where

$$C_{\lambda}^{\kappa}(\tau) = \sum_{\lambda_n < \tau} c_n (\tau - \lambda_n)^{\kappa}. \quad \ldots\ldots (2)$$

We denote this by

$$\Sigma c_n = c \quad |R, \lambda, \kappa|^p.$$

The assumption that $\kappa p > -1$ is essential.‡ Otherwise, on account of

$$C_{\lambda}^{\kappa}(\tau) \sim c_n(\tau-\lambda_n)^{\kappa} \quad \text{as} \quad \tau \to \lambda_n + 0 \qquad (c_n \neq 0),$$

the integral

$$\int | C_{\lambda}^{\kappa}(\tau) - c\tau^{\kappa} |^p \, d\tau,$$

taken over any interval including λ_n, does not exist.

2. Notation and lemmas. In Hölder's inequality q denotes a number conjugate to $p>1$; i.e.

$$\frac{1}{p}+\frac{1}{q}=1 \quad \text{or} \quad q=\frac{p}{p-1}.$$

Moreover we define q to be ∞ if $p=1$.

† A. V. Boyd and J. M. Hyslop [**1**] were the first to study strong Rieszian summability. Their definition, however, is not equivalent to Definition 1. Cf. § 5.

‡ This restriction is identical with that imposed on κ in [**1**], if the results given there are translated into our notation.

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O- and o-symbols will be understood to relate to a variable tending to $+\infty$ continuously (discretely in the proof of Theorem 9 only).

Empty sums, i.e. $\sum_{a \leqslant n < b}$ where $a \geqslant b$, are to be interpreted as zero. The terms of the series are indexed from 1 onwards, and Σ written without limits means $\sum_{n=1}^{\infty}$.

Capital letters, denoting the Rieszian sums (2), correspond to the small letters used for the terms of the series.

The following lemmas will be required.

Lemma 1 [**11**, p. 153]. *If $f(\tau)$ is positive and continuous for $0 \leqslant \tau \leqslant \omega$, then*

$$\lim_{y \to \infty} \left\{ \int_0^\omega (f(\tau))^y \, d\tau \right\}^{1/y} = \max_{0 \leqslant \tau \leqslant \omega} f(\tau).$$

Lemma 2. *If $\chi(\tau) \geqslant 0$ and $\alpha > -1$, then the two assertions*

$$\int_0^\omega \chi(\tau) \, d\tau = o(\omega) \quad \text{......(3)}$$

and

$$\int_0^\omega \tau^\alpha \chi(\tau) \, d\tau = o(\omega^{\alpha+1}) \quad \text{......(4)}$$

are equivalent, it being assumed that both integrals converge at the origin.

Proof. That (3) implies (4) follows on integrating (4) by parts. The converse follows in the same way, if the integrand in (4) is multiplied by $\tau^{-\alpha}$.

Lemma 3 [**4**, Theorem 4]. *Let $p > 1$, $0 < \mu < 1/p$, $p' = p/(1 - \mu p)$, $f(t) \geqslant 0$ and*

$$f_\mu(\tau) = \frac{1}{\Gamma(\mu)} \int_0^\tau f(t)(\tau - t)^{\mu - 1} \, dt.$$

If $f(\tau) \in L^p(0, \omega)$, then $f_\mu(\tau) \in L^{p'}(0, \omega)$ and

$$\int_0^\omega (f_\mu(\tau))^{p'} \, d\tau \leqslant K \left\{ \int_0^\omega (f(\tau))^p \, d\tau \right\}^{p'/p},$$

where K depends on μ and p only.

3. Consistency in p and κ. Multiplication theorem. We now consider some properties of strong Rieszian summability, which concern, among other things, relations between different values of p (or κ), where λ and κ (or p) are fixed.

First of all we note† that the $| R, \lambda, \kappa |^p$-sum of a series is unique.‡ For it follows from the elementary inequality $(a+b)^p \leqslant 2^p(a^p + b^p)$, where $a \geqslant 0$, $b \geqslant 0$, $p > 0$ that, if (1) holds for c and c', we have the relation

$$| c - c' |^p \omega^{\kappa p + 1} = (\kappa p + 1) \int_0^\omega | (C_\lambda^\kappa(\tau) - c\tau^\kappa) - (C_\lambda^\kappa(\tau) - c'\tau^\kappa) |^p \, d\tau$$

$$= O\left(\int_0^\omega | C_\lambda^\kappa(\tau) - c\tau^\kappa |^p \, d\tau + \int_0^\omega | C_\lambda^\kappa(\tau) - c'\tau^\kappa |^p \, d\tau \right) = o(\omega^{\kappa p + 1}).$$

Hence $c = c'$.

In the following proofs we may suppose the sum of the given series to be zero by transforming the terms c_n into

† This result and Theorems 1 and 2 are already known for the $| R, \log n, \kappa |^2$-method ($\kappa \geqslant 0$). Cf. [**10**, § 9].

‡ If nothing else is said about κ and p, the restrictions imposed in Definition 1 apply.

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$$c_n' = \begin{cases} c_1 - c & \text{when } n=1, \\ c_n & \text{when } n \geqslant 2. \end{cases} \qquad \text{(5)}$$

This is a consequence of the simple result that

$$\Sigma a_n = a \quad | R, \lambda, \kappa |^p \qquad \text{and} \qquad \Sigma b_n = b \quad | R, \lambda, \kappa |^p$$

imply that

$$\Sigma c_n = \Sigma(\alpha a_n + \beta b_n) = \alpha a + \beta b \quad | R, \lambda, \kappa |^p.$$

This follows from the relation

$$C_\lambda^\kappa(\tau) = \alpha A_\lambda^\kappa(\tau) + \beta B_\lambda^\kappa(\tau).$$

THEOREM 1. *For* $0 < p' < p$,

$$\Sigma c_n = c \quad | R, \lambda, \kappa |^p$$

implies that

$$\Sigma c_n = c \quad | R, \lambda, \kappa |^{p'}.$$

Proof. Let $p = p' + r$ and $r > 0$; then

$$\frac{p'+r}{p} = \frac{p'}{p} + \frac{r}{p} = 1 \quad \text{and} \quad \frac{p}{p'} > 1.$$

Therefore Hölder's inequality gives

$$\int_0^\omega | C_\lambda^\kappa(\tau) |^{p'} d\tau \leqslant \left\{ \int_0^\omega | C_\lambda^\kappa(\tau) |^{p'(p/p')} d\tau \right\}^{p'/p} \left\{ \int_0^\omega d\tau \right\}^{r/p} = o(\omega^{\kappa p'+1}).$$

THEOREM 2. *For* $p \geqslant 1$ *and* $\kappa' > \kappa$,

$$\Sigma c_n = c \quad | R, \lambda, \kappa |^p$$

implies that

$$\Sigma c_n = c \quad | R, \lambda, \kappa' |^p.$$

Proof. As is well known [**5**, Lemma 6], we have, for $\kappa > -1$ and $\mu > 0$,

$$C_\lambda^{\kappa+\mu}(\tau) = \gamma \int_0^\tau C_\lambda^\kappa(t)(\tau - t)^{\mu-1} dt, \quad \gamma = \frac{\Gamma(\kappa+\mu+1)}{\Gamma(\kappa+1)\Gamma(\mu)}. \qquad \text{(6)}$$

On applying Hölder's inequality, it follows, if $p > 1$ and $\kappa' = \kappa + \mu$, $\mu > 0$, that

$$| C_\lambda^{\kappa'}(\tau) |^p = O\left\{ \int_0^\tau | C_\lambda^\kappa(t) |^p (\tau - t)^{\mu-1} dt \left(\int_0^\tau (\tau - t)^{\mu-1} dt \right)^{p-1} \right\},$$

and, furthermore,

$$\int_0^\omega | C_\lambda^{\kappa'}(\tau) |^p d\tau = O\left\{ \omega^{\mu(p-1)} \int_0^\omega | C_\lambda^\kappa(t) |^p dt \int_t^\omega (\tau - t)^{\mu-1} d\tau \right\} = o(\omega^{\kappa' p+1}).$$

For $p = 1$ we obtain, from (6),

$$\int_0^\omega | C_\lambda^{\kappa'}(t) | d\tau = O\left\{ \int_0^\omega | C_\lambda^\kappa(t) | dt \int_t^\omega (\tau - t)^{\mu-1} d\tau \right\} = o(\omega^{\kappa'+1}).$$

THEOREM 3. *Let p and p' be positive numbers such that* $(1/p) + (1/p') \leqslant 1$. *If*

$$\Sigma a_n = a \quad | R, \lambda, \kappa |^p \quad \textit{and} \quad \Sigma b_n = b \quad | R, \rho, \mu |^{p'}, \qquad \text{(7)}$$

then

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$$\Sigma c_n = ab \quad (R, \nu, \kappa+\mu+1),\dagger$$

where $c_n = \sum_{\lambda_m+\rho_s=\nu_n} a_m b_s$.

Proof. For $\kappa > -1$, $\mu > -1$ the relation [**5**, p. 64]

$$C_\nu^{\kappa+\mu+1}(\omega) = \frac{\Gamma(\kappa+\mu+2)}{\Gamma(\kappa+1)\Gamma(\mu+1)} \int_0^\omega A_\lambda^\kappa(\tau) B_\rho^\mu(\omega-\tau)\, d\tau \quad \text{......................(8)}$$

is well known.

We suppose first that $p' = q$, and consider the series (7) transformed according to (5). Thus, by Hölder's inequality, we obtain from (8)

$$C_\nu'^{\kappa+\mu+1}(\omega) = O\left\{\left(\int_0^\omega |A'^\kappa(\tau)|^p\, d\tau\right)^{1/p}\left(\int_0^\omega |B'^\mu(\omega-\tau)|^q\, d\tau\right)^{1/q}\right\} = o(\omega^{\kappa+\mu+1}). \quad \text{.........(9)}$$

It is easy to see that

$$C_\nu^{\kappa+\mu+1}(\omega) = C_\nu'^{\kappa+\mu+1}(\omega) + ab\omega^{\kappa+\mu+1} + o(\omega^{\kappa+\mu+1}),$$

which combined with (9) proves the assertion in the special case.

The weaker condition on the indices follows from Theorem 1.

4. Connexions with ordinary and absolute Rieszian summability. We have the following familiar definitions of the ordinary and absolute Rieszian summability [**5**, pp. 21–22 ; **9**].

A series Σc_n is said to be summable (R, λ, κ) of order $\kappa \geqslant 0$ and type λ to the sum c, namely

$$\Sigma c_n = c \quad (R, \lambda, \kappa),$$

if

$$C_\lambda^\kappa(\omega) = c\omega^\kappa + o(\omega^\kappa).$$

The same series is summable $|R, \lambda, \kappa|$ of order $\kappa \geqslant 0$ and type λ, if there is a positive number A such that

$$\int_A^\infty \left| d\, \frac{C_\lambda^\kappa(\tau)}{\tau^\kappa} \right| < \infty.$$

It is natural to define strong Rieszian summability for $p = \infty$ as follows :

Definition 2. *A series Σc_n is summable $|R, \lambda, \kappa|^\infty$ of order $\kappa > 0$ and type λ to the sum c, if there exists a number c such that*

$$\lim_{p\to\infty} \left\{\int_0^\omega |C_\lambda^\kappa(\tau) - c\tau^\kappa|^p\, d\tau\right\}^{1/p} = o(\omega^\kappa).$$

The series is summable $|R, \lambda, 0|^\infty$ to the sum c, if it converges to c.

We now prove the following equivalence.

Theorem 4. *The assertions*

$$\Sigma c_n = c \quad |R, \lambda, \kappa|^\infty \quad \text{..(10)}$$

and

$$\Sigma c_n = c \quad (R, \lambda, \kappa) \quad \text{...(11)}$$

are equivalent.

Proof. Since the Rieszian method of order zero means convergence, we need only consider the case $\kappa > 0$. On account of the continuity of $C_\lambda^\kappa(\tau)$ for $\kappa > 0$, Lemma 1 gives, if we again take $c = 0$,

† Compare this notation with the definitions in the following section.

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$$\lim_{p\to\infty}\left\{\int_0^\omega |C_\lambda^\kappa(\tau)|^p\,d\tau\right\}^{1/p}=\max_{0\leqslant\tau\leqslant\omega}|C_\lambda^\kappa(\tau)|. \qquad (12)$$

Hence we see that (11) follows from (10). Conversely, if (11) is assumed, i.e. if, to a given $\varepsilon>0$,

$$|C_\lambda^\kappa(\tau)|<\varepsilon\tau^\kappa \quad \text{for } \tau>\tau_0,$$

then there exists a constant M such that

$$|C_\lambda^\kappa(\tau)|<M+\varepsilon\tau^\kappa \quad \text{for all } \tau\geqslant 0.$$

This inequality together with (12) implies (10).

The results proved above (in § 3) also remain valid for $p=\infty$. Theorem 1 states that summability $|R,\lambda,\kappa|^\infty$ of a series implies its summability $|R,\lambda,\kappa|^p$ for any $p>0$. For the proof the relation $C_\lambda^\kappa(\tau)=c\tau^\kappa+o(\tau^\kappa)$ may be substituted in (1). As Richert has pointed out by an example [**10**, p. 98], the converse of this theorem is not true. Finally we note that the regularity of summability $|R,\lambda,\kappa|^p$ for $\kappa\geqslant 0$ is a consequence of Theorems 1 and 2.

Since in the theory of Dirichlet series the abscissa of summability $|R,\lambda,\kappa|^1$ coincides with that of absolute Rieszian summability of order $\kappa+1$ and type λ,† the question of the equivalence of these methods is of particular interest. We prove the following theorem, the converse of which is not true.

THEOREM 5. *Any series which is summable* $|R,\lambda,\kappa+1|$ *is also summable* $|R,\lambda,\kappa|^1$‡ $(\kappa>-1)$.

Proof. Using the abbreviation

$$\chi(\tau)=\frac{d}{d\tau}\frac{C_\lambda^{\kappa+1}(\tau)}{\tau^{\kappa+1}}=(\kappa+1)\tau^{-\kappa-1}\{C_\lambda^\kappa(\tau)-\tau^{-1}C_\lambda^{\kappa+1}(\tau)\},$$

we have

$$C_\lambda^\kappa(\tau)-c\tau^\kappa=\frac{\tau^{\kappa+1}}{\kappa+1}\chi(\tau)+\tau^{-1}(C_\lambda^{\kappa+1}(\tau)-c\tau^{\kappa+1}),$$

and, furthermore,

$$\int_0^\omega |C_\lambda^\kappa(\tau)-c\tau^\kappa|\,d\tau\leqslant\frac{1}{\kappa+1}\int_A^\omega \tau^{\kappa+1}|\chi(\tau)|\,d\tau+\int_A^\omega \tau^{-1}|C_\lambda^{\kappa+1}(\tau)-c\tau^{\kappa+1}|\,d\tau$$
$$+\int_0^A |C_\lambda^\kappa(\tau)-c\tau^\kappa|\,d\tau=I_1+I_2+I_3,$$

say.

Writing the hypothesis in the form

$$\int_A^\omega |\chi(t)|\,dt=C+o(1)$$

and integrating I_1 by parts, we obtain

$$I_1=\frac{1}{\kappa+1}\omega^{\kappa+1}\int_A^\omega |\chi(t)|\,dt-\int_A^\omega \tau^\kappa\,d\tau\int_A^\tau |\chi(t)|\,dt=o(\omega^{\kappa+1}).$$

† Cf. [**10**, p. 109]. The relations given there hold for every type.

‡ The index 1 cannot be replaced by any $p>1$. Cf. [**6**, p. 20].

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Summability $| R, \lambda, \kappa+1 |$ of a series implies its summability $(R, \lambda, \kappa+1)$ [9, p. 376]. Thus, substituting $C_\lambda^{\kappa+1}(\tau) - c\tau^{\kappa+1} = o(\tau^{\kappa+1})$ in I_2, we have $I_2 = o(\omega^{\kappa+1})$; furthermore $I_3 = O(1)$. Hence the theorem follows.

Fekete† has shown by an example that, if we use the terminology of Rieszian summability, a series summable (R, n, κ), need not be summable $| R, n, \kappa+\eta |$ for any $\eta > 0$. Taking into consideration that the (R, n, κ)-method is weaker than the $| R, n, \kappa |^1$-method, we have

THEOREM 6. *For any p, summability $| R, \lambda, \kappa |^p$ of a series does not imply its summability $| R, \lambda, \kappa+\eta |$, where η is an arbitrary positive number.*

5. Connexions with Boyd and Hyslop's definition of strong Rieszian summability. According to A. V. Boyd and J. M. Hyslop [1] a series Σc_n is summable to the sum c in the sense of strong Rieszian summability, namely

$$\Sigma c_n = c \quad [R\,;\, \kappa, p] \quad (p \geqslant 1,\ p(\kappa-1) > -1),$$

if

$$\Sigma c_n = c \quad (R, n, \kappa)$$

and‡

$$\int_0^\omega \left| \tau \frac{d}{d\tau} \frac{C_n^\kappa(\tau+1)}{\tau^\kappa} \right|^p d\tau = o(\omega).$$

THEOREM 7. *Let $p \geqslant 1$, $\kappa p > -1$. Then the methods $| R, n, \kappa |^p$ and $[R\,;\, \kappa+1, p]$ are equivalent.*

Proof. With the help of Minkowski's inequality and the following identity

$$-\tau \frac{d}{dt} \frac{C_n^{\kappa+1}(\tau+1)}{\tau^{\kappa+1}} = (\kappa+1) \left\{ \frac{C_n^{\kappa+1}(\tau+1)}{\tau^{\kappa+1}} - \frac{C_n^\kappa(\tau+1)}{\tau^\kappa} \right\}$$

it can be seen that the relation

$$\int_0^\omega \tau^{-\kappa p} \,|\, C_n^\kappa(\tau+1) - c\tau^\kappa \,|^p\, d\tau = o(\omega) \quad \ldots\ldots(13)$$

is necessary and sufficient for summability $[R\,;\, \kappa+1, p]$. From (13) and Lemma 2 (putting $\alpha = \kappa p$) we immediately obtain the theorem.

6. Cross-relations between κ and p. By Theorems 1 and 2 we are led to the question, whether in the $| R, \lambda, \kappa |^p$-method κ and p may both be changed simultaneously (either increased or decreased) within certain bounds, without disturbing the summability of the series. The following two theorems deal with this.

THEOREM 8. *Let $\kappa' = \kappa + \mu$, $\mu > 0$. Then*

$$\Sigma c_n = c \quad | R, \lambda, \kappa |^p \quad \ldots\ldots(14)$$

implies that

$$\Sigma c_n = c \quad | R, \lambda, \kappa' |^{p'} \quad \ldots\ldots(15)$$

under the following conditions:

$$1 < p < \infty\,;\ \ \mu = \frac{1}{p} - \frac{1}{p'},\ \ p' < \infty \quad \textit{or} \quad \mu > \frac{1}{p} - \frac{1}{p'},\ \ p' \leqslant \infty,$$

$$p = 1\,;\ \ \mu > 1 - \frac{1}{p'},\ \ p' < \infty \quad \textit{or} \quad \mu \geqslant 1,\ \ p' \leqslant \infty.$$

† [2]. Cf. [7, p. 28].

‡ In the Rieszian sums the argument here is $\tau+1$, because the terms in our notation are indexed from 1, in [1], however, from 0 onwards.

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Proof. We first consider the case $p>1$, $\mu=(1/p)+\varepsilon>1/p$. Since $(\mu-1)q+1=\varepsilon q>0$, we obtain from (6) and (14) the relation

$$C_\lambda^{\kappa+\frac{1}{p}+\varepsilon}(\omega)=O\left\{\left(\int_0^\omega |C_\lambda^\kappa(\tau)|^p\,d\tau\right)^{1/p}\left(\int_0^\omega(\omega-\tau)^{(\mu-1)q}\,d\tau\right)^{1/q}\right\}=o(\omega^{\kappa+\frac{1}{p}+\varepsilon}),$$

i.e. summability $(R,\lambda,\kappa+(1/p)+\varepsilon)$ of the given series.

In the case $p>1$, $\mu<1/p$, Lemma 3, together with (6) and (14), gives

$$\int_0^\omega |C_\lambda^{\kappa+\mu}(\tau)|^{p/(1-\mu p)}\,d\tau=\int_0^\omega\left|\gamma\int_0^\tau C_\lambda^\kappa(t)(\tau-t)^{\mu-1}\,dt\right|^{p/(1-\mu p)}d\tau$$
$$=O\left\{\left(\int_0^\omega |C_\lambda^\kappa(\tau)|^p\,d\tau\right)^{1/(1-\mu p)}\right\}=o(\omega^{(\kappa+\mu)p/(1-\mu p)+1}),$$

i.e. (15) holds for $p'\leqslant p/(1-\mu p)$.

Now let $p>1$, $\mu=1/p$. With regard to the preceding case, for $\mu=(1/p)-\varepsilon$ the value $p'=1/\varepsilon$ is admissible in (15) and, by Theorem 2, this index can also be taken for $\mu=1/p$. Since, by a suitable choice of ε, $1/\varepsilon$ represents a number, arbitrarily large, we see that p' may assume every finite value. As can be inferred from a remark of B. Kuttner [**8**], $p'=\infty$ is not admissible here. Because of its interest, this limiting case may be explained by the following example :†

Let $E_{n-\nu}^{(-\frac{1}{2})}$ denote the binomial coefficient $\binom{n-\nu-\frac{1}{2}}{-\frac{1}{2}}$ and take, for instance, $n_i=4^i$, $n_i'=\frac{1}{2}4^i$, $\varepsilon_\nu=(\log\nu)^{-\frac{1}{4}}$. Then the series, the partial sums of which are

$$s_\nu=\begin{cases}E_{n_i-\nu}^{(-\frac{1}{2})}\left(\dfrac{n_i}{\log n_i}\right)^{\frac{1}{2}}\varepsilon_\nu & (n_i'\leqslant\nu\leqslant n_i,\ i\geqslant 1 \text{ integral}),\\ 0 & (n_{i-1}<\nu<n_i')\end{cases}$$

is not summable $(R,n,\frac{1}{2})$, although it is summable $|R,n,0|^2$.

If $p=1$, $\mu\geqslant 1$, it follows immediately from (6) that $p'\leqslant\infty$. It may be noted that for $\mu<1$ summability $(R,\lambda,\kappa+\mu)$ cannot be deduced from summability $|R,\lambda,\kappa|^1$.‡

Finally there remains the case $p=1$, $\mu<1$. As it suffices to take $p'>1$, we have, from (6) and Hölder's inequality,

$$|C_\lambda^{\kappa+\mu}(\tau)|^{p'}=O\left\{\int_0^\tau |C_\lambda^\kappa(t)|(\tau-t)^{(\mu-1)p'}\,dt\left(\int_0^\tau|C_\lambda^\kappa(t)|\,dt\right)^{p'-1}\right\}$$

and hence

$$\int_0^\omega |C_\lambda^{\kappa+\mu}(\tau)|^{p'}\,d\tau=o\left\{\omega^{(\kappa+1)(p'-1)}\int_0^\omega|C_\lambda^\kappa(t)|\,dt\int_t^\omega(\tau-t)^{(\mu-1)p'}\,d\tau\right\}=o(\omega^{(\kappa+\mu)p'+1})$$

on the assumption that $p'<1/(1-\mu)$.

The prooof of the theorem is now completed.

The analogous statement for a decrease in the order does not hold. We prove

THEOREM 9. *There is no number P such that for any $\delta>0$*

$$\Sigma c_n=c\quad |R,\lambda,\kappa+\delta|^p$$

† I am indebted to A. Peyerimhoff for communicating this example to me.

‡ This follows immediately from Theorem 5 and the fact that a series summable $|R,n,\kappa|$ need not be summable $(R,n,\kappa-\eta)$ for any $\eta>0$, which was proved by E. Kogbetliantz. Cf. [7, p. 28].

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implies that

$$\Sigma c_n = c \quad | R, \lambda, \kappa |^{p'},$$

where $p' \leqslant P < p$.

Proof. Let us first consider the (more important) part of the index interval, i.e. the part which is bounded below by 1. For this we show that there does not exist a number P for any $\kappa+\delta$, and hence, *a fortiori*, there is no number P, which is independent of the order.

Since for $p \geqslant 1$ strong Cesàro summability† $[C; \kappa+1, p]$ is equivalent to the $| R, n, \kappa |^p$-method,‡ it is sufficient to look for an example of strong Cesàro summability.

Let $p \geqslant 1$ and $\kappa > -1/p$. The series

$$\Sigma (-1)^{n-1} \binom{n-1+\kappa}{n-1} \qquad (16)$$

is summable $(C, \kappa+\delta)$ for any $\delta > 0$ [3, Theorem 79] (and thus summable $[C; \kappa+\delta+1, p]$). It therefore satisfies the hypothesis of the theorem.

Let $C_n^{(\kappa)}$ define the nth Cesàro mean of order κ of the series (16) :

$$\binom{n+\kappa}{n} C_n^{(\kappa)} = \sum_{\nu=1}^{n+1} \binom{n+1-\nu+\kappa}{n+1-\nu} c_\nu ;$$

then we deduce from the identity

$$\sum_{n=1}^{\infty} \binom{n-1+\kappa}{n-1} C_{n-1}^{(\kappa)} x^{n-1} = (1-x)^{-\kappa-1} \sum_{n=1}^{\infty} c_n x^{n-1} = (1-x^2)^{-\kappa-1}$$

the following relation

$$C_n^{(\kappa)} = \begin{cases} 0 & (n \quad \text{odd}), \\ \binom{\frac{1}{2}n+\kappa}{\frac{1}{2}n} \Big/ \binom{n+\kappa}{n} = c + o(1) & (c \neq 0,\ n \text{ even}). \end{cases}$$

We now see that the series (16) is not summable $[C; \kappa+1, p']$ for any p' (i.e. not summable $| R, n, \kappa |^{p'}$ for any $p' \geqslant 1$), because

$$\sum_{\nu=0}^{n} | C_\nu^{(\kappa)} - s |^{p'} = \tfrac{1}{2}n \, | s |^{p'} + \tfrac{1}{2}n \, | s - c |^{p'} + o(n) \neq o(n),$$

however s may be chosen.

For a special order we have thus also obtained a statement concerning the whole index interval,—namely that in the case $\kappa = 0$ the series (16) is summable (R, n, δ) for every $\delta > 0$, but not summable $| R, n, 0 |^{p'}$ for any p', if we keep in mind that the methods $[C; 1, p']$ and $| R, n, 0 |^{p'}$ are trivially equivalent for all p'.

7. The case $p < 1$. Strong Rieszian summability for $0 < p < 1$ is quite different from what it is in the case $p \leqslant 1$. Theorem 8 is no longer true and no connexions with higher indices can be established.§ There are series, which are summable $| R, n, 0 |^p$ for $0 < p < 1$, but not summable (R, n, κ), however large κ may be. As a concrete example, the series consisting of the terms

† For strong and ordinary Cesàro summability the notations of [6] are used.

‡ Cf. [1] and § 5.

§ This remark is contained in a much more general theorem of B. Kuttner [8].

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$$c_n = \begin{cases} n^j & \text{when } n = g^\nu, \\ -(n-1)^j & \text{when } n = g^\nu + 1, \\ 0 & \text{otherwise,} \end{cases} \quad (g \geqslant 2 \text{ a fixed integer, } \nu \geqslant 1 \text{ integral, } 1 < j < 1/p),$$

may be taken.

For $0 < p < 1$ the $|R, \lambda, \kappa|^p$-method does not seem to be very important for applications.

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