



RESEARCH ARTICLE

5 × 5-graded Lie algebras, cubic norm structures and quadrangular algebras

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Abstract

We study simple Lie algebras generated by extremal elements, over arbitrary fields of arbitrary characteristic. We show the following: (1) If the extremal geometry contains lines, then the Lie algebra admits a 5×5 -grading that can be parametrized by a cubic norm structure; (2) If there exists a field extension of degree at most 2 such that the extremal geometry over that field extension contains lines, and in addition, there exist symplectic pairs of extremal elements, then the Lie algebra admits a 5×5 -grading that can be parametrized by a quadrangular algebra.

One of our key tools is a new definition of exponential maps that makes sense even over fields of characteristic 2 and 3, which ought to be interesting in its own right.

Not only was Jacques Tits a constant source of inspiration through his work, he also had a direct personal influence, notably through his threat to speak evil of our work if it did not include the characteristic 2 case.

—The Book of Involutions [KMRT98, p. xv]

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Introduction

The classification of simple Lie algebras over algebraically closed fields is a classical subject, and the situation is, in many aspects, well understood. That is not the case at all over arbitrary fields. Typically, this situation is studied by first extending to a splitting field (e.g., the algebraic closure) and then performing Galois descent, using tools such as Galois cohomology.

This is not the only possible approach, however. Inspired by the successful applications (e.g., by Efim Zel'manov and Aleksei Kostrikin [ZK90]) of *sandwich elements* (i.e., nonzero elements x in a Lie algebra L such that $\text{ad}_x^2 = 0$), various people have studied simple Lie algebras through the concept of *extremal elements*. A nonzero element $x \in L$ is called extremal if $[x, [x, L]] \leq kx$ (together with an additional axiom that is relevant only in characteristic 2); notice that this implies $\text{ad}_x^3 = 0$. A typical example of an extremal element is a long root element in a classical Lie algebra. In fact, these extremal elements span a one-dimensional *inner ideal* of the Lie algebra, a notion going back to John Faulkner [Fau73] and Georgia Benkart [Ben74]. The first paper explicitly underlying the importance of one-dimensional inner ideals, that we are aware of, is Alexander Premet's paper [Pre86] from 1986; see, in particular, its main result stating that any finite-dimensional Lie algebra over an algebraically closed field of characteristic $p > 5$ contains a one-dimensional inner ideal (Theorem 1). The idea of extremal elements has also famously been used by Vladimir Chernousov in his proof of the Hasse principle for algebraic groups of type E_8 [Che89]. (He called the extremal elements *root elements* because in his situation, they belong to root groups. A detailed exposition can be found in [PR94, Chapter 6, Section 8]; see, in particular, the discussion about root elements starting on p. 387.)

Around the turn of the century, Arjeh Cohen and his collaborators (Anja Steinbach, Rosane Ushirobira, David Wales, Gábor Ivanyos, Dan Roozemon) obtained the clever insight that the extremal elements have a *geometric* interpretation, and the corresponding one-dimensional inner ideals can serve as the point set of a so-called shadow space of a building [CSUW01, CI06, CI07, CIR08]. That idea led them to the notion of a *root filtration space* and makes it possible to visualize the different relations between extremal elements. In particular, distinct extremal points can be in four possible relations: collinear, symplectic, special or hyperbolic. These notions will play an important role in Sections 2 and 3 of our paper. In particular, a crucial ingredient will be the result by Cohen and Ivanyos that each *hyperbolic* pair of extremal points gives rise to a 5-grading of the Lie algebra.

Since then, extremal elements in Lie algebras, and the corresponding extremal geometry, have been studied by various people; see, for instance, [Ditp08, itpPR09, Roo11, Coh12, CRS15, FL16, CF18, CM21, CF23] and [FL19, Chapter 6]. The origin of the terminology 'extremal elements' remains slightly mysterious. We have learned from Arjeh Cohen that he believes that he picked it up from Jean-Pierre Serre and consequently used it in their paper [CSUW01], but neither Cohen nor we have been able to find a written source confirming this origin, and in fact, Serre informed us that he does not think that he is responsible for this terminology, but 'that it is possible that [he] used it once in a discussion'.

In our paper, we will adopt a different point of view on extremal elements and relate the Lie algebras to two types of exceptional algebraic structures: *cubic norm structures* and *quadrangular algebras*.

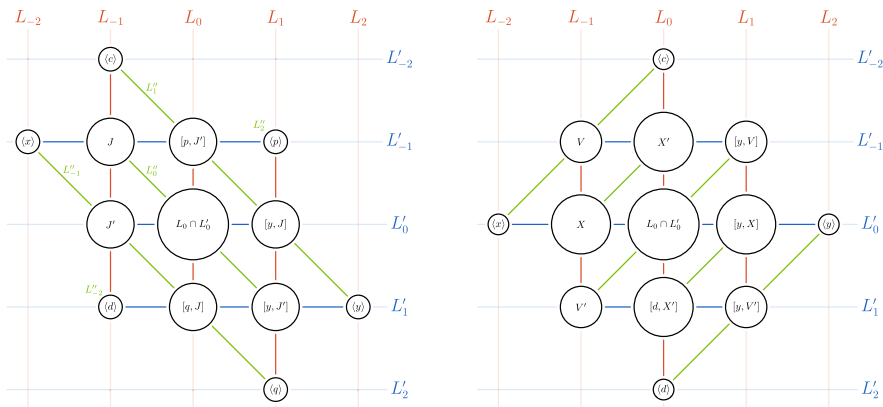
Cubic norm structures are well known: *The Book of Involutions* [KMRT98] spends a whole Chapter IX on them. In particular, §38 and the Notes at the end of Chapter IX contain a lot of additional information and (historical) background. The key players in this theory are Hans Freudenthal [Fre54], Tonny Springer [Spr62], Kevin McCrimmon [McC69], Holger Petersson and Michel Racine [PR84, PR86a, PR86b]. We will also encounter 'twin cubic norm structures' (or 'cubic norm pairs'), a notion that seems to have appeared only once before in the literature in a paper by John Faulkner [Fau01].

Quadrangular algebras are certainly not so well known. They first appeared implicitly in the work of Jacques Tits and Richard Weiss on the classification of Moufang polygons, which are geometric structures (spherical buildings of rank 2) associated to simple linear algebraic groups of relative rank 2 [TW02]. The first explicit definition of these algebras, in the anisotropic case, appeared in Weiss' monograph [Wei06]. The general definition of arbitrary quadrangular algebras is even more recent and is due to Bernhard Mühlherr and Richard Weiss [MW19]. The definition looks rather daunting

(see Definition 1.17 below) mostly because of technicalities in characteristic 2, although the definition itself is characteristic-free.

It is worth pointing out that the anisotropic quadrangular algebras are precisely the parametrizing structures for *Moufang quadrangles* and that the anisotropic cubic norm structures are precisely the parametrizing structures for *Moufang hexagons*. More generally, arbitrary quadrangular algebras and arbitrary cubic norm structures (which might or might not be anisotropic) have been used in the theory of *Tits quadrangles* and *Tits hexagons*, respectively [MW22]. See, in particular, the appendix of [MW19] for the connection between quadrangular algebras and Tits quadrangles, and [MW22, Chapter 2] for the connection between cubic norm structures and Tits hexagons. Notice that Moufang quadrangles are, in fact, buildings of type B_2 or BC_2 , and that Moufang hexagons are buildings of type G_2 . (In the language of linear algebraic groups, this is the type of the *relative root system*.)

In our approach, we will start from a simple Lie algebra admitting extremal elements satisfying two possible sets of assumptions (see Theorems B and C below). In each of these two cases, we will use *two different* hyperbolic pairs of extremal points, giving rise to two different 5-gradings on the Lie algebra, so we obtain a 5×5 -graded Lie algebra. Depending on which of the two cases we are considering, this 5×5 -grading takes a different shape. For the reader's convenience, we have reproduced smaller versions of Figures 1 and 2 of these gradings below, that can be found in full size on pages 32 and 42, respectively.



A closer look at these gradings reveals that, in fact, the 5×5 -grading in Figure 1 is a G_2 -grading, and the 5×5 -grading in Figure 2 is a BC_2 -grading. We will show that in the first case, the Lie algebra is parametrized by a cubic norm pair, in the sense that each of the pieces arising in the decomposition as well as the explicit formulas describing their Lie brackets, are determined by the structure of this cubic norm pair (Theorem B). In the second case, it turns out that the Lie algebra is parametrized by a quadrangular algebra (Theorem C).

The study of Lie algebras *graded by root systems* is not new, and we refer, for instance, to the work of Bruce Allison, Georgia Benkart, Yun Gao and Efim Zel'manov [ABG02, BZ96]. However, their goals have been rather different from ours. Whereas earlier research has focused on classifying all Lie algebras graded by certain root systems in terms of central extensions of TKK-constructions, we focus on *simple* Lie algebras, but we have as goal to understand the *internal structure* of the Lie algebra, via the different pieces of the grading, directly in terms of an algebraic structure.¹ In addition, we do not *assume* the existence of a BC_2 - or G_2 -grading, but it is a consequence of some natural conditions on the extremal elements of the Lie algebra. These conditions are satisfied very often, and in fact, there are many Lie algebras for which *both* sets of conditions are satisfied, and hence admit both a BC_2 - and a G_2 -grading, and hence, they can be parametrized both by a cubic norm structure and by a quadrangular

¹Our description of the resulting internal structure of the Lie algebra is perhaps more in line with Seligman's approach from [Sel76]. For the case of a G_2 -grading, Seligman's Theorem III.6 somewhat resembles our description in terms of a cubic norm pair, but his Theorem III.10 for the BC_2 -case clearly illustrates our point that the connection with quadrangular algebras reveals what is really going on internally.

algebra! It is our hope that this observation will lead to a deeper understanding of connections between different forms of exceptional Lie algebras (and exceptional algebraic groups, as well as their geometric counterparts – namely, the corresponding Moufang polygons and Tits polygons mentioned above). We have indicated a first impression in this direction in Remark 1.2 in the appendix.

We formulate our main results. We emphasize that k is an arbitrary field of arbitrary characteristic.

Theorem A. *Let L be a simple Lie algebra. Assume that for some Galois extension k'/k with $k' \neq \mathbb{F}_2$, $L_{k'} := L \otimes_k k'$ is a simple Lie algebra generated by its pure extremal elements such that the extremal geometry contains lines.*

Consider a hyperbolic pair (x, y) of extremal elements and let $L = L_{-2} \oplus L_{-1} \oplus L_0 \oplus L_1 \oplus L_2$ be the associated 5-grading.

Then for each $l \in L_1$, we can define an ‘ l -exponential automorphism’

$$\alpha: L \rightarrow L: m \mapsto m + [l, m] + q_\alpha(m) + n_\alpha(m) + v_\alpha(m),$$

where $q_\alpha, n_\alpha, v_\alpha: L \rightarrow L$ are maps with

$$q_\alpha(L_i) \subseteq L_{i+2}, \quad n_\alpha(L_i) \subseteq L_{i+3}, \quad v_\alpha(L_i) \subseteq L_{i+4}$$

for all $i \in [-2, 2]$.

This is shown in Theorem 3.16 below, with the essential part being obtained by Theorem 3.13, along with uniqueness results and other auxiliary results. (Notice that α is not unique: there is a one-parameter family of such automorphisms.)

In particular, when $\text{char}(k) \neq 2$, we can always choose $q_\alpha(m) = \frac{1}{2}[l, [l, m]]$, and if, in addition, $\text{char}(k) \neq 3$, then it is a consequence that $n_\alpha(m) = \frac{1}{6}[l, [l, [l, m]]]$ and that $v_\alpha(m) = \frac{1}{24}[l, [l, [l, [l, m]]]]$, so our l -exponential automorphism really are a generalization of the usual exponential automorphisms.

It is worth mentioning that the condition ‘being generated by pure extremal elements’ is easily fulfilled. For instance, the main result of [CIR08] shows that as soon as $\text{char}(k) > 5$, each simple Lie algebra admitting a single pure extremal element is, in fact, generated by its extremal elements. (See also [FL19, Chapter 6].) However, there exist interesting simple Lie algebras not admitting any extremal element, and in that case, studying the *minimal inner ideals* instead is an interesting approach. We refer, for instance, to our earlier work [DMM24] for examples of this situation.

In addition, notice that the existence of a Galois extension k'/k such that the extremal geometry of the Lie algebra over k' contains lines, is also easily fulfilled. Indeed, if $\text{char}(k) \neq 2$, then it follows from [CF23, Theorem 1.1], and the paragraph following this result that unless the Lie algebra is a symplectic Lie algebra (i.e., of the form $\mathfrak{fsp}(V, f)$ for some nondegenerate symplectic space (V, f)), there always even exists an extension k'/k of degree at most 2 fulfilling the requirement that the extremal geometry contains lines. (It is a more subtle fact that the *simplicity* of the Lie algebra is preserved after such a base extension, but we deal with this in Lemma 5.2.)

Our Theorems B and C are very similar in structure. Notice the difference between the conditions between the two results. In Theorem B, the main condition is the fact that we require the existence of lines already over the base field k . In Theorem C, the main condition is the existence of a symplectic pair of extremal elements. (Notice that by the preceding discussion, condition (i) in the statement of Theorem C essentially only excludes symplectic Lie algebras.)

Theorem B. *Let L be a simple Lie algebra defined over a field k with $|k| \geq 4$. Assume that L is generated by its pure extremal elements and that the extremal geometry contains lines.*

Then we can find two hyperbolic pairs (x, y) and (p, q) of extremal elements, with corresponding 5-gradings

$$\begin{aligned} L &= L_{-2} \oplus L_{-1} \oplus L_0 \oplus L_1 \oplus L_2, \\ L &= L'_{-2} \oplus L'_{-1} \oplus L'_0 \oplus L'_1 \oplus L'_2, \end{aligned}$$

such that the gradings intersect as in Figure 1. Moreover, the structure of the Lie algebra induces various maps between the different components of this 5×5 -grading, resulting in a ‘twin cubic norm structure’ on the pair

$$J = L_{-1} \cap L'_{-1}, \quad J' = L_{-1} \cap L'_0.$$

If the norm of this structure is not identically zero, then this results in a genuine cubic norm structure.

Theorem C. Let L be a simple Lie algebra defined over a field k with $|k| \geq 3$. Assume that L is generated by its pure extremal elements and that

- (i) there exists a Galois extension k'/k of degree at most 2 such that the extremal geometry of $L \otimes k'$ contains lines;
- (ii) there exist symplectic pairs of extremal elements.

Then we can find two hyperbolic pairs (x, y) and (p, q) of extremal elements, with corresponding 5-gradings

$$\begin{aligned} L &= L_{-2} \oplus L_{-1} \oplus L_0 \oplus L_1 \oplus L_2, \\ L &= L'_{-2} \oplus L'_{-1} \oplus L'_0 \oplus L'_1 \oplus L'_2, \end{aligned}$$

such that the gradings intersect as in Figure 2. Moreover, the structure of the Lie algebra induces various maps between the different components of this 5×5 -grading, resulting in a quadrangular algebra on the pair

$$V = L_{-1} \cap L'_{-1}, \quad X = L_{-1} \cap L'_0.$$

Theorem B is shown in Section 4 culminating in Theorem 4.23, and Theorem C is shown in Section 5 culminating in Theorem 5.56. For both results, we rely in a crucial way on Theorem A. For Theorem B, we use this result to define the norm N and the adjoint $\sharp$ of the cubic norm structure in Definition 4.12, relying on Proposition 4.11. For Theorem C, we use it to define the quadratic form Q on V in Definition 5.10 (relying on Lemma 5.9) and to define the map $\theta: X \times V \rightarrow V$ in Definition 5.29. Not surprisingly, the corresponding Theorem 3.13 is used in many places of our proof.

1. Preliminaries

Throughout, k is an arbitrary commutative field of arbitrary characteristic. All algebras occurring in this paper are assumed to be k -algebras. This preliminary section consists of three parts. Section 1.1 deals with Lie algebras and introduces the basic theory of extremal elements and the extremal geometry. Most of this material is taken from [CI06]. We then introduce cubic norm structures in Section 1.2 and quadrangular algebras in Section 1.3.

1.1. Lie algebras

Definition 1.1. A $\mathbb{Z}$ -grading of a Lie algebra L is a vector space decomposition $L = \bigoplus_{i \in \mathbb{Z}} L_i$ such that $[L_i, L_j] \leq L_{i+j}$ for all $i, j \in \mathbb{Z}$. If n is a natural number such that $L_i = 0$ for all $i \in \mathbb{Z}$ such that $|i| > n$ while $L_{-n} \oplus L_n \neq 0$, then we call this grading a $(2n + 1)$ -grading. We call L_{-n} and L_n the ends of this grading. The i -component of $x \in L$ is the image of the projection of x onto L_i . We also set $L_{\leq i} = \bigoplus_{j \leq i} L_j$ and $L_{\geq i} = \bigoplus_{j \geq i} L_j$.

Definition 1.2 [CI06, Definition 14]. Let L be a Lie algebra over k .

- (i) A nonzero element $x \in L$ is called *extremal* if there is a map $g_x: L \rightarrow k$, called the *extremal form* on x , such that for all $y, z \in L$, we have

$$[x, [x, y]] = 2g_x(y)x, \quad (1)$$

$$[[x, y], [x, z]] = g_x([y, z])x + g_x(z)[x, y] - g_x(y)[x, z], \quad (2)$$

$$[x, [y, [x, z]]] = g_x([y, z])x - g_x(z)[x, y] - g_x(y)[x, z]. \quad (3)$$

The last two identities are called the *Premet identities*. If the characteristic of k is not 2, then the Premet identities (2) and (3) follow from (1); see [CI06, Definition 14]. Moreover, using the Jacobi identity, (2) and (3) are equivalent if (1) holds.

Note that the extremal form g_x might not be unique if $\text{char}(k) = 2$.

- (ii) We call $x \in L$ a *sandwich* or an *absolute zero divisor* if $[x, [x, y]] = 0$ and $[x, [y, [x, z]]] = 0$ for all $y, z \in L$. An extremal element is called *pure* if it is not a sandwich.
- (iii) The Lie algebra L is *nondegenerate* if it has no nontrivial sandwiches.
- (iv) We denote the set of extremal elements of a Lie algebra L by $E(L)$ or, if L is clear from the context, by E . Accordingly, we denote the set $\{kx \mid x \in E(L)\}$ of *extremal points* in the projective space on L by $\mathcal{E}(L)$ or $\mathcal{E}$.

Remark 1.3.

- (i) If $x \in L$ is a sandwich, then the extremal form g_x can be chosen to be identically zero; we adopt the convention from [CI06] that g_x is *identically zero whenever x is a sandwich in L* .
- (ii) By [CI06, Lemma 16], the existence of two distinct functions g_x and g'_x satisfying the identities (1) to (3) implies that $\text{char}(k) = 2$ and that x is a sandwich. Combined with our convention in (i), this means that g_x is always *uniquely determined*.

Definition 1.4. Let L be a Lie algebra. An *inner ideal* of L is a subspace I of L satisfying $[I, [I, L]] \leq I$.

If $\text{char}(k) \neq 2$, then the 1-dimensional inner ideals of L are precisely the subspaces spanned by an extremal element. We will need inner ideals of dimension > 1 only once, namely in the proof of Theorem 3.8.

Recall that if D is a linear map from a Lie algebra L to itself such that $D^n = 0$ for some $n \in \mathbb{N}$ and $(n-1)! \in k^\times$, then $\exp(D) = \sum_{i=0}^{n-1} \frac{1}{i!} D^i$. A crucial aspect of extremal elements is that they allow for exponential maps in any characteristic.

Definition 1.5 [CI06, p. 444]. Let $x \in E$ be an extremal element, with extremal form g_x . Then we define

$$\exp(x): L \rightarrow L: y \mapsto y + [x, y] + g_x(y)x.$$

We also write $\text{Exp}(x) := \{\exp(\lambda x) \mid \lambda \in k\}$.

Lemma 1.6 [CI06, Lemma 15]. For each $x \in E$, we have $\exp(x) \in \text{Aut}(L)$. Moreover, $\exp(\lambda x) \exp(\mu x) = \exp((\lambda + \mu)x)$ for all $\lambda, \mu \in k$, so $\text{Exp}(x)$ is a subgroup of $\text{Aut}(L)$.

Notice that for an extremal element $x \in E$, we always have $\text{ad}_x^3 = 0$. In particular, if $\text{char}(k) \neq 2$, then $g_x(y)x = \frac{1}{2}[x, [x, y]]$, and we recover the usual exponential map $\exp(x) = \exp(\text{ad}_x)$.

In this paper, we will always assume that our Lie algebra L is generated by its extremal elements. This has some powerful consequences.

Proposition 1.7 [CI06, Proposition 20]. Suppose that L is generated by E . Then

- (i) L is linearly spanned by E .
- (ii) There is a unique bilinear form $g: L \times L \rightarrow k$ such that $g_x(y) = g(x, y)$ for all $x \in E$ and all $y \in L$. The form g is symmetric.
- (iii) The form g associates with the Lie bracket (i.e., $g([x, y], z) = g(x, [y, z])$ for all $x, y, z \in L$).

Proposition 1.8 [CI06, Lemmas 21, 24, 25 and 27]. Suppose that L is generated by E . Let $x, y \in E$ be pure extremal elements. Then exactly one of the following holds:

- (a) $kx = ky$;
- (b) $kx \neq ky$, $[x, y] = 0$, and $\lambda x + \mu y \in E \cup \{0\}$ for all $\lambda, \mu \in k$ (we write $x \sim y$);
- (c) $kx \neq ky$, $[x, y] = 0$, and $\lambda x + \mu y \in E \cup \{0\}$ if and only if $\lambda = 0$ or $\mu = 0$;
- (d) $[x, y] \neq 0$ and $g(x, y) = 0$, in which case $[x, y] \in E$ and $x \sim [x, y] \sim y$;
- (e) $[x, y] \neq 0$ and $g(x, y) \neq 0$, in which case x and y generate an $\mathfrak{sl}_2(k)$ -subalgebra.

Moreover, we have $g(x, y) = 0$ in all four cases (a)–(d).

This gives rise to the following definitions.

Definition 1.9.

- (i) Let $\langle x \rangle, \langle y \rangle \in \mathcal{E}(L)$ be two distinct pure extremal points. Then depending on whether the corresponding elements x, y are in case (b), (c), (d) or (e) of Proposition 1.8, we call the pair $\langle x \rangle, \langle y \rangle$ *collinear*, *symplectic*, *special* or *hyperbolic*, respectively. Following the notation from [CI06], we denote the five possible relations between points of $\mathcal{E}$ by $\mathcal{E}_{-2}$ (equal), $\mathcal{E}_{-1}$ (collinear), $\mathcal{E}_0$ (symplectic), $\mathcal{E}_1$ (special), $\mathcal{E}_2$ (hyperbolic), respectively. We also use $E_{-2}, E_{-1}, E_0, E_1, E_2$ for the corresponding relations between elements of E . Moreover, we write $E_i(x)$ for the set of elements of E in relation E_i with x , and similarly for $\mathcal{E}_i(\langle x \rangle)$.

We refer the reader to Proposition 2.5 for the motivation for this notation.

- (ii) The *extremal geometry* associated with L is the point-line geometry with point set $\mathcal{E}(L)$ and line set

$$\mathcal{F}(L) := \{\langle x, y \rangle \mid \langle x \rangle \text{ and } \langle y \rangle \text{ are collinear}\}.$$

It is a *partial linear space* (i.e., any two distinct points lie on at most one line).

A crucial ingredient for us will be the fact that each hyperbolic pair of extremal elements gives rise to a 5-grading of the Lie algebra; see Proposition 2.4 below.

We finish this section with a subtle but important fact about exponential maps arising from a 5-grading.

Definition 1.10. Assume that $\text{char}(k) \neq 2, 3$ and let L be a 5-graded finite-dimensional Lie algebra over k . We say that L is *algebraic* if for any $(x, s) \in L_{\sigma_1} \oplus L_{\sigma_2}$ (with $\sigma \in \{+, -\}$), the linear endomorphism

$$\exp(\text{ad}(x + s)) = \sum_{i=0}^4 \frac{1}{i!} \text{ad}(x + s)^i$$

of L is a Lie algebra automorphism.

It is not difficult to see that when $\text{char}(k) \neq 2, 3, 5$, then any 5-graded Lie algebra is algebraic; see [BDMS19, Lemma 3.1.7]. If $\text{char}(k) = 5$, this is much more delicate; see [BDMS19, §4.2] and [Sta22, Theorem 2.10].

One of our key tools will precisely be an extension of the existence of such automorphisms to the case where $\text{char}(k)$ is arbitrary, including characteristic 2 and 3; see Definitions 3.12 and 3.13 and Theorems 3.15 and 3.16 below.

1.2. Cubic norm structures

Cubic norm structures have been introduced by Kevin McCrimmon in [McC69]. We follow the approach from [TW02, Chapter 15], which is equivalent but is more practical for our purposes.

Definition 1.11. A *cubic norm structure* is a tuple

$$(J, k, N, \sharp, T, \times, 1),$$

where k is a field, J is a vector space over k , $N: J \rightarrow k$ is a map called the *norm*, $\sharp: J \rightarrow J$ is a map called the *adjoint*, $T: J \times J \rightarrow k$ is a symmetric bilinear form called the *trace*, $\times: J \times J \rightarrow J$ is a symmetric bilinear map called the *Freudenthal cross product*, and 1 is a nonzero element of J called the *identity* such that for all $\lambda \in k$, and all $a, b, c \in J$, we have

- (i) $(\lambda a)^\sharp = \lambda^2 a^\sharp$;
- (ii) $N(\lambda a) = \lambda^3 N(a)$;
- (iii) $T(a, b \times c) = T(a \times b, c)$;
- (iv) $(a + b)^\sharp = a^\sharp + a \times b + b^\sharp$;
- (v) $N(a + b) = N(a) + T(a^\sharp, b) + T(a, b^\sharp) + N(b)$;
- (vi) $T(a, a^\sharp) = 3N(a)$;
- (vii) $(a^\sharp)^\sharp = N(a)a$;
- (viii) $a^\sharp \times (a \times b) = N(a)b + T(a^\sharp, b)a$;
- (ix) $a^\sharp \times b^\sharp + (a \times b)^\sharp = T(a^\sharp, b)b + T(a, b^\sharp)a$;
- (x) $1^\sharp = 1$;
- (xi) $a = T(a, 1)1 - 1 \times a$.

We call this cubic norm structure *nondegenerate* if $\{a \in J \mid N(a) = 0 = T(a, J) = T(a^\sharp, J)\} = 0$.

Definition 1.12. We call $a \in J$ *invertible* if $N(a) \neq 0$. We denote the set of all invertible elements of J by $J^\times$. If all nonzero elements of J are invertible, we call J *anisotropic*; otherwise, we call it *isotropic*.

Lemma 1.13 [TW02, (15.18)]. *In Definition 1.11, condition (iii) is a consequence of (iv) and (v). If $|k| > 3$, then conditions (vi), (viii) and (ix) are a consequence of (i), (ii), (iv), (v) and (vii).*

Remark 1.14. Every cubic norm structure can be made into a (quadratic) Jordan algebra, where the U -operator of the Jordan algebra is given by

$$U_x(y) = T(y, x)x - y \times x^\sharp$$

for all $x, y \in J$.

Definition 1.15.

- (i) Let $(J, k, N, \sharp, T, \times, 1)$ and $(J', k, N', \sharp', T', \times', 1')$ be two cubic norm structures. A vector space isomorphism $\varphi: J \rightarrow J'$ is an *isomorphism* from $(J, k, N, \sharp, T, \times, 1)$ to $(J', k, N', \sharp', T', \times', 1')$ if $\varphi \circ \sharp = \sharp' \circ \varphi$ and $\varphi(1) = 1'$.
- (ii) Let $(J, k, N, \sharp, T, \times, 1)$ be a cubic norm structure and let $d \in J^\times$. We define new maps $N_d, \sharp_d, T_d, \times_d$ and 1_d by

$$\begin{aligned} N_d(a) &:= N(d)^{-1}N(a), \\ a^{\sharp_d} &:= N(d)^{-1}U_d(a^\sharp), \\ T_d(a, b) &:= T(U_{N(d)^{-1}d^\sharp}(a), b), \\ a \times_d b &:= N(d)^{-1}U_d(a \times b), \\ 1_d &:= d, \end{aligned}$$

for all $a, b \in J$. Then $(J, k, N_d, \sharp_d, T_d, \times_d, 1_d)$ is again a cubic norm structure; see [TW02, (29.36)].

Two cubic norm structures $(J, k, N, \sharp, T, \times, 1)$ and $(J', k, N', \sharp', T', \times', 1')$ are called *isotopic* if there exists an isomorphism from $(J', k, N', \sharp', T', \times', 1')$ to $(J, k, N_d, \sharp_d, T_d, \times_d, 1_d)$ for a certain $d \in J^\times$.

1.3. Quadrangular algebras

Quadrangular algebras have been introduced by Richard Weiss in [Wei06] in the anisotropic case, and have been generalized to allow isotropic quadrangular algebras in [MW19]. Our notation follows [MW19], except that we use different letters for some of our spaces and maps.

Definition 1.16. Let V be a vector space over the field k . A *quadratic form* Q on V is a map $Q: V \rightarrow k$ such that

- $Q(\lambda v) = \lambda^2 Q(v)$ for all $\lambda \in k$ and $v \in V$;
- the map $T: V \times V \rightarrow k$ defined by

$$T(u, v) = Q(u + v) - Q(u) - Q(v)$$

for all $u, v \in V$ is bilinear.

We call Q

- *anisotropic* if $Q(v) = 0$ implies $v = 0$;
- *nondegenerate* if $V^\perp := \{v \in V \mid T(v, V) = 0\} = 0$;
- *non-singular* if it is either nondegenerate or $\dim(V^\perp) = 1$ and $Q(V^\perp) \neq 0$;
- *regular* if $\{v \in V \mid Q(v) = 0 = T(v, V)\} = 0$.

A *base point* for Q is an element $e \in V$ such that $Q(e) = 1$. Using this base point, we can define an involution $\sigma: V \rightarrow V$ by

$$v^\sigma = T(v, e)e - v, \quad (4)$$

for all $v \in V$.

Definition 1.17 [MW19, Definition 2.1]. A *quadrangular algebra* is a tuple

$$(k, V, Q, T, e, X, \cdot, h, \theta),$$

where

- k is a field;
- V is a vector space over k ;
- $Q: V \rightarrow k$ is a regular quadratic form, with associated bilinear form T ;
- $e \in V$ is a base point for Q ;
- X is a vector space over k ;
- $\cdot: X \times V \rightarrow X: (a, v) \mapsto a \cdot v$ is a bilinear map;
- $h: X \times X \rightarrow V$ is a bilinear map;
- $\theta: X \times V \rightarrow V$ is a map;

such that

- (i) $a \cdot e = a$ for all $a \in X$;
- (ii) $(a \cdot v) \cdot v^\sigma = Q(v)a$ for all $a \in X$ and all $v \in V$;
- (iii) $h(a, b \cdot v) = h(b, a \cdot v) + T(h(a, b), e)v$ for all $a, b \in X$ and all $v \in V$;
- (iv) $T(h(a \cdot v, b), e) = T(h(a, b), v)$ for all $a, b \in X$ and all $v \in V$;
- (v) For each $a \in X$, the map $v \mapsto \theta(a, v)$ is linear;
- (vi) $\theta(\lambda a, v) = \lambda^2 \theta(a, v)$ for all $a \in X$, $v \in V$ and $\lambda \in k$;
- (vii) There exists a function $\gamma: X \times X \rightarrow k$ such that

$$\theta(a + b, v) = \theta(a, v) + \theta(b, v) + h(a, b \cdot v) - \gamma(a, b)v$$

for all $a, b \in X$ and all $v \in V$;

(viii) There exists a function $\phi: X \times V \rightarrow k$ such that

$$\theta(a \cdot v, w) = \theta(a, w^\sigma)^\sigma Q(v) - T(w, v^\sigma) \theta(a, v)^\sigma + T(\theta(a, v), w^\sigma) v^\sigma + \phi(a, v) w$$

for all $a \in X$ and all $v, w \in V$;

(ix) $a \cdot \theta(a, v) = (a \cdot \theta(a, e)) \cdot v$ for all $a \in X$ and all $v \in V$.

Notation 1.18. We set $\pi(a) = \theta(a, e)$ for all $a \in X$.

Definition 1.19 [MW19, Definition 2.3]. We call a quadrangular algebra *anisotropic* if Q is anisotropic and $\pi(a)$ is a multiple of e if and only if $a = 0$.

Remark 1.20. The existence of $e \in V$ implies $V \neq 0$. However, contrary to the common convention, we allow $X = 0$. In other words, we view quadratic spaces as special (degenerate) examples of quadrangular algebras.

2. Gradings from extremal elements

In this section, we describe the 5-grading associated with any pair of hyperbolic extremal elements, and we collect several properties about this grading that we will need later. We emphasize again that we do not make any assumptions on the field k , so in particular, we allow $\text{char}(k) = 2$ or 3 .

Notation 2.1. We will assume from now on that L is a Lie algebra over k generated by its pure extremal elements. As in Definition 1.2, we will write E for the set of extremal elements of L and $\mathcal{E}$ for the set of extremal points of L . We let g be the symmetric bilinear form from Proposition 1.7. If V is any subspace of L , we write

$$V^\perp := \{u \in L \mid g(u, v) = 0 \text{ for all } v \in V\}.$$

Remark 2.2. If, in addition, L is simple, then our assumptions imply that L does not contain sandwich elements. Indeed, the sandwich elements are contained in the radical of the form g , which is an ideal of L . Since L is generated by its pure extremal elements, the form g cannot be identically zero, so the radical of g must be trivial.

In this section, we do not assume that L is simple (except in Lemma 2.11 and Propositions 2.13 and 2.16), but we will make this important assumption in the next sections.

Definition 2.3. We define the *normalizer* of x as

$$N_L(x) := \{l \in L \mid [x, l] \in \langle x \rangle\}.$$

Proposition 2.4 [CI06, Proposition 22 and Corollary 23]. Let $x, y \in E$ such that $g(x, y) = 1$. Then

(i) L has a $\mathbb{Z}$ -grading

$$L = L_{-2} \oplus L_{-1} \oplus L_0 \oplus L_1 \oplus L_2,$$

with $L_{-2} = \langle x \rangle$, $L_{-1} = [x, U]$, $L_0 = N_L(x) \cap N_L(y)$, $L_1 = [y, U]$ and $L_2 = \langle y \rangle$, where

$$U = \langle x, y, [x, y] \rangle^\perp.$$

(ii) Each L_i is contained in the i -eigenspace of $\text{ad}_{[x, y]}$.

- (iii) ad_x defines a linear isomorphism from L_1 to L_{-1} with inverse $-\text{ad}_y$.
 (iv) The filtration $L_{-2} \leq L_{\leq -1} \leq L_{\leq 0} \leq L_{\leq 1} \leq L$ only depends on x and not on y . More precisely, we have

$$\begin{aligned} L_{\leq 1} &= x^\perp, \\ L_{\leq 0} &= N_L(x), \\ L_{\leq -1} &= kx + [x, x^\perp], \\ L_{-2} &= kx. \end{aligned}$$

- (v) In particular, $g_x(L_{\leq 1}) = 0$ and $g_y(L_{\geq -1}) = 0$.

These gradings are closely related to the five different relations E_i that we have introduced in Definition 1.9:

Proposition 2.5. *Let x, y and L_i be as in Proposition 2.4. For each $i \in [-2, 2]$, we have*

$$E_i(x) = (E \cap L_{\leq i}) \setminus L_{\leq i-1}.$$

Proof. This is shown in the proof of [CI06, Theorem 28]. □

Corollary 2.6. *Let x, y and L_i be as in Proposition 2.4. Then*

$$E_{-1}(x) = \{\lambda x + e \mid \lambda \in k, e \in E \cap L_{-1}\}.$$

Proof. By Proposition 2.5, $E_{-1}(x) = E \cap (kx + L_{-1}) \setminus kx$. Now observe that an element $\lambda x + e$ (with $e \in L_{-1}$) is extremal if and only if e is extremal, because if one of them is extremal, it is collinear with x , and hence, all points on the line through x are extremal points. □

Lemma 2.7. *Let x, y and L_i be as in Proposition 2.4. Let $a \in L_1$ and $b \in L_{-1}$. Then $[a, [y, b]] = g(a, b)y$.*

Proof. By Proposition 2.4(iii), we can write $a = [y, c]$ for some $c \in L_{-1}$. By the Premet identity (2) and Proposition 2.4(v), we then get

$$\begin{aligned} [a, [y, b]] &= [[y, c], [y, b]] \\ &= g_y([c, b])y + g_y(b)[y, c] - g_y(c)[y, b] \\ &= g(y, [c, b])y = g([y, c], b)y = g(a, b)y. \end{aligned}$$
□

The following lemma describes an automorphism that reverses the grading.

Lemma 2.8. *Let x, y and L_i be as in Proposition 2.4. Consider the automorphism $\varphi = \exp(y)\exp(x)\exp(y)$. Then*

$$\begin{aligned} \varphi(x) &= y, \\ \varphi(l_{-1}) &= [y, l_{-1}], \\ \varphi(l_0) &= l_0 + [x, [y, l_0]], \\ \varphi(l_1) &= [x, l_1], \\ \varphi(y) &= x, \end{aligned}$$

for all $l_{-1} \in L_{-1}$, $l_0 \in L_0$, $l_1 \in L_1$.

Proof. Since $g_x(y) = g_y(x) = 1$, we get $\exp(x)(y) = y + [x, y] + x$ and $\exp(y)(x) = x - [x, y] + y$. Since $g_x([x, y]) = 0$ and $g_y([x, y]) = 0$, it now follows that $\varphi(x) = y$ and $\varphi(y) = x$.

Next, let $l_1 \in L_1$. Then $\exp(y)(l_1) = l_1$ and $\exp(x)(l_1) = l_1 + [x, l_1]$ because $g_x(L_1) = 0$ (by Proposition 2.4(v)). By the same Proposition 2.4(v), we also have $g_y(L_{-1}) = 0$, and hence,

$$\varphi(l_1) = \exp(y)([x, l_1] + l_1) = [x, l_1] + (l_1 + [y, [x, l_1]]) = [x, l_1],$$

by Proposition 2.4(iii). Similarly, $\varphi(l_{-1}) = [y, l_{-1}]$ for all $l_{-1} \in L_{-1}$.

Finally, let $l_0 \in L_0$. Let $\lambda, \mu \in k$ be such that $[x, l_0] = \lambda x$ and $[y, l_0] = \mu y$. Then by the fact that g associates with the Lie bracket, we have

$$\mu = g_x(\mu y) = g_x([y, l_0]) = -g_y([x, l_0]) = -g_y(\lambda x) = -\lambda.$$

Hence,

$$\begin{aligned} \varphi(l_0) &= \exp(y) \exp(x)(l_0 - \lambda y) \\ &= \exp(y)(l_0 + \lambda x) - \lambda \varphi(y) \\ &= (l_0 - \lambda y) + \lambda(x - [x, y] + y) - \lambda x \\ &= l_0 - \lambda[x, y] = l_0 + [x, [y, l_0]]. \end{aligned}$$

□

Remark 2.9. Because of Lemma 2.8, many results involving the 5-grading can be ‘swapped around’. We will often do this without explicitly mentioning.

The automorphism group $\text{Aut}(L)$ contains a torus preserving the grading. (This easy fact is, of course, true for any $\mathbb{Z}$ -graded algebra.)

Lemma 2.10. *Let x, y and L_i be as in Proposition 2.4. Let $\lambda \in k^\times$ be arbitrary. Consider the map $\varphi_\lambda: L \rightarrow L$ defined by $\varphi_\lambda(l_i) = \lambda^i l_i$ for all $l_i \in L_i$, with $i \in [-2, 2]$. Then $\varphi_\lambda \in \text{Aut}(L)$.*

Proof. Clearly, φ_λ is bijective. Consider $l_i \in L_i$ and $l_j \in L_j$ arbitrary, with $i, j \in [-2, 2]$. Then

$$\varphi_\lambda([l_i, l_j]) = \lambda^{i+j} [l_i, l_j] = [\lambda^i l_i, \lambda^j l_j] = [\varphi_\lambda(l_i), \varphi_\lambda(l_j)].$$

□

The following lemma will ensure the nondegeneracy of certain maps later on.

Lemma 2.11. *Assume that L is simple, and let x, y and L_i be as in Proposition 2.4. Let $Z_i = \{z \in L_i \mid [z, L_i] = 0\}$ for $i = -1$ and $i = 1$. Then $Z_{-1} = Z_1 = 0$.*

Proof. Let $z \in Z_{-1}$ and $a \in L_{-1}$. Then $[a, z] = 0$ by definition of Z_{-1} , and $g_y(z) = g_y(a) = 0$ by Proposition 2.4(v). By the Premet identities, we then have

$$[[y, a], [y, z]] = g_y([a, z]) + g_y(z)[y, a] - g_y(a)[y, z] = 0.$$

Since $[y, L_{-1}] = L_1$ by Proposition 2.4(iii), this shows that $[y, Z_{-1}] \leq Z_1$. Similarly, $[x, Z_1] \leq Z_{-1}$, but then $Z_{-1} = [x, [y, Z_{-1}]] \leq [x, Z_1] \leq Z_{-1}$, so in fact,

$$[y, Z_{-1}] = Z_1 \quad \text{and} \quad [x, Z_1] = Z_{-1}.$$

Observe now that by the Jacobi identity, $[[L_0, Z_{-1}], L_{-1}] \leq [L_0, [Z_{-1}, L_{-1}]] + [Z_{-1}, [L_0, L_{-1}]] = 0$, and thus, $[L_0, Z_{-1}] \leq Z_{-1}$; similarly, we have $[L_0, Z_1] \leq Z_1$. Moreover,

$$[y, [Z_{-1}, L_1]] = [[y, Z_{-1}], L_1] = [Z_1, L_1] = 0,$$

and hence, the map φ from Lemma 2.8 fixes $[Z_{-1}, L_1]$. However, $\varphi(Z_{-1}) = [y, Z_{-1}] = Z_1$ and $\varphi(L_1) = [x, L_1] = L_{-1}$, and therefore, $[Z_{-1}, L_1] = [Z_1, L_{-1}]$. We can now see that

$$Z_{-1} \oplus [Z_{-1}, L_1] \oplus Z_1$$

is an ideal of L . Because L is simple, we conclude that $Z_{-1} = 0$ and $Z_1 = 0$. $\square$

The idea of looking at two different 5-gradings obtained from hyperbolic pairs is a key ingredient for our results. The following lemma is a very first (easy) instance of this.

Lemma 2.12. *Consider $x, y, a, b \in E$ such that $g_x(y) = 1 = g_a(b)$. Let*

$$\begin{aligned} L &= L_{-2} \oplus L_{-1} \oplus L_0 \oplus L_1 \oplus L_2 \quad \text{and} \\ L &= L'_{-2} \oplus L'_{-1} \oplus L'_0 \oplus L'_1 \oplus L'_2 \end{aligned}$$

be the 5-gradings corresponding to the hyperbolic pairs (x, y) and (a, b) , respectively, as in Proposition 2.4.

If $\varphi \in \text{Aut}(L)$ maps L_{-2} to L'_{-2} and L_2 to L'_2 , then $\varphi(L_i) = L'_i$ for all $i \in [-2, 2]$.

Proof. By assumption, there exist $\lambda, \mu \in k^\times$ such that $\varphi(x) = \lambda a$ and $\varphi(y) = \mu b$. We have $L_0 = N_L(x) \cap N_L(y)$ and $L'_0 = N_L(a) \cap N_L(b)$ by Proposition 2.4, so L_0 is mapped to L'_0 by φ .

Next, let $U = \langle x, y, [x, y] \rangle^\perp$ and $U' = \langle a, b, [a, b] \rangle^\perp$. By Remark 1.3 and Proposition 1.7(ii), the form g is uniquely determined by the Lie algebra L , so $\varphi(U) = U'$. (Recall that ‘ $\perp$ ’ is with respect to g .) It follows that φ maps $L_{-1} = [x, U]$ to $L'_{-1} = [a, U']$ and $L_1 = [y, U]$ to $L'_1 = [b, U']$. $\square$

In Proposition 2.16, we will show that if L is simple and the extremal geometry has lines, we can write x and y as the Lie bracket of two extremal elements in L_{-1} and L_1 , respectively. We first recall some geometric facts from [CI06].

Proposition 2.13. *Assume that L is simple and that the extremal geometry contains lines (i.e., $\mathcal{F}(L) \neq \emptyset$). Let $x, y \in \mathcal{E}$. Then*

- (i) *If $(x, y) \in E_{-1}$, then $E_{-1}(x) \cap E_1(y) \neq \emptyset$.*
- (ii) *If $(x, y) \in E_0$, then $E_0(x) \cap E_2(y) \neq \emptyset$.*
- (iii) *If $(x, y) \in E_1$, then $E_{-1}(x) \cap E_2(y) \neq \emptyset$.*
- (iv) *If $(x, y) \in E_2$, then $E_{-1}(x) \cap E_1(y) \neq \emptyset$, while $E_{-1}(x) \cap E_{\leq 0}(y) = \emptyset$.*

Proof. By Remark 2.2, L does not contain sandwich elements. By [CI06, Theorem 28], $(\mathcal{E}, \mathcal{F})$ is a so-called *root filtration space*, which is either *nondegenerate* or has an empty set of lines. By our assumption, the latter case does not occur.

In particular, the statements of [CI06, Lemmas 1 and 4] and [CI07, Lemma 8] hold. Now claim (i) is [CI06, Lemma 4(ii)] and claim (ii) is [CI07, Lemma 8(ii)]. To show claim (iii), let $z := [x, y]$, so $(z, x) \in E_{-1}$. By (i), we can find some $u \in E_{-1}(x) \cap E_1(u)$. We can now invoke [CI06, Lemma 1(v)] on the pairs (y, x) and (z, u) to conclude that $u \in E_{-1}(x) \cap E_2(y)$.

Finally, to show (iv), we first observe that the nondegeneracy of the root filtration space implies that $E_{-1}(x) \neq \emptyset$ (this is condition (H) on [CI06, p. 439]) and hence, there exists a line ℓ through $\langle x \rangle$. We now use the defining properties (D) and (F) of a root filtration space (see [CI06, p. 435]) to see that $\mathcal{E}_1(y)$ has nonempty intersection with ℓ , as required. The final statement $E_{-1}(x) \cap E_{\leq 0}(y) = \emptyset$ is just the defining property (D) itself. $\square$

Lemma 2.14. *Let x, y and L_i be as in Proposition 2.4. Then $E_{-1}(x) \cap E_1(y) = E \cap L_{-1}$.*

Proof. By Proposition 2.5, we have $E_{-1}(x) = (E \cap L_{\leq -1}) \setminus L_{-2}$. Similarly (e.g., by applying Lemma 2.8; see Remark 2.9), we have $E_1(y) = (E \cap L_{\geq -1}) \setminus L_{\geq 0}$. We conclude that $E_{-1}(x) \cap E_1(y) = E \cap L_{-1}$. $\square$

Lemma 2.15. *Let x, y and L_i be as in Proposition 2.4. Consider $e \in E \cap L_1$. Then $g_e(x) = 0$ for all $x \in L_i$ with $i \neq -1$.*

Proof. By Proposition 2.4(v), we have $g_e(x) = 0$ and $g_e(y) = 0$. Moreover, for any $l \in L_1$, there exists $l' \in L_{-1}$ such that $l = [y, l']$ by Proposition 2.4. Since g associates with the Lie bracket, we have

$$g_e(l) = g_e([y, l']) = g([e, y], l') = 0.$$

Finally, let $l \in L_0$. By Proposition 2.4 again, there exist $e' \in L_{-1}$ such that $e = [y, e']$. Let $\lambda \in k$ such that $[y, l] = \lambda y$. Then

$$g_e(l) = g([y, e'], l) = -g(e', [y, l]) = \lambda g_y(e') = 0. \quad \square$$

Proposition 2.16. *Assume that L is simple and that the extremal geometry of L contains lines. Let x, y and L_i be as in Proposition 2.4. Then there exist $c, d \in E \cap L_{-1}$ such that $[c, d] = x$. Moreover,*

$$[x, y] = [c + d, -[c, y] + [d, y]] \in [L_{-1}, L_1],$$

where L_i is the subspace of L_i spanned by $E \cap L_i$, for $i = \pm 1$.

Proof. By Proposition 2.13(iv), there exists an $a \in E_{-1}(x) \cap E_1(y)$. By Lemma 2.14, $a \in E \cap L_{-1}$.

Now, by Proposition 2.13(i), we can find some $b \in E_{-1}(x) \cap E_1(a)$. By Corollary 2.6, we can write $b = \lambda x + d$ with $d \in E \cap L_{-1}$. Moreover, $[a, d] = [a, b]$, and since the pair (a, b) is special and both a and b are collinear with x , we must have $[a, b] = \lambda x$ for some $\lambda \in k^\times$. Now write $c = \lambda^{-1}a$, and we get $[c, d] = x$, as required.

Next, observe that $g(c, y) = 0$ and $g(d, y) = 0$ because these pairs are special. Using the Jacobi identity, we get

$$\begin{aligned} [c + d, -[c, y] + [d, y]] &= -2g_c(y)c + 2g_d(y)d + [c, [d, y]] - [d, [c, y]] \\ &= [c, [d, y]] + [d, [y, c]] = -[y, [c, d]] \\ &= -[y, x] = [x, y], \end{aligned}$$

as claimed. $\square$

3. l -exponential automorphisms

In this section, we consider a simple Lie algebra over the field $k \neq \mathbb{F}_2$ which is generated by its pure extremal elements. Such a Lie algebra admits hyperbolic pairs of extremal elements, and by Proposition 2.4, each such pair gives rise to a 5-grading $L = L_{-2} \oplus L_{-1} \oplus L_0 \oplus L_1 \oplus L_2$. The main goal of this section is to define automorphisms that behave like exponential automorphisms, with respect to elements of L_1 (or L_{-1}). If $l \in L_1$, then we call the corresponding automorphisms *l -exponential*. We introduce these in Definition 3.12 and prove their existence and uniqueness (up to a one-parameter choice) in Theorems 3.13 and 3.16.

We first need an auxiliary result; namely, we show in Theorem 3.8 that if the extremal geometry of L contains lines, then L_1 is spanned by the extremal elements contained in L_1 (and similarly for L_{-1}). We will need this condition on the existence of lines in Theorem 3.13, but in Theorem 3.16, we will use a Galois descent technique that will allow us to get the same existence and uniqueness results under a much weaker condition – namely, the existence of lines after extending the scalars by a Galois extension.

The existence and uniqueness of these automorphisms will play a key role in Sections 4 and 5.

Assumption 3.1. We assume in this section that L is a simple Lie algebra generated by its set E of pure extremal elements. We also assume that $k \neq \mathbb{F}_2$. Notice that this implies that L does not contain sandwich elements, and by [CI06, Theorem 28], the graph $(\mathcal{E}, \mathcal{E}_2)$ is connected, so L contains many hyperbolic pairs.

Notation 3.2. Let $(x, y) \in E_2$ be a hyperbolic pair and scale x and y so that $g(x, y) = 1$. Let $L = L_{-2} \oplus L_{-1} \oplus L_0 \oplus L_1 \oplus L_2$ be the corresponding 5-grading as in Proposition 2.4, with $L_{-2} = \langle x \rangle$ and $L_2 = \langle y \rangle$.

Let I_{-1} be the subspace of L_{-1} spanned by $E \cap L_{-1}$ and let I_1 be the subspace of L_1 spanned by $E \cap L_1$. Notice that the extremal geometry has no lines if and only if E_{-1} and E_1 are empty or, equivalently, if and only if $I_{-1} = I_1 = 0$.

We begin with a lemma that provides us with a rather ‘small’ generating set for L whenever the extremal geometry has lines.

Lemma 3.3. *If $I_{-1} \neq \emptyset$, then L is generated by y and I_{-1} .*

Proof. Denote the subalgebra generated by y and I_{-1} by I . Since $I_{-1} \neq \emptyset$, there are lines in the extremal geometry. By Proposition 2.16, $x \in [I_{-1}, I_{-1}] \subseteq I$. By Corollary 2.6, all elements in $E_{-1}(x)$ are contained in I .

Now observe that if $a, b \in I$, then also $\exp(a)(b) = b + [a, b] + g(a, b)a \in I$. In particular, the automorphism φ introduced in Lemma 2.8 maps I to itself. This implies that also $I_1 \in I$ and all elements of $E_{-1}(y)$ belong to I .

We can now replace y by any $y' \in E_2(x) \cap E_{-1}(y)$; observe that both y' and the ‘new’ I_{-1} w.r.t. y' are contained in I . By repeating this procedure, we see that

$$\text{all elements of } E_2(x) \text{ connected to } y \text{ are contained in } I. \quad (5)$$

If the relation E_0 is nonempty, then we can invoke [CM21, Lemma 2.20]. It tells us that $E_2(x)$ is connected, and we conclude that $E_2(x) \subseteq I$. If $z \in E_1(x)$, then by Proposition 2.13(iii), we can find an element $y' \in E_{-1}(z) \cap E_2(x)$. Again, $y' \in I$, and we can replace y by y' to see that also the collinear point z is contained in I . Thus, $E_1(x) \subseteq I$. Applying φ , we also get $E_1(y) \subseteq I$. The only extremal elements not considered yet are those in $C := E_0(x) \cap E_0(y)$; notice that $C = E \cap L_0$ by Proposition 2.5. So let $z \in C$ and let ℓ be any line through z not completely contained in C . Now observe that C is a subspace of the extremal geometry (i.e., if two collinear elements are contained in C , then the whole line through these elements lies in C). (This follows, for instance, from the fact that $L_0 = N_L(x) \cap N_L(y)$ by Proposition 2.4). Hence, z is the unique point of ℓ in C . Since we assume $|k| \geq 3$ (see Assumption 3.1), ℓ contains at least 2 other points, which we already know are contained in I , and we conclude that also $z \in I$. Since L is generated by E , this shows that indeed $I = L$ in this case.

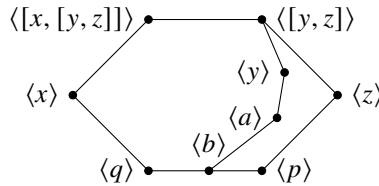
Assume from now on $\mathcal{E}_0 = \emptyset$. In this case, the extremal geometry is a generalized hexagon, and the relations $\mathcal{E}_{-1}$, $\mathcal{E}_1$ and $\mathcal{E}_2$ coincide with the relations ‘distance 1’, ‘distance 2’, and ‘distance 3’, respectively, in the collinearity graph. (Indeed, we have observed in the proof of Proposition 2.13 that $(\mathcal{E}, \mathcal{F})$ is a nondegenerate root filtration space, and it has been observed in [Coh12, Example 3.5] that a nondegenerate root filtration space with $\mathcal{E}_0 = \emptyset$ is necessarily a generalized hexagon.) Recall that in a generalized hexagon, for any point p and any line ℓ , there is a *unique* point on ℓ closest to p , called the *projection* of p on ℓ .

Let $z \in E_2(x)$ be arbitrary; our goal is to show that $z \in I$. If $(y, z) \in E_{\leq -1}$, then it follows from (5) that $z \in I$. Next, assume that $(y, z) \in E_1$, so $\langle y \rangle$ and $\langle z \rangle$ are at distance 2. Recall that the common neighbor of $\langle y \rangle$ and $\langle z \rangle$ is $\langle [y, z] \rangle$. By (5), we are done if $[y, z] \in E_2(x)$, so we may assume that $[y, z] \in E_1(x)$. Now consider any line through $\langle z \rangle$ not containing $\langle [y, z] \rangle$, and let $\langle p \rangle$ be the projection of $\langle x \rangle$ on that line, so $\text{dist}(\langle x \rangle, \langle p \rangle) = 2$; let $q := [p, x]$. Observe now that

$$\langle x \rangle, \langle [x, [y, z]] \rangle, \langle [y, z] \rangle, \langle z \rangle, \langle p \rangle, \langle q \rangle, \langle x \rangle$$

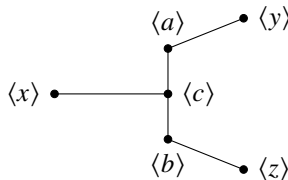
forms an ordinary hexagon. Next, let $\langle b \rangle$ the projection of $\langle y \rangle$ on the line $\langle p, q \rangle$. Notice that $\langle b \rangle$ is different from $\langle p \rangle$ and $\text{dist}(\langle y \rangle, \langle b \rangle) = 2$. If $\langle b \rangle = \langle q \rangle$, we replace y by another y' collinear with y and

$[y, z]$, which will then have a different projection on the line $\langle p, q \rangle$, so we may assume without loss of generality that $\langle b \rangle \neq \langle q \rangle$. Let $a := [b, y]$.



Since all neighbors of $\langle x \rangle$ and $\langle y \rangle$ are contained in I , we see that $q \in I$ and $a \in I$, and hence also their common neighbor $\langle b \rangle$ belongs to I . Now the line $\langle p, q \rangle$ already contains two points in I (namely $\langle b \rangle$ and $\langle q \rangle$), and hence also $p \in I$. Finally, since $\langle p \rangle$ and $\langle [y, z] \rangle$ belong to I , their common neighbor $\langle z \rangle$ belongs to I .

Finally, assume that $(y, z) \in E_2$, so $z \in E_2(x) \cap E_2(y)$. Let $\langle y \rangle, \langle a \rangle, \langle b \rangle, \langle z \rangle$ be a path of length 3. If $\langle a, b \rangle$ is not completely contained in $E_{\leq 1}(x)$, we can apply the previous paragraph twice to obtain $z \in I$. So assume² $a, b \in E_1(x)$ (i.e., both a and b are at distance 2 from x). Since a generalized hexagon does not contain cycles of length ≤ 5 , this can only happen if the line $\langle a, b \rangle$ contains a point $\langle c \rangle$ collinear with $\langle x \rangle$.



Since $\text{Exp}(x)$ acts sharply transitively on the set of points of the line $\langle a, b \rangle$ without $\langle c \rangle$ (see Definition 1.5), we find a unique $\psi \in \text{Exp}(x)$ such that $\psi(a) \in \langle b \rangle$. Let $z' := \psi(y)$, and notice that ψ fixes x . Since $x, y \in I$, also $z' \in I$. Since ψ fixes x , we get $z' \in E_2(x)$. Now $\langle b \rangle = \psi(\langle a \rangle)$ is a neighbor of both $\langle z \rangle$ and of $\langle z' \rangle$, and thus, $z' \in E_{\leq 1}(z)$. Since $z' \in I$, the previous paragraph now implies $z \in I$.

We have now shown that $E_2(x) \subseteq I$, and just as in the case where $\mathcal{E}_0 \neq \emptyset$, we conclude that $I = L$. $\square$

Corollary 3.4. Suppose $I_{-1} \neq \emptyset$. If a nontrivial subspace $I \leq L$ satisfies $[I, y] \leq I$ and $[I, I_{-1}] \leq I$, then $I = L$.

Proof. By Lemma 3.3 and the Jacobi identity, I is an ideal of L . Since L is simple, this implies $I = L$. $\square$

Another ingredient that we will need is the fact from [CI07] that if $\mathcal{E}_0 \neq \emptyset$, then each symplectic pair of points induces a subgeometry of the extremal geometry, known as a *symplecton*. We state the version from [CM21, Proposition 2.12]. Recall that a *polar space* is a partial linear space such that for each point p and each line ℓ , the point p is collinear with either one or all points of ℓ . A polar space is called *nondegenerate* if none of its points is collinear with all other points.

Proposition 3.5 [CM21, Proposition 2.12 and Lemma 2.18]. Assume that the extremal geometry contains lines and that $\mathcal{E}_0 \neq \emptyset$. Then the extremal geometry contains a collection $\mathcal{S}$ of subspaces such that every pair of points x, y with $(x, y) \in \mathcal{E}_0$ is contained in a unique element $S \in \mathcal{S}$.

Moreover, for each $S \in \mathcal{S}$ we have

- (i) S is a nondegenerate polar space (of rank ≥ 2 , i.e., containing lines).
- (ii) For all points $x, y \in S$, we have $(x, y) \in \mathcal{E}_{\leq 0}$.

²In fact, we may assume that this is true for all possible paths of length 3 from $\langle y \rangle$ to $\langle z \rangle$, but this does not seem to lead to a simpler argument.

- (iii) If a point x is collinear to two non-collinear points of S , then $x \in S$.
- (iv) For each point x , the set of points in $\mathcal{E}_{\leq -1}(x) \cap S$ is either empty or contains a line.
- (v) If x is a point with $\mathcal{E}_2(x) \cap S \neq \emptyset$, then $S \cap \mathcal{E}_0(x)$ contains a unique point.

The elements of S are called *symplecta*.

We will need the following property of polar spaces.

Lemma 3.6. *Let S be a nondegenerate polar space. Let x and y be non-collinear points of S . Then S is generated (as a point-line geometry) by x , y and $x^\perp \cap y^\perp$, where $x^\perp$ denotes the set of points collinear with x .*

Proof. Let X be the subspace generated by x , y and $x^\perp \cap y^\perp$. Since every line through x contains a point collinear to y , we have $x^\perp \subseteq X$, and similarly $y^\perp \subseteq X$. Consider $z \in X$ arbitrary. Let ℓ be any line through x . If $x^\perp \cap \ell \neq y^\perp \cap \ell$, then $\ell \subseteq X$. So we may assume that ℓ contains a unique point $a \in x^\perp \cap y^\perp$. By the nondegeneracy, we find $b \in x^\perp \cap y^\perp$ such that a and b are not collinear. Now ℓ and by are opposite lines, and hence, there exists a point c on ℓ distinct from a which is collinear to a point of by distinct from y and b . Hence, c lies on a line m such that $m \cap y^\perp \in by \setminus \{b, y\}$ and thus, $m \cap y^\perp \neq m \cap x^\perp$. As before, $m \subseteq X$. In particular, $c \in X$, so together with $a \in X$ we obtain $z \in X$. $\square$

The next lemma gives a more precise description of the subspace L_S of L spanned by the elements in S .

Lemma 3.7. *Let S be a symplecton containing x . Then*

$$L_S = \langle x \rangle \oplus \langle \{ \langle z \rangle \in S \mid z \in L_{-1} \} \rangle \oplus \langle \{ \langle z \rangle \in S \mid z \in L_0 \} \rangle.$$

Moreover, $\langle \{ \langle z \rangle \in S \mid z \in L_0 \} \rangle$ is 1-dimensional.

Proof. By Proposition 3.5(v), $S \cap \mathcal{E}_0(y)$ consists of a single point $\langle z \rangle$, and since $S \subseteq \mathcal{E}_{\leq 0}(x)$, this implies $z \in L_0$. By Lemma 3.6, L_S is spanned by x , z , and all points $\langle a \rangle$ collinear to both $\langle x \rangle$ and $\langle z \rangle$. For such a point $\langle a \rangle$, we have of course $a \in E_{-1}(x)$. Now $z \in E_{-1}(a) \cap E_0(y)$, so by Proposition 2.13(iv), we have $(a, y) \notin E_2$. Hence, $a \in E_{-1}(x) \cap E_{\leq 1}(y) = L_{-1}$ by Lemma 2.14. $\square$

We are now ready to prove the first theorem of this section.

Theorem 3.8. *Assume that L is a simple Lie algebra over $k \neq \mathbb{F}_2$ generated by its pure extremal elements, and assume that the set of lines $\mathcal{F}$ is nonempty. Consider $x, y \in E$ with $g(x, y) = 1$ and let $L = L_{-2} \oplus L_{-1} \oplus L_0 \oplus L_1 \oplus L_2$ be the associated 5-grading. Then both L_{-1} and L_1 are linearly spanned by the extremal elements contained in it.*

Proof. As before, let I_{-1} and I_1 be the subspaces of L linearly spanned by the extremal elements contained in L_{-1} and L_1 , respectively. Let

$$I = \langle x \rangle \oplus I_{-1} \oplus [I_{-1}, I_1] \oplus I_1 \oplus \langle y \rangle.$$

We will use Corollary 3.4 to show that $I = L$.

We first show that $[I, y] \leq I$. By Proposition 2.16, we know that $[x, y] \in [I_{-1}, I_1]$. Next, if $a \in E \cap L_{-1}$, then $a \in E_1(y)$, and hence, $[a, y] \in I_1$. Since $[L_{\geq 0}, y] \leq L_2 = \langle y \rangle$, the claim $[I, y] \leq I$ is now clear.

It remains to show that $[I, I_{-1}] \leq I$. Obviously, $[\langle x \rangle \oplus I_{-1} \oplus I_1, I_{-1}] \leq I$. Similarly as before, $[\langle y \rangle, I_{-1}] \leq I_1 \leq I$, so the only case left to prove is

$$[[I_{-1}, I_1], I_{-1}] \leq I_{-1}.$$

So consider arbitrary extremal elements $a, b \in E \cap L_{-1}$ and $c \in E \cap L_1$; our goal is to show that

$$[[a, c], b] \in I_{-1}. \quad (6)$$

Observe that $[a, b] \in L_{-2} = \langle x \rangle$, so by the Jacobi identity and the fact that $[x, c]$ is contained in $E \cap L_{-1}$, we have

$$[[a, c], b] \in I_{-1} \iff [[b, c], a] \in I_{-1}.$$

Also observe that $c \in E_{\leq 0}(a)$ implies $[a, c] = 0$ and that $c \in E_{\leq 0}(b)$ implies $[b, c] = 0$. Hence, we may assume from now on that $c \in E_{\geq 1}(a) \cap E_{\geq 1}(b)$.

Since a is extremal, $[[a, c], a] \in \langle a \rangle$ is again extremal, so (6) obviously holds if $(a, b) \in E_{-2}$. If $(a, b) \in E_{-1}$, then by [CM21, Lemma 6.3], $J := \langle a, b \rangle$ is an *inner ideal* (i.e., $[J, [J, L]] \leq J$); in particular, $[b, [a, c]] \in J \leq I_{-1}$. Next, if $(a, b) \in E_0$, then a and b are contained in a symplecton S . The subspace L_S spanned by all elements of S is an inner ideal by [CM21, Lemma 6.5], and hence, $[b, [a, c]] \in L_S$. Since $[b, [a, c]] \in L_{-1}$, Lemma 3.7 now implies that $[b, [a, c]] \in \langle \{ \langle z \rangle \in S \mid z \in L_{-1} \} \rangle \leq I_{-1}$.

Now notice that $(a, b) \in E_2$ cannot occur because both a and b are collinear to x , so we can assume from now on that $(a, b) \in E_1$. In particular, $[a, b] \in \langle x \rangle \setminus \{0\}$.

If $c \in E_1(a)$, then $[a, c] \in E$ and since the extremal form g is associative, we have $g(b, [a, c]) = g([b, a], c) = 0$, since $c \in E_1(x)$. Hence, $[a, c] \in E_{\leq 1}(b)$, so either $[[a, c], b] \in E$ or $[[a, c], b] = 0$. In either case, we obtain $[[a, c], b] \in I_{-1}$. The case $c \in E_1(b)$ is similar.

The only case left to consider is $a \in E_1(b)$ and $c \in E_2(a) \cap E_2(b)$, and this case requires some more effort. We have $g(a, b) = 0$, and by rescaling, we may assume that $g(a, c) = g(b, c) = 1$, and we also have $g(b, [a, c]) = 0$ as before.

Let $\lambda, \mu \in k$ be arbitrary nonzero elements such that

$$\mu(\lambda - 1) = 1, \tag{7}$$

which exist because $k \neq \mathbb{F}_2$, and consider the extremal element

$$d := \exp(\mu c) \exp(\lambda b) \exp(c)(a).$$

Then

$$\begin{aligned} d &= \exp(\mu c) \exp(\lambda b) (a + [c, a] + c) \\ &= \exp(\mu c) \left(\underbrace{\lambda[b, a]}_{\in L_{-2}} + \underbrace{a + \lambda[b, [c, a]] + \lambda^2 b}_{\in L_{-1}} + \underbrace{[c, a] + \lambda[b, c]}_{\in L_0} + \underbrace{c}_{\in L_1} \right). \end{aligned}$$

We claim that $d \in L_{\leq -1}$. This could be checked by a direct computation, but it is easier to reverse the process, by starting from an arbitrary extremal element of the form

$$e := \lambda[b, a] + z \in \langle x \rangle + L_{-1},$$

compute $\exp(-\mu c)(e)$, and verify that with the correct choice of z , this is equal to $\exp(-\mu c)(d)$. We get

$$\exp(-\mu c)(e) = \underbrace{\lambda[b, a]}_{\in L_{-2}} + \underbrace{z - \lambda\mu[c, [b, a]]}_{\in L_{-1}} - \underbrace{\mu[c, z]}_{\in L_0} + \underbrace{\mu^2 g(c, z)c}_{\in L_1}.$$

Comparing the L_{-1} -components, we have no other choice than setting

$$z := a + \lambda^2 b + \lambda[b, [c, a]] + \lambda\mu[c, [b, a]]. \tag{8}$$

Now, using the Premet identity (3) (see Definition 1.2) and the fact that $g(c, [b, a]) = 0$, we get

$$\begin{aligned} [c, z] &= [c, a] + \lambda^2[c, b] + \lambda[c, [b, [c, a]]] + \lambda\mu[c, [c, [b, a]]] \\ &= [c, a] + \lambda^2[c, b] - \lambda([c, b] + [c, a]) \\ &= (1 - \lambda)[c, a] + \lambda(\lambda - 1)[c, b], \end{aligned}$$

and hence by (7), $-\mu[c, z] = [c, a] - \lambda[c, b]$, so the L_0 -components coincide. Finally,

$$\begin{aligned} g(c, z) &= g(c, a) + \lambda^2g(c, b) + \lambda g(c, [b, [c, a]]) + \lambda\mu g(c, [c, [b, a]]) \\ &= 1 + \lambda^2 - \lambda g([c, [c, a]], b) \\ &= 1 + \lambda^2 - 2\lambda g(c, a)g(c, b) = 1 + \lambda^2 - 2\lambda = (1 - \lambda)^2, \end{aligned}$$

so again by (7), $\mu^2g(c, z) = 1$, and hence also the L_1 -components coincide. Therefore, $e = d$, and we conclude that the element $e = \lambda[b, a] + z \in \langle x \rangle + L_{-1}$ is an extremal element.

If $z = 0$, then by (8), $[b, [c, a]]$ can be written as the sum of 3 extremal elements contained in L_{-1} . Indeed, obviously $a, b \in E$, but also $[c, [b, a]] \in E$ since $\langle c \rangle$ and $\langle x \rangle = \langle [b, a] \rangle$ form a special pair.

If $z \neq 0$, then $\langle e, x \rangle$ is a line of the extremal geometry. Since $z \in \langle e, x \rangle$, it is contained in E . So by (8) again, the element $[b, [c, a]]$ can be written as the sum of 4 extremal elements contained in L_{-1} .

This shows that $[b, [c, a]] \in L_{-1}$ in all possible cases. $\square$

Now we will work toward showing the so-called *algebraicity* of the Lie algebra, using the previous theorem. We will give a precise definition of this property in Definition 3.15 below, but loosely speaking, it means that for any element in L_{-1} , there exist automorphisms depending on this element which behave nicely with respect to the 5-grading. Actually, if the characteristic is not 2, 3 or 5, this property is easily shown (recall the discussion following Definition 1.10), so it is not surprising that we sometimes have to handle the small characteristic cases a bit more carefully.

In the next two lemmas, we show that as soon as an extremal element has a certain form, it is contained in the image of a specific element under $\text{Exp}(x)$. We will use this later to show the uniqueness of certain automorphisms.

Lemma 3.9. Assume $\text{char}(k) \neq 3$. Let $l = l_{-2} + l_{-1} + e$ be an extremal element, with $l_i \in L_i$ and $e \in E \cap L_1$. Then $l_{-2} = 0$ and $l_{-1} \in \langle [x, e] \rangle$. In particular, $l = \exp(\lambda x)(e) = e + \lambda[x, e]$ for some $\lambda \in k$.

Proof. By Proposition 2.4(ii), we have

$$[l, [l, [x, y]]] = [l, 2l_{-2} + l_{-1} - e] = -3[l_{-2}, e] - 2[l_{-1}, e] \in L_{-1} + L_0.$$

However, since l is extremal, we have $[l, [l, [x, y]]] \in \langle l \rangle$. By the previous identity, this element must have trivial L_1 -component, but since $e \neq 0$ by assumption, this implies that $[l, [l, [x, y]]] = 0$, so

$$3[l_{-2}, e] = 2[l_{-1}, e] = 0.$$

Since $\text{char}(k) \neq 3$, this implies $[l_{-2}, e] = 0$, and hence, $l_{-2} = 0$ by Proposition 2.4(iii). If $l_{-1} = 0$, then there is nothing to show, so assume $l_{-1} \neq 0$. Then

$$l = l_{-1} + e \in L_{-1} + L_1, \quad l_{-1} \neq 0, \quad e \neq 0.$$

By Proposition 2.5, we have $l \in E_1(y)$, and thus, $[l_{-1}, y] = [l, y] \in E$. By Proposition 2.5 again, $[l_{-1}, y] \in E_1(x)$, and thus, $l_{-1} = [x, [l_{-1}, y]] \in E$. Since l_{-1}, e and $l = l_{-1} + e$ are contained in E , Proposition 1.8 implies that $(l_{-1}, e) \in E_{-1}$. Hence, $\langle l_{-1} \rangle$ is the common neighbor of the special pair $(\langle x \rangle, \langle e \rangle)$, and thus, $l_{-1} \in \langle [x, e] \rangle$. $\square$

Lemma 3.10. *Let $l = l_{-2} + l_{-1} + l_0 + y \in E$ be an extremal element, with $l_i \in L_i$. Then $l_{-1} = 0$ and $l = \exp(\lambda x)(y)$ for some $\lambda \in k$.*

Proof. Let $\lambda, \mu \in k$ be such that $l_{-2} = \lambda x$ and $[l_0, x] = \mu x$. By Proposition 2.4(ii), we have

$$\begin{aligned} [l, [l, x]] &= [l, \mu x + [y, x]] \\ &= \mu[l, x] + [\lambda x + l_{-1} + l_0 + y, [y, x]] \\ &= (\mu^2 x + \mu[y, x]) + (-2\lambda x - l_{-1} + 2y). \end{aligned}$$

Since l is extremal and $g(l, x) = g(y, x) = 1$ by Proposition 2.4(v), we get

$$\mu^2 = 4\lambda, \quad 3l_{-1} = 0, \quad \mu[y, x] = 2l_0. \quad (9)$$

If $\text{char}(k)$ is not 2 or 3, then this already implies that $l = \exp(-\frac{1}{2}\mu x)(y)$.

Assume now $\text{char}(k) = 3$. We claim that also in this case, $l_{-1} = 0$; it then follows again that $l = \exp(-\frac{1}{2}\mu x)(y)$. To show the claim, suppose that $l_{-1} \neq 0$ and let $z := \exp(\frac{1}{2}\mu x)(l)$. Then z is an extremal element, and we compute that $z = l_{-1} + y$. We apply the same technique as in the last paragraph of the proof of Lemma 3.9: By Proposition 2.5, $z \in E_1(y)$, and thus, $[l_{-1}, y] = [z, y] \in E$. By Proposition 2.5 again, $[l_{-1}, y] \in E_1(x)$, and thus, $l_{-1} = [x, [l_{-1}, y]] \in E$. Since l_{-1}, y and $z = l_{-1} + y$ are contained in E , Proposition 1.8 implies that $(l_{-1}, y) \in E_{-1}$, contradicting Proposition 2.5.

Assume, finally, that $\text{char}(k) = 2$. By (9), we have $l_{-1} = 0$, so $l \in L_{-2} + L_0 + L_2$. By Proposition 2.16, there exist $c, d \in E \cap L_{-1}$ such that $[c, d] = x$. By Lemma 2.15, we get for any $e \in E \cap L_{-1}$ that $g(l, e) = 0$, and since $[y, e] \neq 0$, this implies that the pair (l, e) is special; in particular, $[l, e] = [l_0, e] + [y, e] \in L_{-1} + L_1$ is extremal. By Lemma 3.9 applied on $[l, c]$ and $[l, d]$, we find $\lambda, \mu \in k$ such that $[l_0, c] = \lambda[x, [y, c]] = \lambda c$ and $[l_0, d] = \mu[x, [y, d]] = \mu d$. Let $\alpha := \exp(\lambda x)$. Then

$$\alpha([l, c]) = [y, c], \quad \alpha([l, d]) = [y, d] + (\lambda + \mu)d.$$

By the Premet identity (2), we have

$$\begin{aligned} [[l, c], [l, d]] &= g_l([c, d])l + g_l(d)[l, c] + g_l(c)[l, d] = l \quad \text{and} \\ [[y, c], [y, d]] &= g_y([c, d])y + g_y(d)[l, c] + g_y(c)[l, d] = y. \end{aligned}$$

Hence,

$$\begin{aligned} \alpha(l) &= [\alpha([l, c]), \alpha([l, d])] = [[y, c], [y, d] + (\lambda + \mu)d] \\ &= y + (\lambda + \mu)[[y, c], d]. \end{aligned}$$

Now note that by the Premet identity (3), we have

$$[y, [d, [y, c]]] = g_y([d, c])y + g_y(c)[y, d] + g_y(d)[y, c] = y,$$

and hence,

$$[\alpha(l), y] = (\lambda + \mu)y,$$

but then since $\text{char}(k) = 2$ and $\alpha(l)$ is an extremal element, we get

$$0 = [\alpha(l), [\alpha(l), y]] = (\lambda + \mu)^2 y,$$

so $\lambda = \mu$ and therefore $\alpha(l) = y$ and so $l = \alpha(y)$ (since $\alpha^2 = 1$).

□

Definition 3.11. Assume that the extremal geometry of L has lines. We set

$$E_+(x, y) = \langle \exp(e) \mid e \in E \cap L_{\geq 1} \rangle \leq \text{Aut}(L),$$

$$E_-(x, y) = \langle \exp(e) \mid e \in E \cap L_{\leq -1} \rangle \leq \text{Aut}(L).$$

It is important to point out that these groups are *generated* by elements of the form $\exp(e)$ and that an arbitrary element of $E_{\pm}(x, y)$ cannot be written as a single $\exp(e)$ in general. In Definition 3.15 below, we will give a different definition of the groups $E_{\pm}(x, y)$ that also makes sense when the extremal geometry has no lines.

The following theorem is the key tool in our paper. It allows us to deal with exponential maps even in characteristic 2 and 3.

Definition 3.12. Let $\alpha \in \text{Aut}(L)$ and $l \in L_1$. We say that α is an *l -exponential automorphism* if there exist (necessarily unique) maps $q_\alpha, n_\alpha, v_\alpha: L \rightarrow L$ with

$$q_\alpha(L_i) \subseteq L_{i+2}, \quad n_\alpha(L_i) \subseteq L_{i+3}, \quad v_\alpha(L_i) \subseteq L_{i+4}, \quad (10)$$

for all $i \in [-2, 2]$, such that

$$\alpha(m) = m + [l, m] + q_\alpha(m) + n_\alpha(m) + v_\alpha(m) \quad (11)$$

for all $m \in L$.

Theorem 3.13. Assume that L is a simple Lie algebra over $k \neq \mathbb{F}_2$ generated by its pure extremal elements, and assume that $\mathcal{F} \neq \emptyset$. Consider a hyperbolic pair $x, y \in E$ with $g(x, y) = 1$ and let $L = L_{-2} \oplus L_{-1} \oplus L_0 \oplus L_1 \oplus L_2$ be the associated 5-grading. Then

- (i) $\exp(a)\exp(b) = \exp([a, b])\exp(b)\exp(a)$ for all $a, b \in E \cap L_{\geq 1}$.
- (ii) $\exp(a)\exp(b) = \exp(a + b)$ for all $a \in E \cap L_{\geq 1}$ and $b \in L_2$.
- (iii) $[E_+(x, y), E_+(x, y)] = \text{Exp}(y)$ and $[E_+(x, y), \text{Exp}(y)] = 1$.

Assume now that $l \in L_1$.

- (iv) Write $l = e_1 + \cdots + e_n$ with $e_i \in E \cap L_1$. Then

$$\alpha := \exp(e_1) \cdots \exp(e_n) \in E_+(x, y)$$

is an *l -exponential automorphism*.

- (v) If α is as in (iv) and β is another *l -exponential automorphism*, then $\beta = \exp(z)\alpha$ for a unique $z \in L_2$. In particular, $\beta \in E_+(x, y)$.
- (vi) Let α be any *l -exponential automorphism*. Then there exist $e_1, \dots, e_n \in E \cap L_1$ such that $l = e_1 + \cdots + e_n$ and $\alpha = \exp(e_1) \cdots \exp(e_n)$. Moreover,

$$E_+(x, y) = \{\alpha \in \text{Aut}(L) \mid \alpha \text{ is } m\text{-exponential for some } m \in L_1\}.$$

- (vii) Let α be an *l -exponential automorphism*, with maps $q_\alpha, n_\alpha, v_\alpha: L \rightarrow L$. Then for each $\lambda \in k$, the map α_λ defined by

$$\alpha_\lambda(m) = m + \lambda[l, m] + \lambda^2 q_\alpha(m) + \lambda^3 n_\alpha(m) + \lambda^4 v_\alpha(m)$$

for all $m \in L$, is a (λl) -exponential automorphism. In particular, $\alpha_\lambda \in E_+(x, y)$.

- (viii) If α is an l -exponential automorphism and β is an l' -exponential automorphism, then $\alpha\beta$ is an $(l + l')$ -exponential automorphism.
- (ix) If $\text{char}(k) \neq 2$, then there is a unique l -exponential automorphism α with $q_\alpha(m) = \frac{1}{2}[l, [l, m]]$ for all $m \in L$. We denote this automorphism by $e_+(l)$. Its inverse is $e_+(-l)$.

Proof.

- (i) Let $a, b \in E \cap L_{\geq 1}$; notice that $g(a, b) = 0$. Since L is generated by $L_{\geq -1}$, it suffices to check that both sides coincide when applied on an element $l \in L_{\geq -1}$. We have

$$\begin{aligned} \exp(a) \exp(b)(l) &= \exp(a)(l + [b, l] + g(b, l)b) \\ &= l + [a + b, l] + [a, [b, l]] \\ &\quad + g([a, b], l)a + g(b, l)[a, b] + g(a, l)a + g(b, l)b. \end{aligned} \quad (12)$$

Using the Premet identity (3), we have

$$\begin{aligned} [[a, b], [a + b, l]] &= [[a, b], [a, l]] - [[b, a], [b, l]] \\ &= g_a([b, l])a + g_a(l)[a, b] - g_a(b)[a, l] \\ &\quad - g_b([a, l])b - g_b(l)[b, a] + g_b(a)[b, l] \\ &= g([a, b], l)a + g([a, b], l)b + (g(a, l) + g(b, l))[a, b]. \end{aligned} \quad (13)$$

Since $[a, b] \in L_2$, the automorphism $\exp([a, b])$ fixes all elements of $L_{\geq 1}$. Hence, by (12) with the roles of a and b reversed, (13) and the Jacobi identity, we see that

$$\exp(a) \exp(b)(l) = \exp([a, b]) \exp(b) \exp(a)(l),$$

as claimed.

- (ii) We again check that both sides coincide when applied on an element $l \in L_{\geq -1}$. Comparing (12) with the expression for $\exp(a + b)(l)$ and using the fact that $[a, b] = 0$ and that b is a multiple of y , we see that this is equivalent with

$$[a, [y, l]] = g(a, l)y + g(y, l)a.$$

Now both sides are equal to 0 if $l \in L_{\geq 0}$. Furthermore, if $l \in L_{-2}$, then we can write $l = \lambda x$ with $\lambda \in k$ to see that both sides are equal to λa , and if $l \in L_{-1}$, then the equality follows from Lemma 2.7.

- (iii) By (i), we already know that $[E_+(x, y), E_+(x, y)] \leq \text{Exp}(y)$. By Proposition 2.16, the element y can be written as $y = [c, d]$ for certain $c, d \in E \cap L_1$. By (i) again, we see that $\exp([c, d]) \in [E_+(x, y), E_+(x, y)]$, and by rescaling, we get $\text{Exp}(y) \leq [E_+(x, y), E_+(x, y)]$, as required. The second statement follows immediately from (i) since $[a, b] = 0$ if $b \in L_2$.
- (iv) Let $l \in L_1$. By Theorem 3.8, we can write l as the sum of a finite number of extremal elements in L_1 :

$$l = e_1 + \cdots + e_n \quad \text{with } e_i \in E \cap L_1.$$

We will prove, by induction on n , that the automorphism

$$\alpha := \exp(e_1) \cdots \exp(e_n) \in E_+(x, y)$$

satisfies (10) and (11) for the appropriate choices of maps q_α , n_α and v_α . (Notice that α depends on the choice of the elements $e_1, \dots, e_n$ as well as on their ordering.)

If $n = 1$, then we set

$$q_\alpha(m) := g_l(m)l, \quad n_\alpha(m) := 0, \quad v_\alpha(m) := 0,$$

for all $m \in L$. By Definition 1.5, we see that (11) is satisfied. By Lemma 2.15, $q_\alpha(L_i) = 0$ unless $i = -1$, so (10) holds as well.

Now assume $n > 1$, let $e := e_1$ and let $l' = e_2 + \cdots + e_n$, so $l = e + l'$. Let $\alpha' := \exp(e_2) \cdots \exp(e_n)$, so $\alpha := \exp(e)\alpha'$. By the induction hypothesis, there exist maps $q_{\alpha'}$, $n_{\alpha'}$ and $v_{\alpha'}$ satisfying (10) and (11). We set

$$eq_\alpha(m) := q_{\alpha'}(m) + g_e(m)e + [e, [l', m]], \quad (14)$$

$$n_\alpha(m) := n_{\alpha'}(m) + g_e([l', m])e + [e, q_{\alpha'}(m)], \quad (15)$$

$$v_\alpha(m) := v_{\alpha'}(m) + [e, n_{\alpha'}(m)], \quad (16)$$

for all $m \in L$. By Lemma 2.15 and the fact that (10) holds for α' , we get

$$\begin{aligned} [e, v_{\alpha'}(m)] &\in L_{\geq(1+4-2)} = L_{\geq 3} = 0 \quad \text{and} \\ g_e(q_{\alpha'}(m)) &= g_e(n_{\alpha'}(m)) = g_e(v_{\alpha'}(m)) = 0 \end{aligned}$$

for all $m \in L$, and by expanding

$$\alpha(m) = \exp(e)(\alpha'(m)) = \alpha'(m) + [e, \alpha'(m)] + g_e(\alpha'(m))e,$$

using the fact that (11) holds for α' , we now see that (11) is satisfied for α . By Lemma 2.15 again, we see that also (10) holds for α .

- (v) Let $l \in L_1$ and let $\beta \in \text{Aut}(L)$ be any automorphism satisfying (10) and (11). Let α be the automorphism $\alpha = \exp(e_1) \cdots \exp(e_n)$ as constructed in (iv) with respect to some choice $l = e_1 + \cdots + e_n$, and observe that by the same construction,

$$\alpha^{-1} = \exp(-e_n) \cdots \exp(-e_1)$$

then satisfies (10) and (11) for the element $-l = -e_n - \cdots - e_1$.

Now α and β coincide on $L_{\geq 1}$. However, any element of $E_+(x, y)$ preserves the filtration $(L_{\geq i})$, so in particular, $\alpha^{-1}(L_{\geq 1}) = L_{\geq 1}$. Hence, $\alpha\alpha^{-1}$ and $\beta\alpha^{-1}$ coincide on $L_{\geq 1}$, so $\gamma := \beta\alpha^{-1}$ fixes $L_{\geq 1}$.

Moreover, by (11) for β and α^{-1} , we have

$$\gamma(x) = \beta\alpha^{-1}(x) \in \beta(x - [l, x] + L_{\geq 0}) = x + L_{\geq 0},$$

so $\gamma(x)$ is an extremal element with trivial L_{-1} -component. By Lemma 3.10, $\gamma(x) = \exp(z)(x)$ for some unique $z = \lambda y \in L_2$. Notice that also $\exp(z)$ fixes $L_{\geq 1}$. Since L is generated by L_1 and x , we conclude that $\gamma = \exp(z)$.

- (vi) Let $\alpha \in E_+(x, y)$. We claim that α can be written as a product of elements of the form $\exp(a)$ with $a \in E \cap L_1$. Indeed, by definition, it is the product of elements $\exp(l)$ with $l \in E \cap L_{\geq 1}$, but each such l can be written as $l = a + b$ with $a \in E \cap L_1$ and $b \in L_2$. By (ii), $\exp(l) = \exp(a)\exp(b)$, and by the proof of (iii), $\exp(b) = [\exp(c), \exp(d)]$ for certain $c, d \in E \cap L_1$; this proves our claim. Hence, $\alpha = \exp(e_1) \cdots \exp(e_n)$ for certain $e_1, \dots, e_n \in E \cap L_1$. Now let $m := e_1 + \cdots + e_n$; then by (iv), α is an m -exponential automorphism. This shows that any automorphism in $E_+(x, y)$ is m -exponential for some $m \in L_1$; the other inclusion ‘ $\supseteq$ ’ holds by (v).

If α is any l -exponential automorphism, then in particular, $\alpha \in E_+(x, y)$, so by the previous paragraph, α is m -exponential for $m = e_1 + \cdots + e_n$. Then by (11), the L_{-1} -component of $\alpha(x)$ is equal to both $[m, x]$ and $[l, x]$, hence $m = l$, and we have shown the required decomposition.

- (vii) By (vi), we can write $\alpha = \exp(e_1) \cdots \exp(e_n)$ and $l = e_1 + \cdots + e_n$, where $e_i \in E \cap L_1$. Now let $\beta := \exp(\lambda e_1) \cdots \exp(\lambda e_n) \in E_+(x, y)$, with corresponding element $\lambda l = \lambda e_1 + \cdots + \lambda e_n \in L_1$ and corresponding maps $q_\beta, n_\beta, v_\beta: L \rightarrow L$. Observe now that if we replace each e_i by λe_i in the recursive formulas (14) to (16) in the proof of (iv), then this yields

$$q_\beta(m) = \lambda^2 q_\alpha(m), \quad n_\beta(m) = \lambda^3 n_\alpha(m), \quad v_\beta(m) = \lambda^4 v_\alpha(m),$$

for all $m \in L$. We conclude that $\beta = \alpha_\lambda$ and hence $\alpha_\lambda \in E_+(x, y)$, which is then a (λl) -exponential automorphism.

- (viii) By (vi), $\alpha = \exp(e_1) \cdots \exp(e_n)$ and $\beta = \exp(f_1) \cdots \exp(f_m)$ with $e_i, f_i \in E \cap L_1$ and with $l = e_1 + \cdots + e_n$ and $l' = f_1 + \cdots + f_m$. Hence, $\alpha\beta = \exp(e_1) \cdots \exp(e_n) \exp(f_1) \cdots \exp(f_m)$, so by (iv), $\alpha\beta$ is an $(l + l')$ -exponential automorphism.

- (ix) Assume $\text{char}(k) \neq 2$. We first show the existence of such an automorphism. As in the proof of part (iv), we write $l = e_1 + \cdots + e_n$ with $e := e_1$ and $l' := e_2 + \cdots + e_n$, and we proceed by induction on n .

If $n = 1$, then we choose $\alpha = \exp(e)$, which has $q_\alpha(m) = g_l(m)l = \frac{1}{2}[l, [l, m]]$, as required. Now assume $n > 1$. By the induction hypothesis, there exists an l' -exponential automorphism α' with $q_{\alpha'}(m) = \frac{1}{2}[l', [l', m]]$ for all $m \in L$, and we first set $\beta := \exp(e)\alpha'$, so that we can invoke the formulas (14) to (16). In particular, we have

$$q_\beta(m) = \frac{1}{2}[l', [l', m]] + \frac{1}{2}[e, [e, m]] + [e, [l', m]]$$

for all $m \in L$. Now set $\alpha := \exp(-\frac{1}{2}[e, l'])\beta$, and notice that $[e, l'] \in L_2$, so α is again an l -exponential automorphism. We get, using the Jacobi identity,

$$\begin{aligned} q_\alpha(m) &= \frac{1}{2}[l', [l', m]] + \frac{1}{2}[e, [e, m]] + [e, [l', m]] + \frac{1}{2}[m, [e, l']] \\ &= \frac{1}{2}[l' + e, [l' + e, m]] = \frac{1}{2}[l, [l, m]], \end{aligned}$$

so α satisfies the required assumptions, proving the existence.

To prove uniqueness, assume that α and β are two l -exponential automorphisms with $q_\alpha = q_\beta$; then in particular, $q_\alpha(x) = q_\beta(x)$. By (v), we have $\beta = \exp(z)\alpha$ for some $z = \lambda y \in L_2$. By expanding $\exp(z)$ – see also Remark 3.14 below – we get $q_\beta(x) = q_\alpha(x) + [z, x]$, but $[z, x] = \lambda[y, x]$, which is 0 only for $\lambda = 0$, and hence $\beta = \alpha$.

Finally, let $\gamma := e_+(-l)e_+(l)$. By (viii), γ is a 0-exponential automorphism, so by (v), we have $\gamma = \exp(z)$ for some $z = \lambda y \in L_2$ again. However,

$$\begin{aligned} q_\gamma(x) &= q_{e_+(-l)}(x) + q_{e_+(l)}(x) + [-l, [l, x]] \\ &= \frac{1}{2}[-l, [-l, x]] + \frac{1}{2}[l, [l, x]] + [-l, [l, x]] = 0, \end{aligned}$$

so $\lambda = 0$, and hence, $\gamma = \text{id}$. □

Remark 3.14.

- (i) If α is an l -exponential automorphism for some $l \in L_1$, then by Theorem 3.13(v), any other l -exponential automorphism β is of the form $\beta = \exp(z)\alpha$ for some $z = \mu y \in L_2$. By expanding $\exp(z)$, we get the explicit formulas

$$\begin{aligned} q_\beta(m) &= q_\alpha(m) + [z, m], \\ n_\beta(m) &= n_\alpha(m) + [z, [l, m]], \\ v_\beta(m) &= v_\alpha(m) + [z, q_\alpha(m)] + g_z(m)z, \end{aligned}$$

for all $m \in L$. In particular, $\alpha = \beta$ if and only if $q_\alpha(x) = q_\beta(x)$. (See also the second to last paragraph of the proof of Theorem 3.13(ix).)

- (ii) If α is an l -exponential automorphism and β is a l' -exponential automorphism, then we get the following explicit formulas for the $(l+l')$ -exponential automorphism $\gamma := \alpha\beta$ from Theorem 3.13(viii):

$$\begin{aligned} q_\gamma(m) &= q_\alpha(m) + [l, [l', m]] + q_\beta(m), \\ n_\gamma(m) &= n_\alpha(m) + q_\alpha([l', m]) + [l, q_\beta(m)] + n_\beta(m), \\ v_\gamma(m) &= v_\alpha(m) + n_\alpha([l', m]) + q_\alpha(q_\beta(m)) + [l, n_\beta(m)] + v_\beta(m), \end{aligned}$$

for all $m \in L$.

We now turn our attention to the case where the extremal geometry does not necessarily contain lines.

Definition 3.15. Let L be a Lie algebra over k with extremal elements x and y with $g(x, y) = 1$, with corresponding 5-grading as in Proposition 2.4. Define

$$E_+(x, y) = \{\alpha \in \text{Aut}(L) \mid \alpha \text{ is } l\text{-exponential for some } l \in L_1\}.$$

By Theorem 3.13(v), this definition is consistent with our earlier definition when the extremal geometry has lines. We call L *algebraic* (with respect to this 5-grading) if for each $l \in L_1$, there exists an l -exponential automorphism $\alpha \in \text{Aut}(L)$.

Now we can extend the results of Theorem 3.13 to a larger class of Lie algebras, using a Galois descent argument.

Theorem 3.16. Let L be a simple Lie algebra. Assume that for some Galois extension k'/k with $k' \neq \mathbb{F}_2$, $L_{k'} := L \otimes_k k'$ is a simple Lie algebra generated by its pure extremal elements and has $\mathcal{F}(L_{k'}) \neq \emptyset$.

Consider a hyperbolic pair $x, y \in E$ and let $L = L_{-2} \oplus L_{-1} \oplus L_0 \oplus L_1 \oplus L_2$ be the associated 5-grading. Then L is algebraic with respect to this grading.

Proof. Consider $l \in L_1$ arbitrary. To simplify the notation, we will identify each $m \in L$ with $m \otimes 1 \in L \otimes k'$.

By Theorem 3.13(v), we find an l -exponential automorphism α of $L \otimes k'$. Since $[x, y] \neq 0$, we can find a basis $\mathcal{B}$ for L_0 containing $[x, y]$. Now, by (10), we have $q_\alpha(x) \in L_0 \otimes k'$, so we can write

$$q_\alpha(x) = [x, y] \otimes \lambda + b_1 \otimes \lambda_1 + \cdots + b_n \otimes \lambda_n$$

with $b_1, \dots, b_n \in \mathcal{B} \setminus \{[x, y]\}$ and $\lambda, \lambda_1, \dots, \lambda_n \in k'$.

Let $\beta := \exp(\lambda y)\alpha$ and notice that β is again an l -exponential automorphism of $L \otimes k'$. By Remark 3.14, we have $q_\beta = q_\alpha + \text{ad}_{\lambda y}$, and hence,

$$q_\beta(x) = q_\alpha(x) + [y, x] \otimes \lambda = b_1 \otimes \lambda_1 + \cdots + b_n \otimes \lambda_n.$$

Now each $\sigma \in \text{Gal}(k'/k)$ acts on the Lie algebra $L \otimes k'$ by sending each $m \otimes \lambda$ to $m \otimes \lambda^\sigma$. Since $\sigma(l) = l$, we have, for each $m' \in L \otimes k'$, that

$$\begin{aligned} (\sigma \circ \beta \circ \sigma^{-1})(m') &= m' + [l, m'] + (\sigma \circ q_\beta \circ \sigma^{-1})(m') \\ &\quad + (\sigma \circ n_{\beta l} \circ \sigma^{-1})(m') + (\sigma \circ v_{\beta l} \circ \sigma^{-1})(m'), \end{aligned}$$

so $\gamma := \sigma \circ \beta \circ \sigma^{-1}$ is again an l -exponential automorphism of $L \otimes k'$, with corresponding maps $q_\gamma = \sigma \circ q_\beta \circ \sigma^{-1}$, $n_\gamma = \sigma \circ n_\beta \circ \sigma^{-1}$ and $v_\gamma = \sigma \circ v_\beta \circ \sigma^{-1}$. Hence, Theorem 3.13(v) implies that there exists $\mu \in k'$ such that $\gamma = \exp(\mu y)\beta$, and by Remark 3.14 again, we have $q_\gamma = q_\beta + \text{ad}_{\mu y}$. In particular,

$$\begin{aligned} b_1 \otimes \lambda_1^\sigma + \cdots + b_n \otimes \lambda_n^\sigma &= q_\gamma(x) = [y, x] \otimes \mu + q_\beta(x) \\ &= [y, x] \otimes \mu + b_1 \otimes \lambda_1 + \cdots + b_n \otimes \lambda_n. \end{aligned}$$

Since $b_1, \dots, b_n$ and $[x, y]$ are linearly independent by construction, this implies $\mu = 0$, and thus, $\gamma = \beta$. Since $\sigma \in \text{Gal}(k'/k)$ was arbitrary, we get $\beta = \sigma \circ \beta \circ \sigma^{-1}$ for all $\sigma \in \text{Gal}(k'/k)$, and therefore,

$$q_\beta = \sigma \circ q_\beta \circ \sigma^{-1}, \quad n_\beta = \sigma \circ n_\beta \circ \sigma^{-1}, \quad v_\beta = \sigma \circ v_\beta \circ \sigma^{-1},$$

for all $\sigma \in \text{Gal}(k'/k)$. Since k'/k is a Galois extension and thus $\text{Fix}(\text{Gal}(k'/k)) = k$, this implies that the maps q_β, n_β and v_β stabilize L , proving that the restriction of β to L is an automorphism of L . Since β is an l -exponential automorphism, we are done. $\square$

Remark 3.17. Under the assumptions of Theorem 3.16, many of the conclusions of Theorem 3.13 now remain valid, simply by extending the scalars to k' . In particular, we are allowed to use Theorem 3.13(v) and (vii) to (ix) in this more general setting. (However, notice that Theorem 3.13(iv) and (vi) do not necessarily make sense over k because $E \cap L_1$ might be empty.)

Assumption 3.18. For the rest of this section, we assume that L is a simple Lie algebra over k with extremal elements x and y with $g(x, y) = 1$ and that k'/k is a Galois extension such that $L \otimes k'$ is a simple Lie algebra generated by its pure extremal elements, with $\mathcal{F}(L \otimes k') \neq \emptyset$ and $|k'| \geq 3$.

In the next two lemmas, we determine how l -exponential automorphisms commute, extending Theorem 3.13(i).

Lemma 3.19. *Let $l, l' \in L_1$ and let α and β be an l -exponential and l' -exponential automorphism of L , respectively. Then $\alpha\beta = \exp([l, l'])\beta\alpha$.*

Proof. Notice that by Theorem 3.13(viii), both $\alpha\beta$ and $\exp([l, l'])\beta\alpha$ are $(l + l')$ -exponential automorphisms. By the uniqueness statement of Remark 3.14, it remains to show that $q_{\alpha\beta}(x) = q_{\exp([l, l'])\beta\alpha}(x)$. By Remark 3.14(ii), we have

$$q_{\alpha\beta}(x) = q_\alpha(x) + [l, [l', x]] + q_\beta(x),$$

and by Remark 3.14(i), we have

$$q_{\exp([l, l'])\beta\alpha}(x) = q_\beta(x) + [l', [l, x]] + q_\alpha(x) + [[l, l'], x],$$

so by the Jacobi identity, these expressions are indeed equal. $\square$

If $\text{char}(k) \neq 2$, we can be more precise.

Lemma 3.20. *Assume $\text{char}(k) \neq 2$ and let $l, l' \in L_1$. Then*

$$e_+(l)e_+(l') = \exp\left(\frac{1}{2}[l, l']\right)e_+(l + l').$$

Proof. Let $\alpha := e_+(l)$, $\beta := e_+(l')$, $\gamma := e_+(l + l')$ and $z := \frac{1}{2}[l, l']$. Exactly as in the proof of Lemma 3.19, we only have to show that $q_{\alpha\beta}(x) = q_{\exp(z)\gamma}(x)$. Using Remark 3.14, we get

$$q_{\alpha\beta}(x) = q_\alpha(x) + q_\beta(x) + [l, [l', x]] = \frac{1}{2}[l, [l, x]] + \frac{1}{2}[l', [l', x]] + [l, [l', x]],$$

and

$$q_{\exp(z)\gamma}(x) = q_\gamma(x) + \frac{1}{2}[[l, l'], x] = \frac{1}{2}[l + l', [l + l', x]] + \frac{1}{2}[[l, l'], x],$$

and again, the equality of these two expressions follows from the Jacobi identity. $\square$

If $\text{char}(k) \neq 2, 3$, we now get an explicit description of the maps $n_{e_+(l)}$ and $v_{e_+(l)}$. In particular, we recover in this case (and under our running Assumption 3.18) the fact that the Lie algebra is algebraic with respect to the given 5-grading, as defined in Definition 1.10.

Corollary 3.21. Assume that $\text{char}(k) \neq 2$. Let $l \in L_1$ and let $\alpha := e_+(l)$. Then

$$6n_\alpha(m) = [l, [l, [l, m]]] \quad \text{and} \quad 24v_\alpha(m) = [l, [l, [l, [l, m]]]]$$

for all $m \in L$.

Proof. Let $\beta := \alpha_2$ be as in Theorem 3.13(vii) and let $\gamma := e_+(2l)$ as in Theorem 3.13(ix). Then $q_\beta = 2^2 q_\alpha$ while $q_\gamma(m) = \frac{1}{2}[2l, [2l, m]] = 4q_\alpha(m)$ for all $m \in L$, so by the uniqueness part of Theorem 3.13(ix), we have $\beta = \gamma$. However, we have $e_+(2l) = e_+(l + l) = e_+(l)^2$ by Lemma 3.20, so $\alpha_2 = \alpha^2$.

By Theorem 3.13(vii), we have $n_{\alpha_2} = 8n_\alpha$, while by Remark 3.14(ii), we have

$$n_{\alpha^2}(m) = 2n_\alpha(m) + q_\alpha([l, m]) + [l, q_\alpha(m)]$$

for all $m \in L$. Since $q_\alpha(m) = \frac{1}{2}[l, [l, m]]$, we conclude that $6n_\alpha(m) = [l, [l, [l, m]]]$ for all $m \in L$.

Similarly, we let $\delta := e_+(-l)e_+(l)$, and we use the fact that $\alpha_{-1} = \alpha^{-1}$. On the one hand, $\delta = \text{id}$, so $v_\delta = 0$, while on the other hand, by Remark 3.14(ii) again,

$$v_\delta(m) = 2v_\alpha(m) - n_\alpha([l, m]) + q_\alpha(q_\alpha(m)) - [l, n_\alpha(m)]$$

for all $m \in L$. Multiplying by 12, we get $24v_\alpha(m) = [l, [l, [l, [l, m]]]]$, as required. $\square$

Our final result of this section shows that $E_+(x, y)$ acts sharply transitively on the set of all extremal points with nonzero L_{-2} -component.

Proposition 3.22. Let L be a simple Lie algebra. Assume that for some Galois extension k'/k with $k' \neq \mathbb{F}_2$, $L_{k'} := L \otimes_k k'$ is a simple Lie algebra generated by its pure extremal elements and has $F(L_{k'}) \neq \emptyset$.

Consider a hyperbolic pair $x, y \in E$ with $g(x, y) = 1$ and let $L = L_{-2} \oplus L_{-1} \oplus L_0 \oplus L_1 \oplus L_2$ be the associated 5-grading.

Then every extremal element with L_{-2} -component equal to x can be written as $\varphi(x)$ for a unique $\varphi \in E_+(x, y)$.

Proof. Let e be an extremal element with L_{-2} -component equal to x . We first show the existence of an automorphism $\varphi \in E_+(x, y)$ with $\varphi(x) = e$.

Let e_{-1} be the L_{-1} -component of e . By Proposition 2.4(iii), there is a (unique) $a \in L_1$ such that $[a, x] = e_{-1}$. By Theorem 3.16, there exists an a -exponential automorphism $\alpha \in E_+(x, y)$. Then $\alpha^{-1}(e) = x + l_0 + l_1 + l_2$, with $l_i \in L_i$. By Lemma 3.10, there exists $\lambda \in k$ such that $\exp(\lambda y)(x) = \alpha^{-1}(e)$. This shows that $\varphi := \alpha \exp(\lambda y) \in E_+(x, y)$ maps x to e , as required.

To show uniqueness, assume that also $\varphi' \in E_+(x, y)$ satisfies $\varphi'(x) = e = \varphi(x)$. In particular, φ and φ' are l -exponential automorphisms for the same $l \in L_1$. This implies that also $\varphi(m) = m + [l, m] = \varphi'(m)$ for all $m \in L_1$. Since L is generated by $\{x\} \cup L_1$, we conclude that $\varphi = \varphi'$. $\square$

Remark 3.23. Note that we did not make any assumptions on the dimension of the Lie algebra L in this section. In particular, the results also hold for infinite-dimensional Lie algebras.

4. Extremal geometry with lines – recovering a cubic norm structure

In this section, we prove that if L is a simple Lie algebra over a field k with $|k| \geq 4$, which is generated by its pure extremal elements and such that its extremal geometry contains lines, then L_1 can be decomposed as $k \oplus J \oplus J' \oplus k$, for a certain ‘twin cubic norm structure’ (J, J') . If the norm of this twin cubic norm structure is not the zero map, then $J' \cong J$ and J gets the structure of a genuine cubic norm structure (depending on the choice of a ‘base point’). At the end of this section, we also sketch how the Lie algebra L can be completely reconstructed from this cubic norm structure.

Assumption 4.1. Throughout the whole section, we assume that L is a simple Lie algebra defined over a field k with $|k| \geq 4$, such that L is generated by its pure extremal elements and such that the extremal geometry contains lines.

We fix a pair of extremal elements $x, y \in E$ with $g(x, y) = 1$, and we consider the corresponding 5-grading $L = L_{-2} \oplus L_{-1} \oplus L_0 \oplus L_1 \oplus L_2$ as in Proposition 2.4.

Notation 4.2. Since the extremal geometry contains lines, Proposition 2.16 implies that there exist extremal elements c and d contained in L_{-1} such that $[c, d] = x$. Set $p := [y, c]$ and $q := -[y, d]$, so $p, q \in L_1$. By Proposition 2.16 again, we see that $[x, y] = [c + d, p + q] = [c, q] + [d, p]$. (Notice that $[c, p] = 0$ and $[d, q] = 0$.) Observe that both (c, d) and (p, q) are special pairs, with $[c, d] = x$ and $[q, p] = y$.

We start with the following observation, needed to obtain a second grading on the Lie algebra which we will use often in this section.

Lemma 4.3. *We have $g(p, d) = 1$ and $g(q, c) = 1$.*

Proof. We have $g(p, d) = g([y, c], d) = g(y, [c, d]) = g(y, x) = 1$ and $g(q, c) = g(-[y, d], c) = -g(y, [d, c]) = g(y, x) = 1$. $\square$

We are now ready to define a subspace J which will turn out to have the structure of a cubic norm structure. Using the grading related to $(c, q) \in E_2$, we are able to describe a decomposition of L_{-1} into 4 parts.

Notation 4.4. We denote the 5-grading of L obtained by considering the pair $(c, q) \in E_2$ in Proposition 2.4 by

$$L = L'_{-2} \oplus L'_{-1} \oplus L'_0 \oplus L'_1 \oplus L'_2, \quad (17)$$

where $L'_{-2} = \langle c \rangle$ and $L'_2 = \langle q \rangle$. We will occasionally also need a third grading arising from the hyperbolic pair $(d, p) \in E_2$, denoted by

$$L = L''_{-2} \oplus L''_{-1} \oplus L''_0 \oplus L''_1 \oplus L''_2, \quad (18)$$

where $L''_{-2} = \langle d \rangle$ and $L''_2 = \langle p \rangle$.

We set

$$J = L_{-1} \cap L'_{-1}, \quad J' = L_{-1} \cap L'_0.$$

Lemma 4.5. *We have*

$$\begin{aligned} x &\in L_{-2} \cap L'_{-1} \cap L''_{-1}, & y &\in L_2 \cap L'_1 \cap L''_1, \\ c &\in L_{-1} \cap L'_{-2} \cap L''_1, & d &\in L_{-1} \cap L'_1 \cap L''_{-2}, \\ p &\in L_1 \cap L'_{-1} \cap L''_2, & q &\in L_1 \cap L'_2 \cap L''_{-1}. \end{aligned}$$

Proof. By our setup in Notation 4.2, the extremal points $\langle x \rangle, \langle c \rangle, \langle p \rangle, \langle y \rangle, \langle q \rangle, \langle d \rangle, \langle x \rangle$ form an ordinary hexagon in the extremal geometry, where all pairs at distance two are special. Now all containments follow either directly from the grading (for the -2 - and 2 -components), or they follow from Lemma 2.14 (for the -1 - and 1 -components). $\square$

We now obtain a decomposition of L_{-1} into 4 parts, by intersecting with the L'_i -grading (17) arising from the hyperbolic pair (c, q) .

Proposition 4.6. *We have a decomposition*

$$L_{-1} = \langle c \rangle \oplus J \oplus J' \oplus \langle d \rangle.$$

Moreover, $L_{-1} \cap L'_1 = \langle d \rangle$.

Proof. Since $c \in L_{-1}$, we have $g_c(L_{-1}) = 0$ by Lemma 2.15. However, we also have $c \in L'_{-2}$, so we get $L_{-1} \leq L'_{-1}$ by Proposition 2.4(iv).

Now let $l \in L_{-1}$ be arbitrary, and write $l = l'_{-2} + l'_{-1} + l'_0 + l'_1$ with each $l'_i \in L'_i$. Since $L'_{-2} = \langle c \rangle \leq L_{-1}$, we may assume $l'_{-2} = 0$.

Now observe that by Proposition 2.4(ii), $[[c, q], l] = -l'_{-1} + l'_1$. However, $[c, q] \in L_0$, so $[[c, q], l] \in L_{-1}$, and we deduce that

$$-l'_{-1} + l'_1 \in L_{-1}. \quad (19)$$

Next, let $\lambda \in k$ be any nonzero scalar and consider the automorphism $\varphi_\lambda \in \text{Aut}(L)$ defined in Lemma 2.10, but with respect to the 5-grading (17). Since $x \in L'_{-1}$ and $y \in L'_1$, we have $\varphi_\lambda(x) = \lambda^{-1}x$ and $\varphi_\lambda(y) = \lambda y$. Then Lemma 2.12 implies $\varphi_\lambda(L_{-1}) = L_{-1}$. In particular, $\varphi_\lambda(l) = \lambda^{-1}l'_{-1} + l'_0 + \lambda l'_1$ is contained in L_{-1} , so subtracting $l \in L_{-1}$, we see that

$$(\lambda^{-1} - 1)l'_{-1} + (\lambda - 1)l'_1 \in L_{-1}$$

for all nonzero $\lambda \in k$. Combining this with (19), we get $(\lambda^{-1} + \lambda - 2)l'_{-1} \in L_{-1}$ for all nonzero $\lambda \in k$. If $\text{char}(k) \neq 2$, we can take $\lambda = -1$ to get $l'_{-1} \in L_{-1}$, while if $\text{char}(k) = 2$, then $|k| > 2$ implies that we can find λ such that $\lambda + \lambda^{-1} \neq 0$. In both cases, we conclude that $l'_{-1} \in L_{-1}$, and then by (19) again, also $l'_1 \in L_{-1}$. Since $l = l'_{-1} + l'_0 + l'_1$, also $l'_0 \in L_{-1}$. Hence,

$$L_{-1} = \underbrace{(L_{-1} \cap L'_{-2})}_{\langle c \rangle} \oplus \underbrace{(L_{-1} \cap L'_{-1})}_J \oplus \underbrace{(L_{-1} \cap L'_0)}_{J'} \oplus (L_{-1} \cap L'_1).$$

Finally, let $l'_1 \in L_{-1} \cap L'_1$ be arbitrary. By Proposition 2.4(iii), we get $[q, [c, l'_1]] = -l'_1$. Since $[c, l'_1] \in L_{-2} = \langle x \rangle$, this implies that l'_1 is a multiple of $[q, x] = -d$. Hence, $L_{-1} \cap L'_1 = \langle d \rangle$. $\square$

Using this decomposition, we obtain some more information on the Lie bracket.

Corollary 4.7. *We have a decomposition*

$$L_1 = \langle p \rangle \oplus [y, J] \oplus [y, J'] \oplus \langle q \rangle.$$

Moreover,

$$\langle p \rangle = L_1 \cap L'_{-1}, \quad [y, J] = L_1 \cap L'_0, \quad [y, J'] = L_1 \cap L'_1; \quad (20)$$

$$[J, J] = 0, \quad [J, c] = [J, d] = 0, \quad [J, p] = 0; \quad (21)$$

$$[J', J'] = 0, \quad [J', c] = [J', d] = 0, \quad [J', q] = 0; \quad (22)$$

$$[J, [J, q]] \leq J', \quad [J', [J', p]] \leq J. \quad (23)$$

Proof. By Proposition 2.4(iii), the map $\text{ad}_y: L_{-1} \rightarrow L_1$ is an isomorphism of vector spaces, so the decomposition is clear. The identities (20) now follow from this new decomposition.

Next, by combining both gradings, we have

$$\begin{aligned} [J, J] &\leq L_{-2} \cap L'_{-2}, & [J, c] &\leq L_{-2} \cap L'_{-3}, \\ [J, d] &\leq L_{-2} \cap L'_0, & [J, p] &\leq L_0 \cap L'_{-2}, \end{aligned}$$

but since $L_{-2} = \langle x \rangle \leq L'_{-1}$ and $L'_{-2} = \langle c \rangle \leq L_{-1}$, all four intersections are trivial, proving (21). The proof of (22) is similar.

Finally, $[J, [J, q]] \leq L_{-1-1+1} \cap L'_{-1-1+2} = L_{-1} \cap L'_0 = J'$, and similarly, we have $[J', [J', p]] \leq L_{-1} \cap L'_{-1} = J$, showing (23). $\square$

Lemma 4.8. *We have*

$$J = \{l \in L_{-1} \mid [c, l] = [d, l] = [p, l] = 0\}, \quad (24)$$

$$J' = \{l \in L_{-1} \mid [c, l] = [d, l] = [q, l] = 0\}. \quad (25)$$

Proof. By (21), it is clear that J is contained in the set on the right-hand side of (24).

Conversely, let $l \in L_{-1}$ such that $[c, l] = [d, l] = [p, l] = 0$ and use Proposition 4.6 to write $l = \lambda c + j + j' + \mu d$, with $\lambda, \mu \in k$, $j \in J$ and $j' \in J'$. Since $[c, d] \neq 0$, it already follows from $[c, l] = 0$ that $\mu = 0$ and from $[d, l] = 0$ that $\lambda = 0$, so we already have $l = j + j'$, and it remains to show that $j' = 0$.

Since $[p, l] = 0$ and $[p, j] = 0$, we have $[p, j'] = 0$. Recall from Notation 4.2 that $[x, y] = [c, q] + [d, p]$ and $[d, j'] = 0$. Since $j' \in L_{-1} \cap L'_0$, we can use Proposition 2.4(ii) twice to get

$$-j' = [[x, y], j'] = [[c, q], j'] + [[d, p], j'] = [d, [p, j']] = 0,$$

as claimed. The proof of (25) is completely similar. $\square$

The subspace L_0 also has a decomposition, but this time it is a decomposition into 3 parts.

Proposition 4.9. *We have a decomposition*

$$L_0 = (L_0 \cap L'_{-1}) \oplus (L_0 \cap L'_0) \oplus (L_0 \cap L'_1). \quad (26)$$

Moreover,

$$L_0 \cap L'_1 = [q, J] \quad \text{and} \quad L_0 \cap L'_{-1} = [p, J']. \quad (27)$$

Proof. By Lemma 2.15 applied to $c \in L_{-1}$, we get $g_c(L_0) = 0$. By Proposition 2.4(iv), this implies $L_0 \leq L'_{-2} \oplus L'_{-1} \oplus L'_0 \oplus L'_1$. Similarly, $g_q(L_0) = 0$ implies $L_0 \leq L'_{-1} \oplus L'_0 \oplus L'_1 \oplus L'_2$. Hence, $L_0 \leq L'_{-1} \oplus L'_0 \oplus L'_1$.

We can now apply the same technique as in the proof of Proposition 4.6. So let $l \in L_0$ be arbitrary and write $l = l'_{-1} + l'_0 + l'_1$ with each $l'_i \in L'_i$. Again, $[[c, q], l] = -l'_{-1} + l'_1 \in L_0$, and using the automorphisms φ_λ , we get $\lambda_{-1}l'_{-1} + l'_0 + \lambda l'_1 \in L_0$ for all $\lambda \in k^\times$, yielding $l'_i \in L_0$ for all three values of i , exactly as in the proof of Proposition 4.6. This proves (26).

To show (27), notice that the inclusions $[q, J] \leq L_0 \cap L'_1$ and $[p, J'] \leq L_0 \cap L'_{-1}$ are clear from the gradings. Conversely, let $l_0 \in L_0 \cap L'_1$ be arbitrary and let $l := -[c, l_0]$. Then $l \in L_{-1} \cap L'_{-1} = J$. By Proposition 2.4(iii), we have $l_0 = [q, l]$, so indeed, $l_0 \in [q, J]$. Similarly, let $l_0 \in L_0 \cap L'_{-1}$ be arbitrary and let $l := -[d, l_0]$. Then $l \in L_{-1} \cap L'_0 = J'$. By Proposition 2.4(iii) again, we have $l_0 = [p, l]$, so indeed, $l_0 \in [p, J']$. $\square$

In the next lemma, we deduce some connections between the two gradings (17) and (18).

Lemma 4.10. *We have*

$$L_{-1} \cap L''_{-1} = J', \quad L_{-1} \cap L''_0 = J, \quad L_{-1} \cap L''_1 = \langle c \rangle, \quad (28)$$

$$L_0 \cap L''_{-1} = [q, J], \quad L_0 \cap L''_0 = L_0 \cap L'_0, \quad L_0 \cap L''_1 = [p, J']. \quad (29)$$

Proof. Notice that in the setup in Notation 4.2, we can replace c with d , d with $-c$, p with $-q$ and q with p . In the resulting decompositions introduced in Notation 4.4, this interchanges the two gradings (17)

and (18). If we subsequently set $K := L_{-1} \cap L''_{-1}$ and $K' := L_{-1} \cap L''_0$, then it follows from Lemma 4.8 that $K = J'$ and $K' = J$. By Proposition 4.6 applied on this new setup, we get

$$L_{-1} = \langle d \rangle \oplus K \oplus K' \oplus \langle c \rangle$$

with $\langle c \rangle = L_{-1} \cap L''_1$. This shows (28).

By (27) applied on our new setup, we get $L_0 \cap L''_1 = [p, J']$ and $L_0 \cap L''_{-1} = [q, J]$. Finally, consider $l \in L_0 \cap L''_0$. By Proposition 2.4(ii), we have $[[x, y], l] = 0$ and $[[d, p], l] = 0$. Recall from Notation 4.2 that $[x, y] = [c, q] + [d, p]$, so we get

$$0 = [[x, y], l] = [[c, q], l] + [[d, p], l] = [[c, q], l].$$

Hence, l is contained in the 0-eigenspace of $\text{ad}_{[c, q]}$, which is L'_0 by Proposition 2.4(ii) again. Hence, $L_0 \cap L''_0 \leq L_0 \cap L'_0$, and $L_0 \cap L'_0 \leq L_0 \cap L''_0$ follows similarly. This shows (29). $\square$

We have summarized all the information about the intersections of the different gradings in Figure 1.

Now that we have a complete understanding of how the different gradings intersect, we are ready to invoke our results from Section 3. The next result on the uniqueness of certain extremal elements will play a key role in defining the maps involved in the cubic norm structure.

Proposition 4.11.

(i) For every $a \in J$, there is a unique extremal element of the form

$$a_{-2} + a_{-1} + [a, q] + q,$$

with $a_{-2} \in L_{-2}$ and $a_{-1} \in J'$. This element is equal to $\alpha(q)$ for a unique a -exponential automorphism $\alpha \in E_-(x, y)$.

(ii) For every $b \in J'$, there is a unique extremal element of the form

$$b_{-2} + b_{-1} + [b, p] + p,$$

with $b_{-2} \in L_{-2}$ and $b_{-1} \in J'$. This element is equal to $\beta(q)$ for a unique b -exponential automorphism $\beta \in E_-(x, y)$.

Proof. We show (i); the proof of (ii) is completely similar. By Theorem 3.13(iv), there exists an a -exponential automorphism α , so we have a corresponding extremal element

$$\alpha(q) = a_{-2} + a_{-1} + [a, q] + q,$$

with $a_{-1} \in L_{-1}$ and $a_{-2} \in L_{-2}$. Notice that $\alpha(d) = d + [a, d] = d$ because $[J, d] = 0$ by (21). By Proposition 2.4(iv), this implies that α preserves the filtration $L''_{\leq i}$, so in particular, $\alpha(q) \in L''_{\leq -1}$. Hence, $a_{-1} \in L_{-1} \cap L''_{\leq -1} = \langle d \rangle \oplus J'$ (see Figure 1).

Now, since $[x, q] = [x, -[y, d]] = -[[x, y], d] = d$, there is a (unique) $\lambda \in k$ such that $\exp(\lambda x)(\alpha(q))$ has its L_{-1} -component in J' . By replacing α with $\exp(\lambda x)\alpha$, we find the required a -exponential automorphism and the required extremal element.

We now show uniqueness, so suppose that $e = b_{-2} + b_{-1} + [a, q] + q$ is an extremal element such that $b_{-2} \in L_{-2}$ and $b_{-1} \in J'$. Let $\beta := \alpha_{-1}$ as defined in Theorem 3.13(vii), so β is a $(-a)$ -exponential automorphism. In particular, $f := \beta(e)$ has trivial L_0 -component. Moreover, the L_{-1} -component of f is equal to $b_{-1} + a_{-1} - [a, [a, q]]$. By assumption, $a_{-1}, b_{-1} \in J'$ and by (23), also $[a, [a, q]] \in J'$, so the L_{-1} -component of f is contained in $J' \leq L'_0$. Hence,

$$f = \mu x + c_{-1} + q, \quad \text{with } \mu \in k \text{ and } c_{-1} \in J'$$

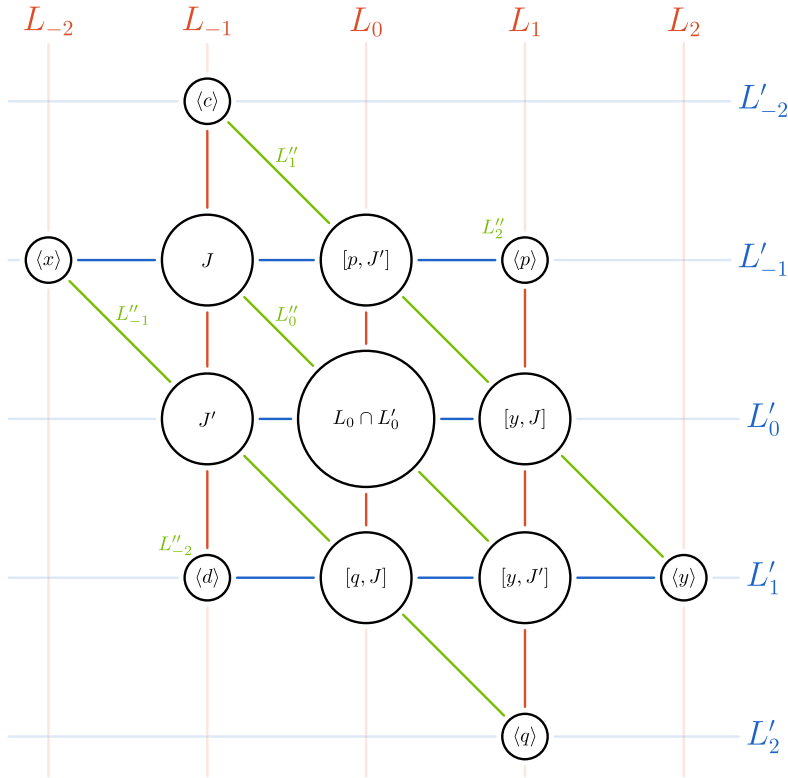


Figure 1. Intersecting gradings for cubic norm structures.

is an extremal element. By Lemma 3.10 applied to the grading (17), we get $\mu = 0$ and $f = \exp(\lambda c)(q)$ for certain $\lambda \in k$. Considering the L_0 -components of these extremal elements yields $0 = \lambda[c, q]$ and thus, $\lambda = 0$ and $q = f = \beta(e)$.

Next, since both β^{-1} and α are a -exponential automorphisms, Theorem 3.13(v) implies $\beta^{-1} = \exp(\nu x)\alpha$ for some $\nu \in k$, so in particular, $e = \exp(\nu x)\alpha(q)$. However, $[x, q] = d \notin J'$, so because the L_{-1} -components of both e and $\alpha(q)$ are contained in J' , this can only happen if $\lambda = 0$, and therefore $e = \alpha(q)$, proving the required uniqueness of the extremal element.

Notice that the uniqueness of the corresponding automorphism $\alpha \in E_-(x, y)$ is now clear because any other $\alpha' \in E_-(x, y)$ with $\alpha'(q) = \alpha(q)$ would again be a -exponential (for the same a) and therefore differ from α by an element of the form $\exp(\lambda x)$, forcing $\lambda = 0$ as before. $\square$

As promised, we can now use Proposition 4.11 to define some maps.

Definition 4.12. We define maps $N: J \rightarrow k$, $\sharp: J \rightarrow J'$, $T: J \times J' \rightarrow k$ and $\times: J \times J \rightarrow J'$ as follows:

$$\begin{aligned} N(a)x &= a_{-2}, \text{ for any } a \in J \text{ with } a_{-2} \text{ as in Proposition 4.11(i);} \\ a^\sharp &= a_{-1}, \text{ for any } a \in J \text{ with } a_{-1} \text{ as in Proposition 4.11(i);} \\ T(a, b)x &= [a, b], \text{ for any } a \in J, b \in J'; \\ a \times b &= [a, [b, q]], \text{ for any } a, b \in J. \end{aligned}$$

Note that T is a bilinear map and that $\times$ is a symmetric bilinear map because $[J, J] = 0$. Similarly, we define maps $N': J' \rightarrow k$, $\sharp': J' \rightarrow J$, $T': J' \times J \rightarrow k$ and $\times': J' \times J' \rightarrow J$:

$$\begin{aligned} N'(b)x &= b_{-2}, \text{ for any } b \in J' \text{ with } b_{-2} \text{ as in Proposition 4.11(ii);} \\ b^{\#'} &= -b_{-1}, \text{ for any } b \in J' \text{ with } b_{-1} \text{ as in Proposition 4.11(ii);} \\ T'(b, a)x &= [a, b], \text{ for any } a \in J, b \in J'; \\ a \times' b &= -[a, [b, p]], \text{ for any } a, b \in J'. \end{aligned}$$

(Notice the two minus signs. Observe that $T'(b, a) = T(a, b)$ for all $a \in J, b \in J'$.)

Using Proposition 4.11, we can deduce some identities.

Lemma 4.13. *Let $a, b \in J$ and $\lambda \in k$. Then*

$$(\lambda a)^{\#} = \lambda^2 a^{\#}; \quad (30)$$

$$N(\lambda a) = \lambda^3 N(a); \quad (31)$$

$$(a + b)^{\#} = a^{\#} + a \times b + b^{\#}; \quad (32)$$

$$N(a + b) = N(a) + T(b, a^{\#}) + T(a, b^{\#}) + N(b); \quad (33)$$

$$(a^{\#})^{\#'} = N(a)a; \quad (34)$$

$$N'(a^{\#}) = N(a)^2. \quad (35)$$

Similar identities hold for $a, b \in J'$.

Proof. Equations (30) and (31) follow immediately from Proposition 4.11 and Theorem 3.13(vii).

Now let $a, b \in J$ and consider the corresponding automorphisms α for a and β for b as in Proposition 4.11. Then $\alpha(q) = N(a)x + a^{\#} + [a, q] + q$ and $\beta(q) = N(b)x + b^{\#} + [b, q] + q$. The elements $\alpha\beta(q)$ and $\beta\alpha(q)$ are also extremal, and

$$\alpha\beta(q) = (N(b)x + T(a, b^{\#})x + q_{\alpha}([b, q]) + N(a)x) + (b^{\#} + a \times b + a^{\#}) + [b + a, q] + q,$$

$$\beta\alpha(q) = (N(a)x + T(b, a^{\#})x + q_{\beta}([a, q]) + N(b)x) + (a^{\#} + b \times a + b^{\#}) + [a + b, q] + q.$$

Since $a^{\#} + a \times b + b^{\#} \in J'$, we can use the uniqueness in Proposition 4.11 applied on $a + b \in J$ to obtain (32) and

$$\begin{aligned} N(a + b)x &= N(a)x + T(b, a^{\#})x + q_{\beta}([a, q]) + N(b)x \\ &= N(a)x + T(a, b^{\#})x + q_{\alpha}([b, q]) + N(b)x. \end{aligned}$$

In particular, we have

$$T(a, b^{\#})x - q_{\beta}([a, q]) = T(b, a^{\#})x - q_{\alpha}([b, q]). \quad (36)$$

Since $b \in J$ is arbitrary, we can replace it by λb for arbitrary $\lambda \in k$. By Theorem 3.13(vii), (30), and the linearity of T , the left-hand side of (36) is quadratic in λ , while the right-hand side is linear in λ . Since $|k| \geq 3$, this implies that both the left- and the right-hand side of (36) are 0. So we obtain (33).

We now show (34) and (35) simultaneously. Let $a \in J$ be arbitrary and consider the corresponding extremal element $e = N(a)x + a^{\#} + [a, q] + q$. Since $a^{\#} \in J' \leq L'_0$ and $p \in L'_{-1}$, we get $g(p, a^{\#}) = 0$ by Lemma 2.15. Notice that $e \in L''_{-1}$, so $(e, p) \in E_1$ because of the grading (18). It follows that

$$[e, p] = -N(a)c + [a^{\#}, p] + [[a, q], p] + y \in E.$$

Notice that $[[a, q], p] = [a, y]$ by the Jacobi identity, since $[a, p] \in [J, p] = 0$. Applying the automorphism φ from Lemma 2.8, we get

$$x + [x, [a, y]] + \varphi([a^\sharp, p]) - N(a)p \in E.$$

Recall that $\varphi(l_0) = l_0 + [x, [y, l_0]]$ for all $l_0 \in L_0$. However, $[y, [a^\sharp, p]] = 0$ because of the grading (see again Figure 1), so $\varphi([a^\sharp, p]) = [a^\sharp, p]$. Combined with $[x, [a, y]] = a$ by Proposition 2.4(iii), we get

$$x + a + [a^\sharp, p] - N(a)p \in E.$$

First, assume $N(a) \neq 0$. Applying the automorphism φ_λ from Lemma 2.10 with $\lambda = -N(a)^{-1}$, we get

$$N(a)^2 x - N(a)a + [a^\sharp, p] + p \in E.$$

By definition of $\sharp'$, this implies $(a^\sharp)^{\sharp'} = N(a)a$ and $N'(a^\sharp) = N(a)^2$, so we obtain (34) and (35) if $N(a) \neq 0$.

Now assume $N(a) = 0$. If $a^\sharp = 0$, then (34) and (35) are trivially satisfied, so assume $a^\sharp \neq 0$. Then

$$e = a^\sharp + [a, q] + q.$$

In particular, $e \in L_{\geq -1} \setminus L_{\geq 0}$, so by Proposition 2.5, we get $(e, y) \in E_1$ and hence, $[a^\sharp, y] = [e, y] \in E$. It follows that also $a^\sharp = [x, [a^\sharp, y]] \in E$. Since $g(p, a^\sharp) = 0$, it follows that $\exp(a^\sharp)(p) = [a^\sharp, p] + p \in E$, and thus, $(a^\sharp)^{\sharp'} = 0$ and $N'(a^\sharp) = 0$, proving (34) and (35) also in this case. $\square$

Lemma 4.14. *For all $a, b, c \in J$, we have*

$$T(a, a^\sharp) = 3N(a); \quad (37)$$

$$a \times a = 2a^\sharp; \quad (38)$$

$$T(c, a \times b) = T(a, b \times c); \quad (39)$$

$$a^\sharp \times' (a \times b) = N(a)b + T(b, a^\sharp)a; \quad (40)$$

$$a^\sharp \times' b^\sharp = -(a \times b)^\sharp + T(b, a^\sharp)b + T(a, b^\sharp)a. \quad (41)$$

Similar identities hold for $a, b, c \in J'$.

Proof. By (30) to (35), this follows in exactly the same way as in [TW02, (15.16), (15.18)]. (Recall from Assumption 4.1 that we assume $|k| \geq 4$.) $\square$

Remark 4.15. The identities (30) to (35) and (37) to (41) we have shown so far may serve as an axiom system for ‘cubic norm pairs’ (J, J') . Although this seems to be a natural counterpart for cubic norm structures in the context of *Jordan pairs* (rather than Jordan algebras), this notion seems to have appeared only once in the literature in a paper by John Faulkner [Fau01], although the setup of a paired structure also appears already in [Spr62]. (We thank Holger Petersson for pointing this out to us.) A recent contribution to the theory of cubic norm pairs, also related to G_2 -graded Lie algebras, is Michiel Smet’s preprint [Sme25].

At this point, we have, in principle, gathered enough information to *reconstruct* the Lie algebra from our data. Since this is not our main focus, we will only sketch the procedure. We will need the following concrete computations, providing us with information about the Lie bracket with the ‘middle part’ $L_0 \cap L'_0$, which is the most difficult piece to control. (Notice that $[J, [y, J']]$ and $[J', [y, J]]$ belong to this middle part.)

Lemma 4.16. *Let $a, b, e \in J$ and $a', b', e' \in J'$ be arbitrary. Then*

$$[a, [b, [y, e]]] = T(a, b \times e)c; \quad (42)$$

$$[a', [b', [y, e']]] = T(b' \times' e', a')d; \quad (43)$$

$$[a, [b', [y, e']]] = -a \times (b' \times' e'); \quad (44)$$

$$[a', [b, [y, e]]] = a' \times' (b \times e); \quad (45)$$

$$[a, [b', [y, e]]] = b' \times' (a \times e) - T(a, b')e; \quad (46)$$

$$[a', [b, [y, e']]] = -b \times (a' \times' e') + T(b, a')e; \quad (47)$$

$$[a, [b, [y, e']]] = e' \times' (a \times b) - T(a, e')b - T(b, e')a; \quad (48)$$

$$[a', [b', [y, e]]] = -e \times (a' \times' b') + T(e, a')b' + T(e, b')a'. \quad (49)$$

Proof. By the grading (see Figure 1), we have $[a, [b, [y, e]]] \in L_{-1} \cap L'_{-2} = \langle c \rangle$, so $[a, [b, [y, e]]] = \lambda c$ for some $\lambda \in k$. Since $[c, d] = x$ and $[J, d] = 0$, we get

$$\lambda x = [d, [a, [b, [e, y]]]] = [a, [b, [e, [d, y]]]] = [a, [b, [e, q]]] = [a, b \times e] = T(a, b \times e)x,$$

and hence, (42) holds. The proof of (43) is similar.

Next, since $y = [q, p]$, $[q, e'] = 0$ and $[q, b'] = 0$, we get

$$\begin{aligned} [a, [b', [y, e']]] &= [a, [b', [[q, p], e']]] \\ &= [a, [b', [q, [p, e']]]] \\ &= [a, [q, [b', [p, e']]]] = -a \times (b' \times' e'), \end{aligned}$$

proving (44). The proof of (45) is similar.

Now, by the Jacobi identity and (45), we have

$$[a, [b', [y, e]]] = [b', [a, [y, e]]] + [[a, b'], [y, e]] = b' \times' (a \times e) + [T(a, b')x, [y, e]],$$

so we get (46). The proof of (47) is similar.

Finally, by applying the Jacobi identity twice and using $[a, [b, e']] \in L_{-3} = 0$ and (46), we get

$$\begin{aligned} [a, [b, [y, e']]] &= [y, [a, [b, e']]] + [a, [[b, y], e']] + [[a, y], [b, e']] \\ &= 0 + [a, [e', [y, b]]] + [[a, y], T(b, e')x] \\ &= e' \times' (a \times b) - T(a, e')b - T(b, e')a, \end{aligned}$$

showing (48). The proof of (49) is similar. □

Here is the promised sketch of the reconstruction result.

Corollary 4.17. *The Lie bracket on L can be completely recovered from the maps T , $\times$ and $\times'$ alone.*

Proof. Notice that, by Figure 1, the algebra L has a decomposition into 13 pieces. Now let

$$K_0 := \langle [x, y], [q, c] \rangle \oplus [J, [y, J']],$$

and let K be the subspace of L spanned by K_0 together with the 12 remaining pieces of L . It is not difficult to check (but it requires some case analysis) that K is an ideal of L . Since L is simple, it follows that $K = L$, so in particular, $K_0 = L_0 \cap L'_0$. Now observe that we can identify K_0 with the Lie algebra $\tilde{K}_0$ consisting of the corresponding inner derivations ad_l (for each $l \in K_0$) restricted to the sum

of the 12 remaining pieces; see, for example, [BDMS19, Construction 4.1.2]. (Since L is simple, the corresponding Lie algebra homomorphism $K_0 \rightarrow \tilde{K}_0$ is injective and hence an isomorphism.)

The Lie algebra L can now be reconstructed from six one-dimensional subspaces, three copies of J , three copies of J' , and a copy of $\tilde{K}_0$, assembled exactly as in Figure 1. It is now a routine (but somewhat lengthy) verification, relying on Lemma 4.16, that the Lie bracket between every two of the 13 pieces is completely determined by the maps T , $\times$ and $\times'$. $\square$

So far, we have been able to endow the pair (J, J') with a ‘twin cubic norm structure’ using the various maps between them. (See also Remark 4.15.) When the norm N is not identically zero, we can go one step further, by selecting a ‘base point’ to identify the two parts of this twin structure, which then results in a genuine cubic norm structure.

Lemma 4.18. *If $N \neq 0$, then we can re-choose c and d in Notation 4.2 in such a way that $N(z) = 1$ for some $z \in J$. We call z a base point for N .*

Proof. Let $z \in J$ such that $N(z) \neq 0$. If we replace c by $\tilde{c} = N(z)c$ and d by $\tilde{d} = N(z)^{-1}d$ in Notation 4.2, and correspondingly replace p by $\tilde{p} = N(z)p$ and q by $\tilde{q} = N(z)^{-1}q$, then all conditions in Notation 4.2 remain satisfied. Since we have only replaced these elements by a scalar multiple, this does not affect any of the decompositions.

Now denote the norm obtained from Definition 4.12 with these new choices by $\tilde{N}$. Since $N(z)x + z^\# + [z, q] + q \in E$, we get $x + N(z)^{-1}z^\# + [z, \tilde{q}] + \tilde{q} \in E$, and it follows that indeed $\tilde{N}(z) = 1$. $\square$

Assume for the rest of this section that $N \neq 0$ and that $z \in J$ is a base point for N .

Definition 4.19. We define maps $\sigma: J \rightarrow J'$ and $\sigma': J' \rightarrow J$ by

$$\begin{aligned}\sigma(a) &= T(a, z^\#)z^\# - z \times a, \\ \sigma'(b) &= T(z, b)z - b \times' z^\#, \end{aligned}$$

for all $a \in J$ and all $b \in J'$.

Lemma 4.20. *We have $\sigma' \circ \sigma = \text{id}_J$ and $\sigma \circ \sigma' = \text{id}_{J'}$.*

Proof. Let $a \in J$ be arbitrary. Then by (34) and (37) to (40) and since $N(z) = 1$,

$$\begin{aligned}\sigma'(\sigma(a)) &= T(z, T(a, z^\#)z^\#)z - T(z, z \times a)z - T(a, z^\#)z^\# \times' z^\# + (z \times a) \times' z^\# \\ &= T(a, z^\#)T(z, z^\#)z - T(a, z \times z)z - 2T(a, z^\#)(z^\#)^\# + N(z)a + T(a, z^\#)z \\ &= 3T(a, z^\#)z - 2T(a, z^\#)z - 2T(a, z^\#)z + a + T(a, z^\#)z \\ &= a. \end{aligned}$$

The proof of the second equality is similar. $\square$

Lemma 4.21. *We have $\sigma' \circ \# = \# \circ \sigma$.*

Proof. Let $a \in J$ be arbitrary. On the one hand, using (41), we get

$$\begin{aligned}\sigma'(a^\#) &= T(z, a^\#)z - a^\# \times' z^\# \\ &= T(z, a^\#)z + (a \times z)^\# - T(z, a^\#)z - T(a, z^\#)a \\ &= (a \times z)^\# - T(a, z^\#)a. \end{aligned}$$

On the other hand, using $N(z) = 1$, (30), (32), (34) and (40), we have

$$\begin{aligned}\sigma(a)^{\sharp'} &= (T(a, z^{\sharp})z^{\sharp} - z \times a)^{\sharp'} \\ &= T(a, z^{\sharp})^2(z^{\sharp})^{\sharp'} - T(a, z^{\sharp})z^{\sharp} \times' (z \times a) + (z \times a)^{\sharp'} \\ &= T(a, z^{\sharp})^2 z - T(a, z^{\sharp})a - T(a, z^{\sharp})^2 z + (z \times a)^{\sharp'} \\ &= (a \times z)^{\sharp'} - T(a, z^{\sharp})a.\end{aligned}$$

□

We are now ready to define the cubic norm structure and prove the required defining identities; see Definition 1.11.

Definition 4.22. We define maps $T_J : J \times J \rightarrow k$, $\times_J : J \times J \rightarrow J$ and $\sharp_J : J \rightarrow J$ by setting

$$T_J(a, b) = T(a, \sigma(b)), \quad (50)$$

$$a \times_J b = \sigma'(a \times b) = \sigma(a) \times' \sigma(b), \quad (51)$$

$$a^{\sharp_J} = \sigma'(a^{\sharp}) = \sigma(a)^{\sharp'}, \quad (52)$$

for all $a, b \in J$. (The rightmost equalities in (51) and (52) hold by Lemma 4.21.)

Theorem 4.23. Let L be as in Notation 4.2 and J be as in Notation 4.4. Assume that the map N defined in Definition 4.12 is not identically zero. Then the data $(J, k, N, \sharp_J, T_J, \times_J, z)$ forms a nondegenerate cubic norm structure.

Proof. Recall that by Lemma 1.13, we only have to show that the defining identities (i), (ii), (iv), (v), (vii), (x) and (xi) of Definition 1.11 are satisfied.

First, note that $\times_J$ is symmetric and bilinear by construction. For T_J , we have

$$T_J(a, b) = T(a, T(b, z^{\sharp})z^{\sharp}) - T(a, z \times b) = T(a, z^{\sharp})T(b, z^{\sharp}) - T(z, a \times b)$$

by (39), and hence, T_J is symmetric and bilinear.

Identity (i) follows from the linearity of σ' and (30). Identity (ii) is precisely (31). Identity (iv) follows from the linearity of σ' and (32). Identity (v) follows from Lemma 4.20 and (33). Identity (vii) follows from Lemmas 4.20 and 4.21 and (34). Identity (x) follows from

$$\sigma'(z^{\sharp}) = T(z, z^{\sharp})z - z^{\sharp} \times' z^{\sharp} = 3N(z)z - 2N(z)z = z,$$

using (34), (37) and (38) and $N(z) = 1$. Identity (xi) follows from

$$\begin{aligned}z \times_J a &= \sigma'(z \times a) = T(z, z \times a)z - z^{\sharp} \times' (z \times a) \\ &= T(a, z \times z)z - N(z)a - T(a, z^{\sharp})z \\ &= T(a, z^{\sharp})z - a = T_J(a, z) - a\end{aligned}$$

for all $a \in J$, using (38) to (40), $\sigma(z) = z^{\sharp}$ and $N(z) = 1$. Finally, the cubic norm structure is nondegenerate: if $a \in J$ is such that $T_J(a, J) = 0$, then we have $T(a, J') = 0$ (i.e., $[a, J'] = 0$), but it then follows from the grading that $[a, L_{-1}] = 0$; Lemma 2.11 then implies that $a = 0$. □

Remark 4.24. A different choice of base point would give rise to a different cubic norm structure, but it is not hard to show (and not surprising) that the resulting cubic norm structures are *isotopic*; see Definition 1.15.

Remark 4.25. If the cubic norm structure is *anisotropic* (see Definition 1.12), then the extremal geometry is, in fact, a generalized hexagon. (The key point here is the fact that the extremal geometry is a generalized hexagon if and only if there are no symplectic pairs, a fact that we have exploited already

in the proof of Lemma 3.3. The existence of symplecta turns out to be equivalent to the existence of extremal elements in $L'_{-1} \cap L_{-1} = J$, and such elements always have norm 0.)

With some more effort, we can use the techniques developed in this section to show that this generalized hexagon is isomorphic to the Moufang hexagon obtained from the cubic norm structure as in [TW02, (16.8)], and we can determine the root groups and the commutator relations explicitly in terms of the Lie algebra. Since this result is not so surprising but still requires a substantial amount of work, we omit the details. See [Meu21, §4.4.2].

Remark 4.26. If the map N is identically zero, then also the maps $\sharp$, $\times$, N' , $\sharp'$ and $\times'$ are identically zero. In this case, it can be shown that the extremal geometry is isomorphic to a geometry of the form $\Gamma(V, W^*)$ as defined in [CM21, Example 2.8]. Note that these examples also occur when the Lie algebra is infinite-dimensional. In the finite-dimensional case, these are precisely so-called *root shadow spaces* of type $A_{n, \{1, n\}}$.

5. Extremal geometry with symplectic pairs – recovering a quadrangular algebra

In this section, we continue to assume that L is a simple Lie algebra generated by its pure extremal elements, but we make different assumptions on the existence of certain extremal elements and extremal lines.

Assumption 5.1. We assume that L is a simple Lie algebra, defined over a field k with $|k| > 2$, such that L is generated by its set E of pure extremal elements, and such that

- (i) there exists a Galois extension k'/k of degree at most 2 such that the extremal geometry of $L \otimes k'$ contains lines;
- (ii) there exist symplectic pairs of extremal elements.

This time, we will show that L_{-1} can be decomposed as $V \oplus X \oplus V'$ into 3 parts, where $V' \cong V$, and that (V, X) can be made into a quadrangular algebra. This will require a substantial amount of work, mostly because we are including the case $\text{char}(k) = 2$. Notice, however, that we will never require a case distinction, not even in the computations in our proofs. For convenience, we have included the $\text{char}(k) \neq 2$ description of θ in Proposition 5.30, but we never use this result.

We first point out that L remains simple after base extension to k' .

Lemma 5.2. *The Lie algebra $L_{k'} := L \otimes k'$ is simple.*

Proof. Essentially, this is shown in the proof of [CM21, Theorem 5.7]. However, the statement in *loc. cit.* assumes $\text{char}(k) \neq 2$, and extending it to all characteristics requires some subtle changes in the final step, so we give a complete proof.

Let σ be the unique nontrivial element of $\text{Gal}(k'/k)$. We first recall³ that, by *Speiser's Lemma* (see, for example, [GS17, Lemma 2.3.8]), each σ -invariant subspace W of $L_{k'}$ can be written as $W = V \otimes_k k'$ for some subspace V of L .

Now let I be a nontrivial ideal of $L_{k'}$ and let $a \in I$ with $a \neq 0$. Then, as the extremal elements in E linearly span L , they also span $L_{k'}$ over k' , so we can write

$$a = x_1 \otimes \lambda_1 + \cdots + x_k \otimes \lambda_k,$$

where $x_j \in E$ are linearly independent and each $\lambda_j \in k'^{\times}$. After replacing a with a scalar multiple, we can assume $\lambda_1 + \lambda_1^{\sigma} \neq 0$. Then $a + a^{\sigma}$ is a nontrivial element in $I + I^{\sigma}$ fixed by σ . Therefore, the subspace spanned by the elements in $I + I^{\sigma}$ fixed by σ forms a nontrivial ideal of L , which by simplicity of L equals L . This implies that $I + I^{\sigma} = L_{k'}$.

³This fact is reproven in [CM21, Lemma 6.9] in this specific situation with an ad-hoc argument, but it holds in a much more general setting.

Since both I and I^σ are ideals, $I \cap I^\sigma$ is an ideal which is stabilized by σ . By Speiser's Lemma, $I \cap I^\sigma = V \otimes_k k'$ for some subspace V of L , but then V is an ideal of L . By the simplicity of L , this implies that either $I = L_{k'}$, or $I \cap I^\sigma = 0$. So we may assume the latter, and therefore, $L_{k'} = I \oplus I^\sigma$. In particular, $[I, I^\sigma] \leq I \cap I^\sigma = 0$, and thus,

$$[L_{k'}, L_{k'}] = [I, I] \oplus [I^\sigma, I^\sigma]. \quad (53)$$

Since L is simple, $L = [L, L]$ and hence $L_{k'} = [L_{k'}, L_{k'}]$, so it follows from (53) that $[I, I] = I$.

Now let $x \in E$ be arbitrary. By the Premet identity (2) with $y, z \in I$, we get $g_x([y, z])x \in I$ for all $y, z \in I$. Since $I = [I, I]$, it follows that $g_x(a)x \in I$ for all $a \in I$. It thus follows that $x \in I$ if we can find an $a \in I$ with $g(x, a) \neq 0$. This is indeed always possible: since g is nondegenerate, there exists a $y \in L$ with $g(x, y) \neq 0$, but since $L_{k'} = I + I^\sigma$, we either find $a \in I$ with $g(x, a) \neq 0$ or we find $b \in I^\sigma$ with $g(x, b) \neq 0$. In the latter case, however, we simply apply σ , and we find $g(x, b^\sigma) \neq 0$ with $b^\sigma \in I$.

Since $x \in E$ was arbitrary, we have shown that E is contained in I . However, E generates $L_{k'}$, so we get that $I = L_{k'}$ as required. (In fact, this is now a contradiction to the fact that $I \cap I^\sigma = 0$.) $\square$

Proposition 5.3. *There exist extremal elements x, y, c and d such that $g(x, y) = g(c, d) = 1$ and $(x, c), (c, y), (y, d), (d, x) \in E_0$.*

Proof. Assume first that the extremal geometry of L contains lines. By Assumption 5.1(ii), there exists an $x \in E$ and a symplecton S of the extremal geometry containing $\langle x \rangle$. Recall from Assumption 3.1 that $(\mathcal{E}, \mathcal{E}_2)$ is connected, so there exist $y \in E$ such that $g(x, y) = 1$. By Proposition 3.5(v), there exists $c \in E$ such that $\langle c \rangle \in S$ and $(c, y) \in E_0$. Since $S \subseteq \mathcal{E}_{\leq 0}(x)$, Proposition 2.13(iv) implies that also $(x, c) \in E_0$.

By Proposition 2.13(ii) applied on (x, c) , there exists a symplecton T containing x and containing an extremal point hyperbolic with $\langle c \rangle$. By Proposition 3.5(v) again, we find $d \in E$ such that $\langle d \rangle \in T$ with $(d, y) \in E_0$, and again, Proposition 2.13(iv) implies that also $(d, x) \in E_0$. It remains to show that $(c, d) \in E_2$. Let $A := T \cap \mathcal{E}_{\leq 0}(c)$. By Proposition 2.13(iv), $A = T \cap \mathcal{E}_0(c)$. By Proposition 3.5(v) now, A consists of the single point $\langle x \rangle$. We can now invoke [CM21, Lemma 2.17] to see that the point $\langle d \rangle$ is indeed contained in $\mathcal{E}_2(c)$. By rescaling c , we obtain $g(c, d) = 1$.

Assume now that the extremal geometry of L does not contain lines. Then $\mathcal{E} \times \mathcal{E} = \mathcal{E}_{-2} \cup \mathcal{E}_0 \cup \mathcal{E}_2$. By Assumption 5.1(i), the extremal geometry $\Gamma' = (\mathcal{E}', \mathcal{F}')$ of $L_{k'}$ contains lines. Recall from Lemma 5.2 that $L_{k'}$ is again simple, so it satisfies Assumption 3.1. Let $\text{Gal}(k'/k) = \langle \sigma \rangle$. Then the involution σ also acts on Γ' and fixes all points of $\mathcal{E}$.

We start now with a pair of extremal elements $(a, b) \in E_0$. Recall from Assumption 3.1 that $(\mathcal{E}, \mathcal{E}_2)$ is connected, so there exist $e \in E$ such that $(a, e) \in E_2$. Notice that by Proposition 3.5, there is a unique symplecton S through a and b in Γ' , but since a and b are fixed by σ , also S is fixed (setwise) by σ . By Proposition 3.5(v), there is a unique point $\langle x \rangle \in S$ in relation $\mathcal{E}'_0$ with $\langle e \rangle$. Then also $\langle x \rangle^\sigma$ has this property, so we must have $\langle x \rangle^\sigma = \langle x \rangle$. By Hilbert's Theorem 90, we may replace x by a k' -multiple of x to get $x^\sigma = x$ and therefore $x \in E$. Notice that now $(x, a) \in E_0$ and that S is the unique symplecton in Γ' through $\langle x \rangle$ and $\langle a \rangle$. We also have $(x, e) \in E_0$; let T be the unique symplecton in Γ' through $\langle x \rangle$ and $\langle e \rangle$. Notice that $S \cap T = \langle x \rangle$.

We now apply the connectedness of $(\mathcal{E}, \mathcal{E}_2)$ again to find an element $y \in E_2(x)$. Then again, there is a unique point $\langle c \rangle \in S$ in relation $\mathcal{E}'_0$ with y , and similarly, there is a unique point $\langle d \rangle \in T$ in relation $\mathcal{E}'_0$ with y . As before, we get $\langle c \rangle^\sigma = \langle c \rangle$ and $\langle d \rangle^\sigma = \langle d \rangle$, and we may rescale c and d so that $c \in E$ and $d \in E$. Now observe that $(c, e) \notin E_0$, since otherwise both $\langle x \rangle$ and $\langle c \rangle$ would be in relation $\mathcal{E}'_0$ with $\langle e \rangle$, contradicting Proposition 3.5(v) applied on S and e . Since $\mathcal{E} \times \mathcal{E} = \mathcal{E}_{-2} \cup \mathcal{E}_0 \cup \mathcal{E}_2$, this implies that $(c, e) \in E_2$. Similarly, it now follows that $(c, d) \notin E_0$, since otherwise both $\langle x \rangle$ and $\langle d \rangle$ would be in relation $\mathcal{E}'_0$ with $\langle c \rangle$, again contradicting Proposition 3.5(v), now applied on T and c . We conclude that $(c, d) \in E_2$. It now only remains to rescale the elements to see that x, y, c and d satisfy all the required assumptions. $\square$

Notation 5.4. Let x, y, c, d be as in Proposition 5.3. We denote the 5-grading on L associated with the hyperbolic pair (x, y) as in Proposition 2.4 by

$$L_{-2} \oplus L_{-1} \oplus L_0 \oplus L_1 \oplus L_2, \quad (54)$$

with $L_{-2} = \langle x \rangle$ and $L_2 = \langle y \rangle$. Similarly, we denote the 5-grading on L associated with the hyperbolic pair (c, d) as in Proposition 2.4 by

$$L'_{-2} \oplus L'_{-1} \oplus L'_0 \oplus L'_1 \oplus L'_2, \quad (55)$$

with $L'_{-2} = \langle c \rangle$ and $L'_2 = \langle d \rangle$.

Now set

$$V := L_{-1} \cap L'_{-1}, \quad X := L_{-1} \cap L'_0, \quad V' := L_{-1} \cap L'_1, \quad X' := L_0 \cap L'_{-1}.$$

Lemma 5.5. *We have $c, d \in L_0$ and $x, y \in L'_0$.*

Proof. Since $(x, c), (c, y) \in E_0$, it follows immediately from Proposition 2.5 that $c \in L_0$. The proof of the other three statements is similar. $\square$

Lemma 5.6. *We have $g_c(L_{-2} \oplus L_{-1} \oplus L_1 \oplus L_2) = 0$ and $g_d(L_{-2} \oplus L_{-1} \oplus L_1 \oplus L_2) = 0$.*

Proof. We show the claim for g_c . Since $(c, x), (c, y) \in E_0$, we get $g_c(x) = 0$ and $g_c(y) = 0$. Next, let $l \in L_{-1}$ be arbitrary and recall from Proposition 2.4(iii) that $[x, [y, l]] = -l$. Using the associativity of g , we get

$$-g_c(l) = g(c, [x, [y, l]]) = g([c, x], [y, l]) = g(0, [y, l]) = 0,$$

so $g_c(L_{-1}) = 0$. Similarly, $g_c(L_1) = 0$. $\square$

Proposition 5.7. *We have decompositions*

$$\begin{aligned} L_{-1} &= V \oplus X \oplus V', \\ L_1 &= [y, V] \oplus [y, X] \oplus [y, V'], \\ L'_{-1} &= V \oplus X' \oplus [y, V], \\ L'_1 &= V' \oplus [d, X'] \oplus [y, V']. \end{aligned}$$

Proof. By Lemma 5.6, we get $g_d(L_{-1}) = 0$ and hence $L_{-1} \leq L'_{\geq -1}$ by Proposition 2.4(v) applied to the grading (55). Similarly, it follows from $g_c(L_{-1}) = 0$ that $L_{-1} \leq L'_{\leq 1}$. Hence, $L_{-1} \leq L'_{-1} \oplus L'_0 \oplus L'_1$.

The rest of the proof of the decomposition of L_{-1} now follows exactly the same method as in the proof of Proposition 4.6, using the grading element $[c, d] \in L_0$, so we can safely omit the details. The second decomposition follows from the first by applying the isomorphism ad_y from L_{-1} to L_1 ; see Proposition 2.4(iii).

The other two decompositions follow in a similar fashion by interchanging the roles of the two gradings (54) and (55). $\square$

We have summarized all the information about the intersections of the two gradings in Figure 2. Notice that the four subspaces $V, V' = [d, V], [y, V]$ and $[y, V'] = [y, [d, V]]$ are all isomorphic, by Proposition 2.4(iii). (See also the proof of Corollary 5.8 below.) For the same reason, $X \cong [y, X]$ and $X' \cong [d, X']$. We will see later that also the subspaces X and X' are isomorphic; see Corollary 5.24. Notice that in Figure 2, we have also identified a third grading (diagonally), but this grading is not directly necessary for our purposes. See, however, Remark 5.60 below.

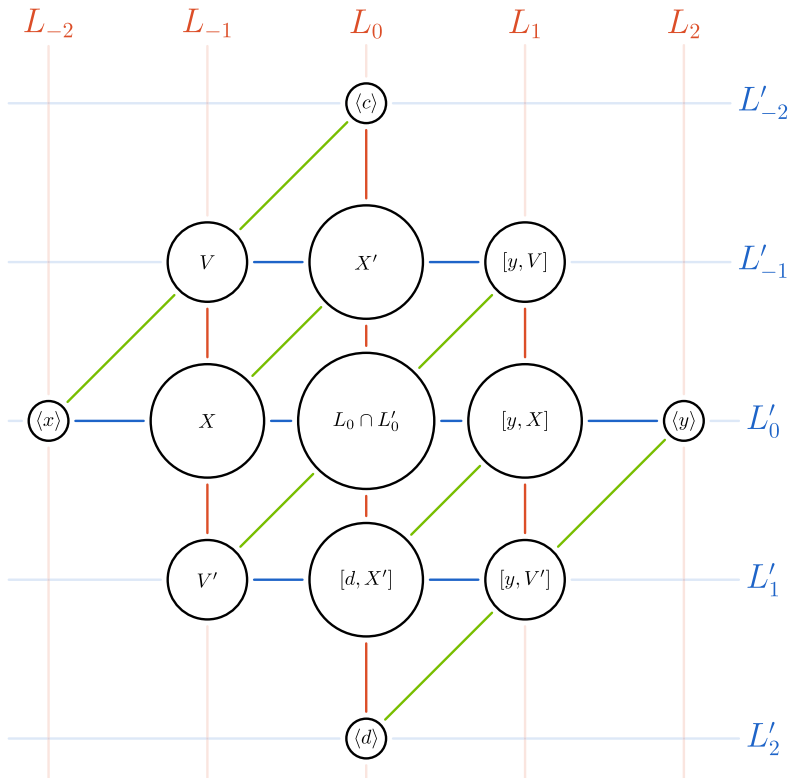


Figure 2. Intersecting gradings for quadrangular algebras.

By combining both gradings, we already obtain some more information on the Lie bracket.

Corollary 5.8. *The following identities and inclusions hold:*

$$[c, V'] = V, \quad [d, V] = V', \quad (56)$$

$$[c, V] = 0, \quad [d, V'] = 0, \quad (57)$$

$$[c, X] = 0, \quad [d, X] = 0, \quad (58)$$

$$[V, V] = 0, \quad [V', V'] = 0, \quad [V, X] = 0, \quad [V', X] = 0, \quad (59)$$

$$[X, [X, [y, V]]] \leq V, \quad [X, [X, [y, V']]] \leq V', \quad (60)$$

$$[X, [V, [y, V']]] \leq X, \quad [X, [V', [y, V]]] \leq X. \quad (61)$$

Proof. By Proposition 2.4(iii) applied on the grading (55), the map ad_d is an isomorphism from L'_{-1} to L'_1 with inverse $-\text{ad}_c$. Since $c, d \in L_0$, these maps preserve the grading (54). It follows from Proposition 5.7 that ad_d induces an isomorphism from V to V' , with inverse $-\text{ad}_c$. This shows (56).

The relations (57) to (61) all follow immediately from combining the two gradings, as can be seen from Figure 2. $\square$

We will now exploit the uniqueness of certain extremal elements to define a quadratic form on the vector space V .

Lemma 5.9. *For every $v \in V$, there is a unique $\lambda \in k$ such that*

$$l_v := \lambda x + [v, d] + d \quad (62)$$

is an extremal element. Moreover, $l_v = l_{v'}$ if and only if $v = v'$, where $v, v' \in V$.

Proof. Since $v \in L_{-1}$, Theorem 3.16 implies that there exists a v -exponential automorphism $\alpha \in \text{Aut}(L)$. Then $\alpha(d)$ is an extremal element, and we have $\alpha(d) = q_\alpha(d) + [v, d] + d$ with $q_\alpha(d) \in L_{-2} = \langle x \rangle$, showing the existence.

Assume now that there are two distinct scalars such that (62) holds. Then the 2-dimensional subspace $\langle x, [v, d] + d \rangle$ contains three distinct extremal elements, so Proposition 1.8 implies that this is a line of the extremal geometry. This contradicts Corollary 2.6.

Finally, if $l_v = l_{v'}$ for certain $v, v' \in V$, then $[v, d] = [v', d]$. Proposition 2.4(iii) applied on the grading (55) then implies that $v = v'$. $\square$

Definition 5.10. Define the map $Q: V \rightarrow k$ by setting $Q(v)$ to be the unique $\lambda \in k$ from Lemma 5.9 – that is, such that the element

$$l_v = Q(v)x + [v, d] + d \quad (63)$$

is an extremal element. Define the bilinear form $T: V \times V \rightarrow k$ by

$$T(u, v)x = [u, [v, d]]$$

for all $u, v \in V$.

Remark 5.11. The uniqueness aspect, although easy to prove, is somewhat subtle, because the v -exponential automorphism α that we have used in the proof of Lemma 5.9 is not unique. Notice, however, that we will be able to single out a unique v -exponential automorphism in Definition 5.15 below.

Lemma 5.12. *The map Q is a quadratic form on V , with corresponding bilinear form T .*

Proof. Let $\lambda \in k$ and $u, v \in V$. Let α, β be a u -exponential and a v -exponential automorphism, respectively. Then $q_\alpha(d) + [u, d] + d \in E$ and $q_\alpha(d) = Q(u)x$. By Theorem 3.13(vii), $\lambda^2 q_\alpha(d) + \lambda[u, d] + d \in E$. Hence, $Q(\lambda u) = \lambda^2 Q(u)$.

Next, we have

$$\begin{aligned} \alpha(\beta(d)) &= \alpha(Q(v)x + [v, d] + d) \\ &= Q(v)x + [v, d] + [u, [v, d]] + Q(u)x + [u, d] + d \\ &= (Q(v) + T(u, v) + Q(u))x + [u + v, d] + d \in E, \end{aligned}$$

and hence, $Q(u + v) = Q(v) + T(u, v) + Q(u)$. The map T is bilinear by construction; hence, Q is a quadratic form on V with corresponding bilinear form T . $\square$

We will see in Lemma 5.19 below that Q is not the zero map. The following lemma is a first step, useful in its own right.

Lemma 5.13. *Let $v \in V \setminus \{0\}$ be such that $Q(v) = 0$. Then v , $[v, d]$, $[y, v]$ and $[y, [v, d]]$ are extremal elements.*

Proof. By definition of Q and Lemma 5.12, we have $\lambda[v, d] + d \in E$ for all $\lambda \in k$. Note that $[v, d] \neq 0$ since ad_d induces an isomorphism from V to V' . By Proposition 1.8, this implies that $[v, d] \in E$. Since ad_c maps extremal elements in L'_1 to extremal elements in L'_{-1} (by Lemma 2.8), it follows that also $v = [c, [v, d]] \in E$. Similarly, ad_y maps extremal elements in L_{-1} to extremal elements in L_1 , and thus also $[y, v]$ and $[y, [v, d]]$ are extremal elements. $\square$

Lemma 5.14. *Let $v \in V$. Then*

$$Q(v)x + v + c \in E, \quad x + v + Q(v)c \in E, \quad (64)$$

$$Q(v)x + [v, d] + d \in E, \quad x + [v, d] + Q(v)d \in E, \quad (65)$$

$$Q(v)y + [y, v] + c \in E, \quad y + [y, v] + Q(v)c \in E, \quad (66)$$

$$Q(v)y + [y, [v, d]] + d \in E, \quad y + [y, [v, d]] + Q(v)d \in E. \quad (67)$$

Moreover, in each case, $Q(v)$ is the unique scalar that makes the statement true.

Proof. Assume first that $Q(v) = 0$. Then by Lemma 5.13, we have $v \in E$. By Corollary 2.6, this element is collinear with c and with x , so in particular, $v + c \in E$ and $x + v \in E$. The proof of the other 6 statements is similar.

Assume now that $Q(v) \neq 0$. By definition, $Q(v)x + [v, d] + d \in E$. By replacing v with $Q(v)^{-1}v$, we also get $Q(v)^{-1}x + Q(v)^{-1}[v, d] + d \in E$, and multiplying by $Q(v)$ now gives $x + [v, d] + Q(v)d \in E$. This proves (65).

Next, let $\varphi \in \text{Aut}(L)$ be the automorphism obtained from Lemma 2.8 applied to the grading (55). In particular, $\varphi(d) = c$ and $\varphi(x) = x + [c, [d, x]] = x$, so we get

$$\varphi(Q(v)x + [v, d] + d) = Q(v)x + [c, [v, d]] + c = Q(v)x + v + c \in E.$$

We can again replace v by $Q(v)^{-1}v$ to get $x + v + Q(v)c \in E$ as well. This proves (64).

We now apply the automorphism $\psi \in \text{Aut}(L)$ obtained from Lemma 2.8 applied to the grading (54). Then in exactly the same way as in the previous paragraph, applying ψ on (64) yields (66) and applying ψ on (65) yields (67).

Finally, the uniqueness statement follows in each case in exactly the same way as in the proof of Lemma 5.9. $\square$

The previous lemma will, in particular, allow us to single out a specific v -exponential automorphism, as announced in Remark 5.11 above.

Definition 5.15. Let $v \in V$. Then by Lemma 5.14 applied on $-v$, we have $y + [v, y] + Q(v)c \in E$. By Proposition 3.22, there exists a *unique* automorphism $\alpha_v \in E_-(x, y)$ such that

$$\alpha_v(y) = y + [v, y] + Q(v)c. \quad (68)$$

By Theorem 3.13(vi), α_v is an ℓ -exponential automorphism for some $\ell \in L_{-1}$, but then $[\ell, y] = [v, y]$, and therefore, by applying ad_x , we get $\ell = v$, so α_v is a v -exponential automorphism. Notice that by Lemma 5.9 and Definition 5.10, we also have

$$\alpha_v(d) = d + [v, d] + Q(v)x. \quad (69)$$

In a similar way, we can define a unique $[y, v]$ -exponential automorphism for each $v \in V$, this time with respect to the L'_i -grading (55). We will need these automorphisms in Lemma 5.53 and Proposition 5.55 below.

Definition 5.16. Let $v \in V$ and consider $[y, v] \in [y, V]$. By Lemma 5.14, we have $d + [y, [v, d]] + Q(v)y \in E$. By Proposition 3.22, there exists a *unique* automorphism $\beta_v \in E_-(c, d)$ such that

$$\beta_v(d) = d + [y, [v, d]] + Q(v)y = d + [[y, v], d] + Q(v)y.$$

In particular, by Theorem 3.13(vi), β_v is an ℓ -exponential automorphism for some $\ell \in L'_{-1}$, but then $[\ell, d] = [[y, v], d]$, so by applying ad_c , we get $\ell = [y, v]$, so β_v is a $[y, v]$ -exponential automorphism

with respect to the L'_i -grading (55). Notice that $\beta_v(x) = x + [[y, v], x] + \lambda c \in E$ for some $\lambda \in k$, so $x + v + \lambda c \in E$, and hence by the uniqueness part of Lemma 5.14, we have

$$\beta_v(x) = x + v + Q(v)c.$$

It is a slightly subtle fact that α_v is not only v -exponential with respect to the L_i -grading (54), but also with respect to the L'_i -grading (55), as we now show.

Proposition 5.17. *Let $v \in V = L_{-1} \cap L'_{-1}$. Then α_v is also v -exponential with respect to the L'_i -grading (55).*

Proof. First apply Assumption 5.1(i) and extend k to a larger field k' such that the extremal geometry $E(L \otimes k')$ contains lines. Let α_v be the v -exponential automorphism with respect to the L_i -grading (54) as in Definition 5.15. Moreover, let α'_v be the unique v -exponential automorphism with respect to the L'_i -grading (55) such that

$$\alpha'_v(d) = d + [v, d] + Q(v)x,$$

similarly to what we have done in Definition 5.16, and notice that also

$$\alpha'_v(y) = y + [v, y] + Q(v)c = \alpha_v(y).$$

We claim that α_v and α'_v also coincide on $L_{-1} = V \oplus X \oplus V'$; it will then follow from Lemma 3.3 that $\alpha_v = \alpha'_v$. Our claim is obvious by the grading for elements of $V \oplus X$, so it only remains to show that $\alpha_v([d, w]) = \alpha'_v([d, w])$ for all $w \in V$. This is also clear, however, because α_v and α'_v coincide on both $\langle d \rangle$ and V .

We conclude that indeed, $\alpha_v = \alpha'_v$, so in particular, α_v is also v -exponential with respect to the L'_i -grading (55). $\square$

Lemma 5.18. *We have $V \neq 0$ and $V' \neq 0$. Moreover, the bilinear form $T: V \times V \rightarrow k$ is nondegenerate.*

Proof. Notice that by (56), we have $V \neq 0$ if and only if $V' \neq 0$. Suppose now that $V = V' = 0$; then $L_{-1} = X$ and $L_1 = [y, X]$ (see Figure 2), so

$$L_{-2} \oplus L_{-1} \oplus L_1 \oplus L_2 \leq L'_0.$$

In particular, by Proposition 2.4(ii), $\text{ad}_{[c, d]}$ maps $L_{-2} \oplus L_{-1} \oplus L_1 \oplus L_2$ to 0. Since L is generated by $L_{-2} \oplus L_{-1} \oplus L_1 \oplus L_2$, this implies that the map $\text{ad}_{[c, d]}$ is zero on all of L , a contradiction. Hence, $V \neq 0$ and $V' \neq 0$.

Now let $v \in V \setminus \{0\}$ be arbitrary, and assume that $T(v, u) = 0$ for all $u \in V$. Then by definition of T , this implies that $[v, V'] = 0$, but since also $[v, X] = 0$ and $[v, V] = 0$ by the grading, we then have $[v, L_{-1}] = 0$. This contradicts Lemma 2.11. $\square$

Lemma 5.19. *There exists $v \in V$ such that $Q(v) \neq 0$.*

Proof. Assume that $Q(v) = 0$ for all $v \in V$. By Lemma 5.13, there exist elements in $E \cap L_{-1}$, so by Corollary 2.6, the extremal geometry contains lines. In fact, each nonzero $v \in V$ is an extremal element collinear with x and, by Lemma 5.14, also collinear with c . Moreover, it now follows from the grading and Corollary 2.6 that $E_{-1}(x) \cap E_{-1}(c) = V \setminus \{0\}$.

By Proposition 3.5, however, there is a (unique) symplecton S containing $\langle x \rangle$ and $\langle c \rangle$. By Proposition 3.5(i), S is a nondegenerate polar space of rank ≥ 2 , so we can find two distinct non-collinear points $\langle u \rangle$ and $\langle v \rangle$ both collinear to both $\langle x \rangle$ and $\langle c \rangle$. By the previous paragraph, u and v belong to $V \setminus \{0\}$. However, this implies that also $u + v \in V \setminus \{0\}$, so by assumption, $Q(u + v) = 0$, so also $u + v$ is an extremal element. This contradicts Proposition 1.8 because u and v are not collinear. $\square$

In a similar way as in Lemma 4.18, we can now ensure that Q has a base point (i.e., $Q(v) = 1$ for some $v \in V$).

Lemma 5.20. *We can re-choose x in y in Notation 5.4 in such a way that $Q(e) = 1$ for some $e \in V$. We call e a base point for Q .*

Proof. Consider $e \in V$ such that $Q(e) \neq 0$. Set $\tilde{x} := Q(e)x$ and $\tilde{y} := Q(e)^{-1}y$. Then $\tilde{x}$, $\tilde{y}$, c and d still satisfy the conclusions from Proposition 5.3. Since we have only replaced these elements by a scalar multiple, this does not affect any of the decompositions.

Now denote the map obtained as in Definition 5.10 from these new choices by $\tilde{Q}$. Since $\tilde{x} + [e, d] + d = Q(e)x + [e, d] + d \in E$, it follows that indeed $\tilde{Q}(e) = 1$. $\square$

Definition 5.21. Assume from now on that $e \in V$ is a base point for Q .

(i) We set

$$e' := [d, e] \in V', \quad f := [y, e] \in [y, V], \quad f' := [y, e'] \in [y, V'].$$

Notice that $f' = [d, f]$ because $[d, y] = 0$.

(ii) Since T is nondegenerate by Lemma 5.18, we can fix an element $\delta \in V$ such that $T(e, \delta) = 1$. If $\text{char}(k) \neq 2$, we will assume that $\delta = \frac{1}{2}e$.

We may assume, in addition, that $Q(\delta) \neq 0$. Indeed, this holds automatically if $\text{char}(k) \neq 2$, and if $\text{char}(k) = 2$ and $Q(\delta) = 0$, then we choose some $\lambda \in k$ with $\lambda^2 \neq \lambda$ (which exist because we assume $|k| > 2$; see Assumption 5.1) and we replace δ by $\lambda e + \delta$.

Remark 5.22. This definition of δ corresponds to the definition of δ for quadrangular algebras in [Wei06, Definition 4.1] and [MW19, Definition 7.1]. In fact, we will use this δ in Proposition 5.28 below to ensure that our quadrangular algebra will be δ -standard; see Definition 5.29(iii).

Lemma 5.23. *For any $v \in V$, we have*

$$Q(v)([c, d] - [x, y]) = [[v, d], [v, y]]; \quad (70)$$

$$Q(v)([c, d] + [x, y]) = [v, [y, [d, v]]]. \quad (71)$$

In particular,

$$[e', f] = [c, d] - [x, y] \quad \text{and} \quad [e, f'] = [c, d] + [x, y]. \quad (72)$$

Proof. Let $v \in V$ and let $\alpha_v \in \text{Aut}(L)$ be as in Definition 5.15. Since $[d, y] = 0$ and $\alpha_v \in \text{Aut}(L)$, we also get $[\alpha_v(d), \alpha_v(y)] = 0$, so by (68) and (69), we get

$$[d + [v, d] + Q(v)x, y + [v, y] + Q(v)c] = 0.$$

Expanding this – or in fact, only expanding the $(L_0 \cap L'_0)$ -component suffices – yields (70).

Next, observe that $[v, [v, d]] = T(v, v)x = 2Q(v)x$ by Lemma 5.12. Hence,

$$[[v, d], [v, y]] = [v, [[v, d], y]] - 2Q(v)[x, y],$$

so together with (70), this yields (71).

Finally, (72) now follows by plugging in $v = e$ and using $Q(e) = 1$. $\square$

Corollary 5.24. *The restriction of ad_f to X defines a linear isomorphism from X to X' with inverse $\text{ad}_{e'}$.*

Proof. Let $a \in X$ and $a' \in X'$. Then using (72) and Proposition 2.4(ii), we get

$$\begin{aligned} [e', [f, a]] &= [a, [f, e']] = [[c, d] - [x, y], a] = a, \\ [f, [e', a']] &= [a', [e', f]] = [-[c, d] + [x, y], a'] = a'. \end{aligned}$$

□

We can now also determine the Lie bracket between V and $[y, V]$ explicitly.

Proposition 5.25. *Let $v, w \in V$. Then $[[y, v], w] = T(v, w)c$.*

Proof. Since $V = L_{-1} \cap L'_{-1}$, it follows from Proposition 2.4(ii) that

$$[[c, d] - [x, y], w] = 0.$$

By (70), therefore, we have $[[[v, d], [v, y]], w] = 0$, and hence,

$$[[v, d], [[v, y], w]] = [[v, y], [[v, d], w]].$$

By the grading, we know that $[[y, v], w] = \mu c$ for some $\mu \in k$. Hence, by Definition 5.10, the previous equality reduces to

$$[[v, d], -\mu c] = [[v, y], -T(v, w)x].$$

By Proposition 2.4(iii), this can be rewritten as $\mu v = T(v, w)v$, so $\mu = T(v, w)$. □

Definition 5.26. We define maps $h: X \times X \rightarrow V$ and $\cdot: X \times V \rightarrow X: (a, v) \mapsto a \cdot v$ by setting

$$h(a, b) := [a, [b, f]], \quad (73)$$

$$a \cdot v := [a, [e', [v, y]]] = [[a, [y, v]], e'], \quad (74)$$

for all $a, b \in X$ and all $v \in V$. Note that the image of these maps is indeed contained in V and X , respectively, by (60) and (61) (see again Figure 2). Also notice that the second equality in (74) holds because $[V', X] = 0$.

Lemma 5.27. *Let $a, b \in X$ and $v \in V$. Then*

- (i) $a \cdot e = a$,
- (ii) $[a \cdot v, f] = [a, [y, v]] = [[a, y], v]$,
- (iii) $h(a, b \cdot v) = [a, [b, [y, v]]]$,
- (iv) $[d, h(a, b)] = [a, [b, f']]$.

Proof.

- (i) Using Definition 5.21 and (72), we have

$$a \cdot e = [a, [e', [e, y]]] = -[a, [e', f]] = [[c, d] - [x, y], a] = a$$

since $a \in X = L_{-1} \cap L'_0$.

- (ii) The first equality follows from (74) and Corollary 5.24, and the second equality follows from the fact that $[X, V] = 0$.
- (iii) This follows from (73) and (ii).
- (iv) Since $[d, X] = 0$, we have

$$[d, h(a, b)] = [d, [a, [b, f]]] = [a, [b, [d, f]]] = [a, [b, f']].$$

□

Proposition 5.28. *Let $a \in X \leq L_{-1}$. Then there exists a unique a -exponential automorphism $\alpha \in \text{Aut}(L)$ such that*

$$[q_\alpha(f), [d, \delta]] = 0. \quad (75)$$

Moreover, α preserves the L'_i -grading (55) of L , so in particular, $q_\alpha([y, v]) \in V$ for all $v \in V$.

Proof. By Theorem 3.16, there exists an a -exponential automorphism $\alpha \in \text{Aut}(L)$. We first show that α preserves the L'_i -grading (55) of L . Since $c \in L_0$, we can write $\alpha(c) = c + [a, c] + q_\alpha(c)$. Now $[a, c] = 0$, and if $q_\alpha(c) \neq 0$, then $\alpha(c)$ is an extremal element of the form $c + \lambda x$, which is impossible. Hence, $q_\alpha(c) = 0$, and similarly $\alpha(d) = d$. By Lemma 2.12, this implies that α preserves the L'_i grading (55).

In particular, $\alpha(f) \in L'_{-1}$, so we can write

$$\alpha(f) = f + [a, f] + q_\alpha(f)$$

with $q_\alpha(f) \in L_{-1} \cap L'_{-1} = V$. By Theorem 3.13(v), any other such a -exponential automorphism α' can be written as $\alpha' = \exp(\lambda x)\alpha$ for some $\lambda \in k$. By Remark 3.14(i), we then have $q_{\alpha'}(f) = q_\alpha(f) + [\lambda x, f] = q_\alpha(f) - \lambda e$. Notice now that $[q_\alpha(f), [d, \delta]] \in \langle x \rangle$. Since $-[e, [d, \delta]] = T(e, \delta)x = x$ by Definitions 5.10 and 5.21, there is indeed a unique choice of λ such that (75) holds. $\square$

Definition 5.29.

- (i) For each $a \in X$, we denote the a -exponential automorphism $\alpha \in \text{Aut}(L)$ satisfying (75) by Θ_a . The corresponding (uniquely defined) maps q_α , n_α and v_α will be denoted by q_a , n_a and v_a , respectively.
- (ii) For each $a \in X$ and each $v \in V$, we define $\theta(a, v) := q_a([y, v]) \in V$. In particular, we have

$$\Theta_a([y, v]) = [y, v] + [a, [y, v]] + \theta(a, v) = [y, v] + [a \cdot v, f] + \theta(a, v), \quad (76)$$

where the second equality holds by Lemma 5.27(ii). Notice that for each $\lambda \in k$, the automorphism $\Theta_{\lambda a}$ coincides with $(\Theta_a)_\lambda$ as defined in Theorem 3.13(vii); it follows that $\theta(\lambda a, v) = \lambda^2 \theta(a, v)$.

- (iii) For each $a \in X$, we define $\pi(a) := \theta(a, e) = q_a(f) \in V$. In particular, we have

$$\Theta_a(f) = f + [a, f] + \pi(a).$$

Notice that (75) tells us that

$$T(\pi(a), \delta) = 0, \quad (77)$$

which corresponds precisely to condition (iii) in [Wei06, Definition 4.1] and [MW19, Definition 7.1].

When $\text{char}(k) \neq 2$, we can describe θ and π in terms of h and $\cdot$; cfr. [Wei06, Proposition 4.5(i) and Remark 4.8].

Proposition 5.30. *If $\text{char}(k) \neq 2$, then $\Theta_a = e_-(a)$ as in Theorem 3.13(ix). Moreover, we have $\pi(a) = \frac{1}{2}h(a, a)$ and $\theta(a, v) = \frac{1}{2}h(a, a \cdot v)$.*

Proof. By definition, we have

$$\begin{aligned} \Theta_a(f) &= f + [a, f] + \pi(a), \\ e_-(a)(f) &= f + [a, f] + \frac{1}{2}[a, [a, f]] = f + [a, f] + \frac{1}{2}h(a, a). \end{aligned}$$

We claim that $T(\frac{1}{2}[a, [a, f]], \delta) = 0$. Recall that $\delta = \frac{1}{2}e$ since $\text{char}(k) \neq 2$. Using twice the fact that $[a, e'] = 0$ and using (72), we get

$$\begin{aligned} T([a, [a, f]], e) &= [[a, [a, f]], [e, d]] = [e', [a, [a, f]]] = [a, [e', [a, f]]] \\ &= [a, [a, [e', f]]] = [a, [a, [c, d] - [x, y]]] = [a, a] = 0, \end{aligned}$$

proving our claim. This shows that $e_-(a)$ satisfies the assumption (75), so it coincides with Θ_a . In particular, $\pi(a) = \frac{1}{2}h(a, a)$. Moreover,

$$\Theta_a([y, v]) = e_-(a)([y, v]) = [y, v] + [a, [y, v]] + \frac{1}{2}[a, [a, [y, v]]],$$

and hence, $\theta(a, v) = \frac{1}{2}[a, [a, [y, v]]] = \frac{1}{2}h(a, a \cdot v)$ by Lemma 5.27(iii). $\square$

Next, we return to the case where $\text{char}(k)$ is arbitrary, and we recover identity (vii) from Definition 1.17.

Proposition 5.31. *There exists a function $\gamma: X \times X \rightarrow k$ such that*

$$\theta(a + b, v) = \theta(a, v) + \theta(b, v) + h(a, b \cdot v) - \gamma(a, b)v$$

for all $a, b \in X$ and all $v \in V$. Moreover, we have $\gamma(a, b) = T(h(a, b), \delta)$ for all $a, b \in X$.

Proof. Let $a, b \in X$. By Theorem 3.13(viii), the composition $\Theta_a \Theta_b$ is an $(a + b)$ -exponential automorphism of L , so by Theorem 3.13(v), there exists some $\lambda \in k$ such that

$$\Theta_a \Theta_b = \exp(\lambda x) \Theta_{a+b}. \quad (78)$$

We write $\gamma(a, b) := -\lambda$.

Observe now that $[b, f] \in [X, f] = X'$. Since Θ_a preserves the L'_i -grading by Proposition 5.28, we thus get

$$\Theta_a([b, f]) = [b, f] + [a, [b, f]] = [b, f] + h(a, b). \quad (79)$$

Now let $v \in V$; then using (76) and (79), we get

$$\begin{aligned} \Theta_a \Theta_b([y, v]) &= \Theta_a([y, v] + [b \cdot v, f] + \theta(b, v)) \\ &= [y, v] + [a \cdot v, f] + \theta(a, v) + [b \cdot v, f] + h(a, b \cdot v) + \theta(b, v). \end{aligned}$$

However,

$$\Theta_{a+b}([y, v]) = [y, v] + [(a + b) \cdot v, f] + \theta(a + b, v),$$

and since $[x, [y, v]] = -v$, we get

$$\exp(\lambda x) \Theta_{a+b}([y, v]) = [y, v] + [(a + b) \cdot v, f] + \theta(a + b, v) - \lambda v.$$

Since $\lambda = -\gamma(a, b)$, the required expression for $\theta(a + b)$ now follows from (78).

Next, if we set $v = e$ in this expression, then we get, using Lemma 5.27(i),

$$\pi(a + b) = \pi(a) + \pi(b) + h(a, b) - \gamma(a, b)e. \quad (80)$$

By (77), this implies that $T(h(a, b) - \gamma(a, b)e, \delta) = 0$, and since $T(e, \delta) = 1$, this implies that $\gamma(a, b) = T(h(a, b), \delta)$, as claimed. $\square$

Recall from Definition 1.16 that the quadratic form Q with base point e has an associated involution

$$\sigma: V \rightarrow V: v \mapsto v^\sigma := T(v, e)e - v.$$

Lemma 5.32. *Let $a \in X$ and $v \in V$. Then $a \cdot v^\sigma = [a, [v, f']] = [v, [a, f']]$.*

Proof. Notice that

$$\begin{aligned} [v, e'] &= [v, [d, e]] = -T(v, e)x \quad \text{and} \\ [v, f'] &= [v, [y, e']] = -[[v, e'], y] - [e', [v, y]] = T(v, e)[x, y] - [e', [v, y]]. \end{aligned}$$

Since $X \leq L_{-1}$, it follows that

$$[a, [v, f']] = T(v, e)a - [a, [e', [v, y]]] = T(v, e)a - a \cdot v = a \cdot v^\sigma.$$

The second equality follows from the fact that $[a, v] = 0$. □

Proposition 5.33. *Let $a \in X$ and $v \in V$. Then*

$$(a \cdot v) \cdot v^\sigma = Q(v)a.$$

Proof. Since $[d, a \cdot v] \in [d, X] = 0$, we have, using Lemma 5.27(ii), that

$$[a \cdot v, f'] = [a \cdot v, [d, f]] = [d, [a \cdot v, f]] = [d, [a, [y, v]]]. \quad (81)$$

By Lemma 5.32, we then get

$$\begin{aligned} (a \cdot v) \cdot v^\sigma &= [a \cdot v, [v, f']] \\ &= [v, [a \cdot v, f']] && \text{since } [X, V] = 0 \\ &= [v, [d, [a, [y, v]]]] && \text{by (81)} \\ &= [v, [a, [d, [y, v]]]] && \text{since } [d, X] = 0 \\ &= [a, [v, [d, [y, v]]]] && \text{since } [X, V] = 0 \\ &= [[v, [d, [v, y]]], a]. \end{aligned}$$

Now $[[v, d], [v, y]] = [v, [d, [v, y]]] - [d, [v, [v, y]]] = [v, [d, [v, y]]] + \lambda[c, d]$ for some $\lambda \in k$ because $[v, [v, y]] \in \langle c \rangle$ by the grading. By Proposition 2.4(ii), however, $[[c, d], a] = 0$, and hence,

$$(a \cdot v) \cdot v^\sigma = [[v, d], [v, y], a].$$

The result now follows from (70) together with Proposition 2.4(ii). □

Lemma 5.34. *Let $a, b \in X$. Then $[a, b] = -T(h(a, b), e)x$. In particular, the bilinear map*

$$X \times X \rightarrow k: (a, b) \mapsto T(h(a, b), e)$$

is nondegenerate.

Proof. Using the definitions of T and h , we get

$$T(h(a, b), e)x = [h(a, b), [e, d]] = [[a, [b, f]], -e'] = [e', [a, [b, f]]].$$

Since $[e', a] = [e', b] = 0$, it follows that

$$T(h(a, b), e)x = [a, [b, [e', f]]].$$

Once again using (72) and Proposition 2.4(ii), we get $[b, [e', f]] = -b$, and the required identity follows.

Next, suppose that $a \in X$ is such that $T(h(a, b), e) = 0$ for all $b \in X$. Then $[a, b] = 0$ for all $b \in X$. Since also $[a, V] = 0$ and $[a, V'] = 0$ by the grading, this implies that $[a, L_{-1}] = 0$. By Lemma 2.11, however, this implies $a = 0$, as required. $\square$

Corollary 5.35. *Let $a, b \in X$. Then $h(a, b)^\sigma = -h(b, a)$.*

Proof. Since $[x, f] = -e$, it follows from Lemma 5.34 that

$$T(h(a, b), e)e = [[a, b], f] = [a, [b, f]] - [b, [a, f]] = h(a, b) - h(b, a).$$

Since $h(a, b)^\sigma = T(h(a, b), e)e - h(a, b)$, the result follows. $\square$

Proposition 5.36. *Let $a, b \in X$ and $v \in V$. Then*

$$T(h(a, b), v) = T(h(a \cdot v, b), e) = T(h(a, b \cdot v^\sigma), e).$$

Proof. Using the definition of T , Lemma 5.27(iv) and the fact that $[V, X] = 0$, we get

$$T(h(a, b), v)x = [v, [h(a, b), d]] = -[v, [a, [b, f']]] = -[a, [b, [v, f']]].$$

By Lemma 5.32, we have $[b, [v, f']] = b \cdot v^\sigma$, so together with Lemma 5.34, we get

$$T(h(a, b), v)x = -[a, b \cdot v^\sigma] = -T(h(a, b \cdot v^\sigma), e)x.$$

Next, since σ is an involutory isometry of $\mathcal{Q}$, we also have, using Corollary 5.35,

$$\begin{aligned} T(h(a, b), v)x &= T(h(a, b)^\sigma, v^\sigma)x = -T(h(b, a), v^\sigma)x = -T(h(b, a \cdot v), e)x \\ &= -T(h(b, a \cdot v)^\sigma, e^\sigma)x = T(h(a \cdot v, b), e)x. \end{aligned}$$

$\square$

Proposition 5.37. *Let $a, b \in X$ and $v \in V$. Then*

$$h(a, b \cdot v) = h(b, a \cdot v) + T(h(a, b), e)v.$$

Proof. By Proposition 5.31, we have

$$h(a, b \cdot v) - \gamma(a, b)v = h(b, a \cdot v) - \gamma(b, a)v.$$

Moreover, we have

$$\gamma(a, b) - \gamma(b, a) = T(h(a, b) - h(b, a), \delta),$$

so by Corollary 5.35, we get, using $T(e, \delta) = 1$, that

$$\gamma(a, b) - \gamma(b, a) = T(T(h(a, b), e)e, \delta) = T(h(a, b), e).$$

The result follows. $\square$

We now introduce some auxiliary notation.

Notation 5.38. Let $a \in X$ and let Θ_a be as in Definition 5.29. Then we can decompose

$$\Theta_a(y) = y + [a, y] + \ell_a + \tilde{a} + \lambda_a x$$

with $\ell_a \in L_0 \cap L'_0$, $\tilde{a} \in X$ and $\lambda_a \in k$.

It turns out that understanding the adjoint action of ℓ_a explicitly on the various components in the 5×5 -grading gives a lot of insight, so this is what we will examine in the next few results.

Lemma 5.39. *Let $a \in X$. Then $[\ell_a, c] = 0$ and $[\ell_a, d] = 0$.*

Proof. Since $\Theta_a(c) = c$ and $[y, c] = 0$, we have $[\Theta_a(y), c] = 0$, so in particular, $[\ell_a, c] = 0$. The proof of $[\ell_a, d] = 0$ is similar. $\square$

Lemma 5.40. *Let $a \in X$ and $v \in V$. Then*

$$\begin{aligned} [\ell_a, v] &= \theta(a, v) \quad \text{and} \\ [\ell_a, [d, v]] &= [d, \theta(a, v)]. \end{aligned}$$

Proof. Since $\Theta_a(v) = v$, we have

$$\begin{aligned} \Theta_a([y, v]) &= [\Theta_a(y), \Theta_a(v)] = [y + [a, y] + \ell_a + \tilde{a} + \lambda_a x, v] \\ &= [y, v] + [[a, y], v] + [\ell_a, v]. \end{aligned}$$

However, recall from (76) that

$$\Theta_a([y, v]) = [y, v] + [a \cdot v, f] + \theta(a, v).$$

Hence, $[\ell_a, v] = \theta(a, v)$. Next, consider the automorphism φ from Lemma 2.8 with respect to the L'_i -grading (55). Then by Lemmas 2.8 and 5.39,

$$\begin{aligned} \varphi(\ell_a) &= \ell_a + [c, [d, \ell_a]] = \ell_a, \\ \varphi(v) &= [d, v], \\ \varphi(\theta(a, v)) &= [d, \theta(a, v)], \end{aligned}$$

so the second identity follows from the first by applying φ . $\square$

Proposition 5.41. *Let $a \in X$ and $v \in V$. Then*

$$\begin{aligned} [\ell_a, x] &= T(\pi(a), e)x, \\ [\ell_a, y] &= -T(\pi(a), e)y \end{aligned}$$

and

$$T(\theta(a, v), v) = Q(v)T(\pi(a), e). \quad (82)$$

Proof. Since $\Theta_a(x) = x$, we have

$$\Theta_a([x, y]) = [x, y + [a, y] + \ell_a + \tilde{a} + \lambda_a x] = [x, y] + a - [\ell_a, x]. \quad (83)$$

We also have $\Theta_a([c, d]) = [\Theta_a(c), \Theta_a(d)] = [c, d]$ and $\Theta_a([d, v]) = [d, v]$. We now apply Θ_a to (70), so we get, using (76) and (83),

$$Q(v)([c, d] - [x, y] - a + [\ell_a, x]) = [[d, v], [y, v] + [a \cdot v, f] + \theta(a, v)].$$

In particular, the L_{-2} -components of both sides are equal, so we get

$$Q(v)[\ell_a, x] = [[d, v], \theta(a, v)] = T(\theta(a, v), v).$$

If we set $v = e$, then we get $[\ell_a, x] = T(\pi(a), e)$. Substituting this again in the identity for general v then yields the third identity. Finally, to get the second identity, write $[\ell_a, y] = \mu y$. Since $[\ell_a, [x, y]] = 0$ by Proposition 2.4(ii), we have $[[\ell_a, x], y] = [[\ell_a, y], x]$ and hence $\mu = -T(\pi(a), e)$. $\square$

Corollary 5.42. *Let $a \in X$ and $v, w \in V$. Then*

$$T(\theta(a, v), w) + T(\theta(a, w), v) = T(v, w)T(\pi(a), e)$$

and

$$\theta(a, v^\sigma)^\sigma = \theta(a, v) - T(e, v)\pi(a) + T(\pi(a), v)e.$$

Proof. The first identity follows by linearizing (82). If we choose $w = e$, then we get

$$T(\theta(a, v), e) + T(\pi(a), v) = T(v, e)T(\pi(a), e).$$

The second identity then follows from the definition of σ together with this identity. $\square$

Lemma 5.43. *Let $a, b \in X$. Then*

$$[\ell_a, [b, f]] = -[a \cdot h(a, b), f] \quad (84)$$

and

$$\theta(a, h(a, b)) + h(\tilde{a}, b) = 0. \quad (85)$$

Proof. We have $[y, [b, f]] = 0$ by the grading, and hence, $[\Theta_a(y), \Theta_a([b, f])] = 0$. Recall from (79) that

$$\Theta_a([b, f]) = [b, f] + h(a, b) \in X' \oplus V.$$

Combining this with Notation 5.38, we get

$$[y + [a, y] + \ell_a + \tilde{a} + \lambda_a x, [b, f] + h(a, b)] = 0.$$

In particular, the L_0 -component and L_{-1} -component of the left-hand side are 0, so

$$\begin{aligned} [[a, y], h(a, b)] + [\ell_a, [b, f]] &= 0, \\ [\ell_a, h(a, b)] + [\tilde{a}, [b, f]] &= 0. \end{aligned}$$

Applying Lemma 5.27(ii) on the first identity now yields (84), and applying Lemma 5.40 and (73) on the second identity yields (85). $\square$

Lemma 5.44. *Let $a \in X$ and $v \in V$. Then*

$$h(a, a \cdot v) + T(\pi(a), e)v = 2\theta(a, v). \quad (86)$$

In particular, $\pi(a)^\sigma = \pi(a) - h(a, a)$. Moreover, we have $T(\pi(a), e) = \gamma(a, a)$ and $2\gamma(a, a) = 0$ (i.e., $\gamma(a, a) = 0$ when $\text{char}(k) \neq 2$).

Proof. By Proposition 2.4(ii), we have $[[x, y], y] = 2y$. Applying Θ_a and using Notation 5.38, Proposition 5.41, and (83) then gives

$$[[x, y] + a - T(\pi(a), e)x, y + [a, y] + \ell_a + \tilde{a} + \lambda_a x] = 2(y + [a, y] + \ell_a + \tilde{a} + \lambda_a x).$$

In particular, the L_0 -components of both sides are equal, so using $[[x, y], \ell_a] = 0$ (by Proposition 2.4(ii) again), we get

$$[a, [a, y]] - T(\pi(a), e)[x, y] = 2\ell_a.$$

Now notice that by Lemma 5.27(iii) and using $[a, v] = 0$, we have $[a, [a, y], v] = h(a, a \cdot v)$. Moreover, $[[x, y], v] = -v$ by Proposition 2.4(ii) again, and $[\ell_a, v] = \theta(a, v)$ by Lemma 5.40, so we get (86). In particular, we have $h(a, a) + T(\pi(a), e)e = 2\pi(a)$, so

$$\pi(a)^\sigma = T(\pi(a), e)e - \pi(a) = \pi(a) - h(a, a).$$

Moreover, by Proposition 5.31 with $a = b$ and $v = e$, we have

$$2\pi(a) = h(a, a) - \gamma(a, a)e,$$

so by comparing with the previous identity, we get $T(\pi(a), e) = -\gamma(a, a)$.

Finally, observe that when $\text{char}(k) \neq 2$, then $\delta = \frac{1}{2}e$, so by Proposition 5.31, $\gamma(a, a) = T(h(a, a), \frac{1}{2}e)$, which is 0 by Lemma 5.34. $\square$

Proposition 5.45. *Let $a \in X$ and $v \in V$. Then $\tilde{a} = -a \cdot \pi(a)^\sigma$ and*

$$a \cdot \theta(a, v) = a \cdot \pi(a) \cdot v. \quad (87)$$

Moreover, we have

$$\lambda_a v = \theta(a, \theta(a, v)) - h(a \cdot \pi(a)^\sigma, a \cdot v). \quad (88)$$

Proof. We have $[y, [y, v]] = 0$ by the grading, and hence, $[\Theta_a(y), \Theta_a([y, v])] = 0$. By (76) and Notation 5.38, we get

$$[y + [a, y] + \ell_a + \tilde{a} + \lambda_a x, [y, v] + [a \cdot v, f] + \theta(a, v)] = 0. \quad (89)$$

In particular, the L_0 -component of the left-hand side is 0, so

$$[[a, y], \theta(a, v)] + [\ell_a, [a \cdot v, f]] + [\tilde{a}, [y, v]] = 0.$$

Now using Lemma 5.27(ii) for the first and third term and using (84) for the second term, we get

$$[a \cdot \theta(a, v) - a \cdot h(a, a \cdot v) + \tilde{a} \cdot v, f] = 0,$$

and hence, by Corollary 5.24,

$$a \cdot \theta(a, v) - a \cdot h(a, a \cdot v) + \tilde{a} \cdot v = 0.$$

By (86), this can be rewritten as

$$\tilde{a} \cdot v = a \cdot \theta(a, v) - T(\pi(a), e)a \cdot v.$$

If we set $v = e$, then we get $\tilde{a} = -a \cdot \pi(a)^\sigma$. Substituting this again in the previous identity, we get, using $\pi(a)^\sigma = T(\pi(a), e)e - \pi(a)$, that

$$a \cdot \theta(a, v) = -a \cdot \pi(a)^\sigma \cdot v + T(\pi(a), e)a \cdot v = a \cdot \pi(a) \cdot v.$$

To prove the second identity, we again start from (89), but this time we compute the L_{-1} -component, so we get

$$[\ell_a, \theta(a, v)] + [\tilde{a}, [a \cdot v, f]] + [\lambda_a x, [y, v]] = 0.$$

Using $\tilde{a} = -a \cdot \pi(a)^\sigma$, (73) and Lemma 5.40, we get (88). $\square$

Lemma 5.46. *Let $a \in X$ and $v \in V$. Then*

$$[\ell_a, [y, v]] = [y, \theta(a, v) - T(\pi(a), e)v].$$

In particular, $[\ell_a, f] = -[y, \pi(a)^\sigma]$.

Proof. Using Lemma 5.40 and Proposition 5.41, we get

$$[\ell_a, [y, v]] = [[\ell_a, y], v] + [y, [\ell_a, v]] = -T(\pi(a), e)[y, v] + [y, \theta(a, v)].$$

The second identity follows by setting $v = e$ since $f = [y, e]$. □

Lemma 5.47. *Let $a, b \in X$. Then*

$$[\ell_a, b] = b \cdot \pi(a)^\sigma - a \cdot h(a, b).$$

Proof. Using (84), Lemma 5.46 and Lemma 5.27(iii), we get

$$\begin{aligned} [[\ell_a, b], f] &= [\ell_a, [b, f]] - [b, [\ell_a, f]] = -[a \cdot h(a, b), f] + [b, [y, \pi(a)^\sigma]] \\ &= [-a \cdot h(a, b) + b \cdot \pi(a)^\sigma, f]. \end{aligned}$$

The result now follows from Corollary 5.24. □

We can now explicitly determine the value of λ_a introduced in Notation 5.38.

Proposition 5.48. *Let $a \in X$. Then $\lambda_a = Q(\pi(a))$.*

Proof. Since $\tilde{a} = -a \cdot \pi(a)^\sigma$ by Proposition 5.45, it follows from (85) with $b = a$ that $h(a \cdot \pi(a)^\sigma, a) = \theta(a, h(a, a))$, so by (87), we get

$$a \cdot h(a \cdot \pi(a)^\sigma, a) = a \cdot \pi(a) \cdot h(a, a). \quad (90)$$

By (88) with $v = e$, we have $\lambda_a e = \theta(a, \pi(a)) - h(a \cdot \pi(a)^\sigma, a)$, and hence,

$$\lambda_a a = a \cdot \theta(a, \pi(a)) - a \cdot h(a \cdot \pi(a)^\sigma, a).$$

By (87) again and by (90), we can rewrite this as

$$\lambda_a a = a \cdot \pi(a) \cdot \pi(a) - a \cdot \pi(a) \cdot h(a, a).$$

Since $\pi(a) - h(a, a) = \pi(a)^\sigma$ by Lemma 5.44, we get, using Proposition 5.33, that

$$\lambda_a a = a \cdot \pi(a) \cdot \pi(a)^\sigma = Q(\pi(a))a. \quad \square$$

We can now summarize our information about $\Theta_a(y)$.

Corollary 5.49. *Let $a \in X$. Then*

$$\Theta_a(y) = y + [a, y] + \ell_a - a \cdot \pi(a)^\sigma + Q(\pi(a))x.$$

Proof. This follows from Notation 5.38 and Propositions 5.45 and 5.48. □

We deduce another interesting identity, which is not required for the proof of our main result (Theorem 5.56), but which is interesting in its own right, and relevant also in the reconstruction process (see Remark 5.59 below).

Proposition 5.50. *Let $a, b, c \in X$. Write $V_{a,b} := [a, [y, b]] \in L_0 \cap L'_0$. Then*

$$[V_{a,b}, c] = a \cdot h(b, c) + b \cdot h(a, c) + c \cdot h(b, a).$$

Proof. We apply (78) on y and invoke Corollary 5.49 to get

$$\begin{aligned} \Theta_a(y + [b, y] + \ell_b - b \cdot \pi(b)^\sigma + Q(\pi(b))x) \\ = \exp(-\gamma(a, b)x)(y + [a + b, y] + \ell_{a+b} - (a + b) \cdot \pi(a + b)^\sigma + Q(\pi(a + b))x). \end{aligned}$$

Comparing the L_0 -component of both sides yields

$$\ell_a + [a, [b, y]] + \ell_b = \ell_{a+b} - \gamma(a, b)[x, y].$$

Taking the Lie bracket with $c \in X \leq L_{-1}$ now gives

$$[\ell_a, c] - [V_{a,b}, c] + [\ell_b, c] = [\ell_{a+b}, c] + \gamma(a, b)c.$$

We now apply Lemma 5.47, and we get

$$\begin{aligned} [V_{a,b}, c] &= (c \cdot \pi(a)^\sigma - a \cdot h(a, c)) + (c \cdot \pi(b)^\sigma - b \cdot h(b, c)) \\ &\quad - (c \cdot \pi(a + b)^\sigma - (a + b) \cdot h(a + b, c)) - \gamma(a, b)c. \end{aligned} \quad (91)$$

By (80) and Corollary 5.35, we have

$$c \cdot \pi(a + b)^\sigma = c \cdot \pi(a)^\sigma + c \cdot \pi(b)^\sigma - c \cdot h(b, a) - \gamma(a, b)c.$$

Substituting this expression in (91), the result follows. $\square$

Remark 5.51. The notation $V_{a,b}$ that we use is not a coincidence: the expression for $[V_{a,b}, c]$ is exactly the expression for the V -operators in the theory of structurable algebras (in characteristic $\neq 2, 3$) related to quadrangular algebras. See [BDM13, Theorem 5.4].

We have now arrived at the most difficult identity, which we will prove in Proposition 5.55 below. The crucial ingredient will be Lemma 5.53, proving the equality of certain products of automorphisms, which is, in fact, a typical ‘long commutator relation’ in disguise.

Similarly to Proposition 5.28 and Definition 5.29, we will need exponential maps for elements of X' , with respect to the L'_i -grading (55).

Definition 5.52.

- (i) For each $[a, f] \in X' \leq L'_{-1}$, we write $\hat{\Theta}_a$ for the unique $[a, f]$ -exponential automorphism $\alpha \in \text{Aut}(L)$ such that $[q_\alpha(e'), [d, \delta]] = 0$. The corresponding (uniquely defined) maps q_α , n_α and v_α will be denoted by $\hat{q}_a$, $\hat{n}_a$ and $\hat{v}_a$, respectively.
- (ii) For each $a \in X$ and each $v \in V$, we define $\hat{\theta}(a, v) := \hat{q}_a([d, v]) \in V$. In particular, we have

$$\hat{\Theta}_a([d, v]) = [d, v] + [[a, f], [d, v]] + \hat{\theta}(a, v) = [d, v] + a \cdot v^\sigma + \hat{\theta}(a, v), \quad (92)$$

where the second equality holds by Lemma 5.32 since $f' = [d, f]$.

For the next lemma, recall that $\alpha_{\theta(a,v)}$ and β_v have been introduced in Definitions 5.15 and 5.16, respectively.

Lemma 5.53. *Let $a \in X$ and $v \in V$. Then there is a (unique) $\mu =: \mu(a, v) \in k$ such that*

$$\exp(-\mu c) \Theta_a \beta_v = \beta_v \hat{\Theta}_{a \cdot v} \alpha_{\theta(a,v)} \Theta_a.$$

Proof. Consider the automorphism $\beta := \Theta_a \beta_v \Theta_a^{-1}$ and recall that β_v is a $[y, v]$ -exponential automorphism with respect to the L'_i -grading (55). Write

$$\beta_v = \text{id} + \text{ad}_{[y,v]} + q_{[y,v]} + n_{[y,v]} + v_{[y,v]}.$$

Now let $l_i \in L'_i$. Since Θ_a preserves the L'_i -grading (55), also $\Theta_a^{-1}(l_i) \in L'_i$, so we can write

$$\beta_v \Theta_a^{-1}(l_i) = \underbrace{\Theta_a^{-1}(l_i)}_{\in L'_i} + \underbrace{[[y, v], \Theta_a^{-1}(l_i)]}_{\in L'_{i-1}} + \underbrace{q_{[y,v]}(\Theta_a^{-1}(l_i))}_{\in L'_{i-2}}.$$

Hence,

$$\beta(l_i) = \underbrace{l_i}_{\in L'_i} + \underbrace{[\Theta_a([y, v]), l_i]}_{\in L'_{i-1}} + \underbrace{\Theta_a(q_{[y,v]}(\Theta_a^{-1}(l_i)))}_{\in L'_{i-2}}.$$

This shows that β is an l -exponential automorphism for $l = \Theta_a([y, v])$, with respect to the L'_i -grading (55). Now

$$l = [y, v] + [a \cdot v, f] + \theta(a, v),$$

so by Theorem 3.13(viii), also $\beta_v \hat{\Theta}_{a \cdot v} \alpha_{\theta(a,v)}$ is an l -exponential automorphism. (Notice that we have implicitly used Proposition 5.17 for $\alpha_{\theta(a,v)}$ here.) The result now follows from Theorem 3.13(v). $\square$

Definition 5.54. Let $a \in X$ and $v \in V$. Then we set

$$\phi(a, v) := \mu(a, v) - \mu(a \cdot v, e).$$

Proposition 5.55. *Let $a \in X$ and $v, w \in V$. Then*

$$\hat{\theta}(a, v) = \theta(a, v^\sigma)^\sigma + \mu(a, e)v$$

and

$$\theta(a \cdot v, w^\sigma)^\sigma = Q(v)\theta(a, w) - T(v, w)\theta(a, v) + T(\theta(a, v), w)v + \phi(a, v)w.$$

Proof. We apply both sides of Lemma 5.53 to $[d, w] \in V'$, and we extract the V -component of the resulting equality (so there are many terms that we do not need to compute explicitly). We first compute

$$\begin{aligned} \beta_v([d, w]) &= [\beta_v(d), \beta_v(w)] \\ &= [d + [[y, v], d] + Q(v)y, w + [[y, v], w]] \\ &= [d + [[y, v], d] + Q(v)y, w + T(v, w)c] \quad (\text{by Proposition 5.25}) \\ &= [d, w] + [[y, v], [d, w]] + [[y, v], d], T(v, w)c + [Q(v)y, w] \\ &= [d, w] + [[y, v], [d, w]] + [y, Q(v)w - T(v, w)v]. \end{aligned}$$

After applying $\exp(-\mu(a, v)c)\Theta_a$ on $\beta_v([d, w])$, we see from the grading in Figure 2 that only two terms contribute to the V -component, namely

$$\exp(-\mu(a, v)c)([d, w]) \quad \text{and} \quad \Theta_a([y, Q(v)w - T(v, w)v]),$$

and the resulting V -component is equal to

$$\mu(a, v)w + Q(v)\theta(a, w) - T(v, w)\theta(a, v). \quad (93)$$

We now compute the V -component of

$$\begin{aligned} \beta_v \hat{\Theta}_{a \cdot v} \alpha_{\theta(a, v)} \Theta_a([d, w]) &= \beta_v \hat{\Theta}_{a \cdot v}([d, w] + [\theta(a, v), [d, w]]) \\ &= \beta_v \hat{\Theta}_{a \cdot v}([d, w] - T(\theta(a, v), w)x). \end{aligned}$$

Again, only two terms contribute, namely

$$\hat{\Theta}_{a \cdot v}([d, w]) \quad \text{and} \quad \beta_v(-T(\theta(a, v), w)x),$$

and the resulting V -component is equal to

$$\hat{\theta}(a \cdot v, w) - T(\theta(a, v), w)v. \quad (94)$$

Since the expressions in (93) and (94) coincide, we get

$$\hat{\theta}(a \cdot v, w) = Q(v)\theta(a, w) - T(v, w)\theta(a, v) + T(\theta(a, v), w)v + \mu(a, v)w. \quad (95)$$

Setting $v = e$ and invoking Corollary 5.42 then gives

$$\begin{aligned} \hat{\theta}(a, w) &= \theta(a, w) - T(e, w)\pi(a) + T(\pi(a), w)e + \mu(a, e)w \\ &= \theta(a, w^\sigma)^\sigma + \mu(a, e)w. \end{aligned}$$

Finally, substituting $a \cdot v$ for a in this identity gives

$$\hat{\theta}(a \cdot v, w) = \theta(a \cdot v, w^\sigma)^\sigma + \mu(a \cdot v, e)w.$$

Comparing this with (95) gives the required identity. $\square$

We have now assembled all that is required to prove our main result of this section.

Theorem 5.56. *Let L be as in Assumption 5.1, let V and X be as in Notation 5.4, let Q and T be as in Definition 5.10, let e and δ be as in Definition 5.21, let h and $\cdot$ be as in Definition 5.26 and let θ be as in Definition 5.29.*

Then the system $(k, V, Q, T, e, X, \cdot, h, \theta)$ is a quadrangular algebra, which is δ -standard and for which both T and the bilinear map $X \times X \rightarrow k: (a, b) \mapsto T(h(a, b), e)$ are nondegenerate (so in particular, h is nodegenerate).

Proof. We first observe that $Q: V \rightarrow k$ is a regular quadratic form with associated bilinear form T , by Definition 5.10 and Lemma 5.18, so in fact, T is nondegenerate. By Definition 5.21, $e \in V$ is a base point for Q . By definition, the maps $\cdot: X \times V \rightarrow X$ and $h: X \times X \rightarrow V$ defined in Definition 5.26 are bilinear maps. Finally, the map $\theta: X \times V \rightarrow V$ has been introduced in Definition 5.29.

We now verify each of the defining axioms (i)–(ix) from Definition 1.17.

- (i) This is Lemma 5.27(i).
- (ii) This is Proposition 5.33.
- (iii) This is Proposition 5.37.

- (iv) This is Proposition 5.36.
- (v) This follows from the definition of θ in Definition 5.29 because for each $a \in X$, Θ_a is a linear map.
- (vi) This was observed already in Definition 5.29.
- (vii) This is Proposition 5.31.
- (viii) This is Proposition 5.55.
- (ix) This is Proposition 5.45.

This shows that $(k, V, Q, T, e, X, \cdot, h, \theta)$ is a quadrangular algebra. By Definition 5.21 and (77), it is δ -standard. Finally, the bilinear map $X \times X \rightarrow k: (a, b) \mapsto T(h(a, b), e)$ is nondegenerate by Lemma 5.34. $\square$

Remark 5.57. The conditions on the quadrangular algebra in the statement of Theorem 5.56 imply that if $X \neq 0$, then either the quadrangular algebra is ‘generic’ (as in [MW19, §8]) or it is ‘of split type F_4 ’ (as in [MW19, §10]). (We ignore the detail that $|K| > 4$ is required in the generic case in [MW19, Hypothesis 8.1] and $|K| > 3$ is required in the split F_4 -case in [MW19, Hypothesis 10.1].) Notice, however, that we allow the situation where $X = 0$, in which case the quadrangular algebra is nothing more than a nondegenerate quadratic space with base point.

Remark 5.58. When the quadrangular algebra is anisotropic, it ought to be possible to show that the resulting *inner ideal geometry* is a Moufang generalized quadrangle and that it is isomorphic to the Moufang quadrangle obtained from the quadrangular algebra as in [Wei06]. (Notice that in this case, the *extremal geometry* has no lines. The inner ideal geometry is the geometry with point set $\mathcal{E}(L)$ and as line set the collection of inner ideals containing at least two extremal points and minimal with respect to this property.)

The explicit computations might be lengthy in general, but in the case that $\text{char}(k) \neq 2$, this has been worked out in [Meu21, §4.5.2].

Remark 5.59. We are confident that, similarly to Corollary 4.17, the Lie algebra can be completely reconstructed from the quadrangular algebra, but we have not worked out the details. This time, we let

$$K_0 := \langle [x, y], [c, d] \rangle \oplus [X, [y, X]] \oplus [V, [y, V']],$$

and we follow exactly the same procedure as in the proof of Corollary 4.17. However, working out all the different cases of Lie brackets between the 13 different pieces in the grading requires substantially more computational effort than in the case of Section 4, and one crucial ingredient seems to be Proposition 5.50. We leave the details to the courageous reader.

Remark 5.60. As we can see, there is a third interesting grading on the Lie algebra L that we obtain from the diagonals in Figure 2. (There is, of course, also a fourth grading from the other diagonals.) Notice that this time, the ends of the grading are not one-dimensional! More precisely, we have

$$L = L''_{-2} \oplus L''_{-1} \oplus L''_0 \oplus L''_1 \oplus L''_2$$

with

$$\begin{aligned} L''_{-2} &= \langle x \rangle \oplus V \oplus \langle c \rangle \\ L''_{-1} &= X \oplus X' \\ L''_0 &= V' \oplus (L_0 \cap L'_0) \oplus [y, V] \\ L''_1 &= [d, X'] \oplus [y, X] \\ L''_2 &= \langle d \rangle \oplus [y, V'] \oplus \langle y \rangle. \end{aligned}$$

This grading has been used implicitly in [BDM15] in the context of J -ternary algebras. See in particular its Theorem 3.5, where the Jordan algebra J is of the form $k \oplus V \oplus k$, and where the space called X plays the role of $L''_{\pm 1}$ and admits a Peirce decomposition into two isomorphic parts $X = X_0 \oplus X_1$ (introduced in its Lemma 3.4).

A. Tables

In this appendix, we present an overview of the possible Lie algebras of *exceptional type* that arise in this context. For each of the two situations (cubic norm structures vs. quadrangular algebras), we present two tables:

- We first list the dimensions of the different pieces in the decomposition of the Lie algebra, depending only on the absolute type of the Lie algebra.
- We then present a more detailed table with the possible Tits indices along with the corresponding precise form of the algebraic structure.

We do not provide proofs, and we rely instead on the corresponding information for Tits hexagons (taken from [MW22]) and Tits quadrangles (taken from [MW19]), respectively. Providing a direct connection between Tits hexagons and Tits quadrangles of index type and our Lie algebras is an interesting project in its own right. (See also Remarks 4.25 and 5.58.)

A.1. Cubic norm structures

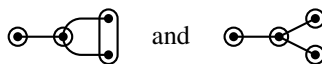
In the case of cubic norm structures, corresponding to the G_2 -grading as in Figure 1 on p. 32, we see that

$$\dim L = 6 + 6 \dim J + \dim(L_0 \cap L'_0).$$

We have listed the six possibilities in Table 1.

In Table 2, we rely on [MW22, Theorem 2.5.22], and we adopt its notation $\mathcal{H}(C, k)$ for the Freudenthal algebra constructed from the composition algebra C over k . (See also [MW22, Notation 4.1.72].)

Remark A.1. The example of type D_4 in the second row of Table 2 also occurs in non-triality forms, which we have not included in the list because they are not exceptional. These correspond to the Tits indices



and have corresponding 3-dimensional J of the form $J = k \times E$, where E/k is a separable quadratic field extension, for the first case, and of the form $J = k \times k \times k$, a split cubic étale extension, for the second case.

A.2. Quadrangular algebras

In the case of quadrangular algebras, corresponding to the BC_2 -grading as in Figure 2 on p. 42, we see that

$$\dim L = 4 + 4 \dim X + 4 \dim V + \dim(L_0 \cap L'_0).$$

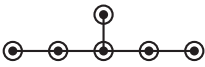
We have listed the four possibilities in Table 3.

By Remark 5.57, our quadrangular algebras are either generic or of split type F_4 . By [MW19, Theorems 8.16 and 10.16], these are either *special* (i.e., not exceptional), or they are anisotropic, or they are isotopic to $\mathcal{Q}_2(C, k)$ for some octonion algebra C or to $\mathcal{Q}_4(C, k)$ for some composition algebra C . (The definition of these quadrangular algebras can be found in [MW19, Notations 4.12 and 4.14].)

Table 1. Dimensions of the pieces for the G_2 -grading.

	$\dim L$	$\dim J$	$\dim(L_0 \cap L'_0)$
G_2	14	1	2
D_4	28	3	4
F_4	52	6	10
E_6	78	9	18
E_7	133	15	37
E_8	248	27	80

Table 2. Exceptional Tits indices with G_2 -graded Lie algebra

Tits index	rel. type	$\dim J$	Cubic norm structure (up to isotopy)
	G_2	1	$J = k$
	G_2	3	J/k sep. cubic field ext.
	F_4	6	$\mathcal{H}(k, k)$
	G_2	9	cubic division algebra
	G_2	9	cubic division algebra over E/k with involution of the second kind
	F_4	9	$\mathcal{H}(E, k)$, E/k sep. quadr. field ext.
	E_6	9	$M_3(k)$
	F_4	15	$\mathcal{H}(\mathcal{Q}, k)$, $\mathcal{Q}$ quaternion division
	E_7	15	$\mathcal{H}(\mathcal{Q}, k)$, $\mathcal{Q}$ quaternion split
	G_2	27	Albert division algebra
	F_4	27	$\mathcal{H}(\mathcal{O}, k)$, $\mathcal{O}$ octonion division
	E_8	27	$\mathcal{H}(\mathcal{O}, k)$, $\mathcal{O}$ octonion split

In other words, we get all cases from [MW19, Table 1] except the last two rows. We have essentially reproduced this table in Table 4.

Remark A.2. Comparing Tables 2 and 4, we see that the Tits indices that occur in *both* tables are the following:

- (i) The split forms of type F_4 , E_6 , E_7 and E_8 ;
- (ii) The forms of *relative* type F_4 (and of *absolute* type F_4 , E_6 , E_7 and E_8).

Table 3. Dimensions of the pieces for the BC_2 -grading.

	$\dim L$	$\dim V$	$\dim X$	$\dim(L_0 \cap L'_0)$
F_4	52	4	5	12
E_6	78	6	8	18
E_7	133	8	16	33
E_8	248	12	32	68

Table 4. Exceptional Tits indices with BC_2 -graded Lie algebra

Tits index	rel.type	$\dim V$	$\dim X$	Quadrangular algebra (up to isotopy)
	F_4	4	5	$\mathcal{Q}_4(k, k)$
	BC_2	6	8	anisotropic of type E_6
	F_4	6	8	$\mathcal{Q}_4(E, k)$, E/k sep. quadr. field ext.
	E_6	6	8	$\mathcal{Q}_4(k \times k, k)$
	BC_2	8	16	anisotropic of type E_7
	C_3	8	16	$\mathcal{Q}_2(\mathcal{O}, k)$, $\mathcal{O}$ octonion division
	F_4	8	16	$\mathcal{Q}_4(Q, k)$, Q quaternion division
	E_7	8	16	$\mathcal{Q}_4(Q, k)$, Q quaternion split
	BC_2	12	32	anisotropic of type E_8
	F_4	12	32	$\mathcal{Q}_4(\mathcal{O}, k)$, $\mathcal{O}$ octonion division
	E_8	12	32	$\mathcal{Q}_4(\mathcal{O}, k)$, $\mathcal{O}$ octonion split

In other words, these seven types of Lie algebras are precisely the ones admitting both a G_2 -grading and a BC_2 -grading, and hence, they are parametrized both by a cubic norm structure and by a quadrangular algebra. As we have alluded to in the introduction, it is an interesting question to investigate the connection between these two different algebraic structures for each of these seven types.

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The first author has presented⁴ a preliminary version of these results at the Collège de France on the occasion of the ‘Colloque pour honorer la mémoire de Jacques Tits’ (11–13 December 2023). We thank the organizers and the Fondation Hugo du Collège de France for the invitation.

⁴A video of this presentation is available at https://www.youtube.com/watch?v=fr3JaJ_ZHsw.

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Competing interest. The authors have no competing interests to declare.

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