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A SPECTRAL THEOREM FOR HERMITIAN OPERATORS OF MEROMORPHIC TYPE ON BANACH SPACES

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1. **Introduction.** It is well known that if T is a compact self-adjoint operator on a Hilbert space whose distinct non-zero eigenvalues $\{\lambda_n\}$ are arranged so that $|\lambda_n|\geq|\lambda_{n+1}|$ for $n=1, 2.\ldots$ and if E_n in the spectral projection corresponding to λ_n, then $T=\sum_{n=1}^{\infty}\lambda_nE_n$, with convergence in the uniform operator topology. With the generalisation of self-adjoint operators on Hilbert spaces to Hermitian operators on Banach spaces by Vidav and Lumer, Bonsall gave a partial analogue of this result for Banach spaces when he proved the following theorem. Let T be a compact Hermitian operator on a Banach space. Let $\{\lambda_n\}$ be an enumeration of the distinct non-zero eigenvalues of T such that $|\lambda_n|\geq|\lambda_{n+1}|$ for $n=1, 2, \ldots$. Let E_n be the spectral projection corresponding to λ_n. Then $T=\sum_{n=1}^{\infty}\lambda_nE_n$, with convergence in the uniform operator topology if each E_n is Hermitian or if $\lambda_n=0(1/n)$. Since no example of a compact Hermitian operator on a Banach space is known for which the expansion does not hold, it is natural to ask if the expansion might not indeed hold for all compact Hermitian operators on Banach spaces. We prove the following result. Let T be a Hermitian operator of meromorphic type on a complex Banach space X. Let $\{\lambda_n\}$ be an enumeration of its distinct non-zero eigenvalues. Let E_n be the spectral projection corresponding to λ_n. If T has minimal uniform index 1 relative to $\{\lambda_n\}$ (a theorem by Derr and Taylor shows that this requirement is equivalent to the convergence of the series $\sum_{n=1}^{\infty}\lambda_nE_n$ in the uniform operator topology), we have

(i) $T^2=\sum_{n=1}^{\infty}\lambda_n^2E_n$;
(ii) $T=\sum_{n=1}^{\infty}\lambda_nE_n-\lim_{\alpha\to 0}(i\alpha)\sum_{n=1}^{\infty}\lambda_nE_n/(ia-\lambda_n)$
(iii) $T=\sum_{n=1}^{\infty}\lambda_nE_n$, if X is reflexive.

The key to all these results is theorem 3.4: Let T be a Hermitian operator on a Banach space. If $T=A+B$, with $AB=BA=0$ and B quasi-nilpotent, then $B^2=0$.

2. Notation, definitions and preliminary results

2.1 *Notation.* If X denotes a complex Banach space, then $B(X)$ will denote the algebra of bounded operators on X. Also $\rho(T)$, $\sigma(T)$ will denote the resolvent set

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and spectrum respectively of an operator T, and $R(\lambda; T)=(\lambda I-T)^{-1}$ will denote the resolvent of T. If λ_n is an isolated singularity of $R(\lambda; T)$, then E_n will denote the residue of $R(\lambda; T)$ at λ_n.

2.2 *Semi-inner product spaces.* A *semi-inner product* on a linear space Y is a mapping $[.,.]$ from $Y\times Y$ into the complex field, such that

2.2.1 $[x+y, z]=[x, z]+[y, z]$

$[\lambda x, y]=\lambda[x, y]$, for all complex λ and all $x, y, z \in Y$.

2.2.2 $[x, x]>0$ if $x\neq 0$, $x \in Y$.

2.2.3 $|[x, y]|^2\leq [x, x][y, y]$, x and $y \in Y$.

A semi-inner product space is a linear space on which a semi-inner product is defined. These definitions were introduced by Lumer [6] who also proved the following results:

2.2.4 Every semi-inner product space is normed by $\|x\|=[x, x]^{1/2}$
2.2.5 Every normed linear space can be made into a semi-inner product space in such a way that the norm induced by the semi-inner product on the space coincides with the norm on the space [in general this can be done in infinitely many ways].

2.3 *The numerical range of an operator T on a Banach Space.* Let T be an operator on a Banach space X. The set of complex numbers

$$W(T) = \{[Tx, x]:[x, x] = 1, x \in X\},$$

where $[.,.]$ is some semi-inner product on X, is called the *numerical range* of T with respect to that semi-inner product. This definition is due to Lumer, who also proved the following results:

2.3.1 The closed convex hull of the numerical range of an operator on a Banach space is independent of the semi-inner product used as long as it is consistent with the norm on the space.
2.3.2 In particular, if the numerical range of an operator is real for any one such semi-inner product, then it is real for all others.

2.4 *Hermitian operators on a Banach space.* An operator T on a complex Banach space X is said to be *hermitian* if and only if its numerical range with respect to some semi-inner product on X consistent with the norm is real. By 2.3.2, this concept of hermiticity does not depend on the semi-inner product. This definition is due to Lumer who also proved the following result:

2.4.1 An operator T on a Banach space is hermitian if and only if $\|I+i\alpha T\|=1+0(\alpha)$ as $\alpha\to 0$, where α is real and I is the identity operator on X. This result shows the equivalence of Lumer's definition of hermiticity and that given by Vidav [13].

2.5 *Operators of meromorphic type.* Let X be a complex Banach space and let T be a bounded linear operator on X such that $\sigma(T)$ is a denumerable set of points with $\lambda=0$ as the only point of accumulation. If every non-zero point of $\sigma(T)$ is a pole of $R(\lambda; T)$, then T is called *an operator of meromorphic type.* For the basic properties of such operators, see [10], [11] and [12].

3. Expansion in terms of spectral projections

3.1. THEOREM. *Let T be a hermitian operator on a Banach space. Then $\|(s+ir-T)^{-1}\|\leq 1/|r|$ for all real s and all real non-zero r.*

Proof. This result follows from the Hille-Yosida-Phillips theorem [3, p. 626], but we give here a simple proof in terms of semi-inner products, which is implicit in [7]. Let $t=s+ir$, $r\neq 0$ be a complex number. Since T is hermitian, its spectrum is on the real axis so that $t=s+ir$ is in the resolvent set of T. Let $(t-T)^{-1}y=z$, $\|y\|=1$. Then $y=(t-T)z$ and so

$$\begin{aligned}\|z\|\cdot\|y\| &= \|(t-T)z\|\cdot\|z\| \\ &\geq |[(t-T)z, z]| \\ &= \|z\|^2\left|t-\frac{1}{\|z\|^2}[Tz, z]\right|\end{aligned}$$

Thus

$$\begin{aligned}\|y\| &\geq \|z\|\left(\left(s-\frac{1}{\|z\|^2}[Tz, z]\right)^2+r^2\right)^{1/2} \\ &\geq \|z\|\cdot|r|.\end{aligned}$$

Hence

$$\|z\| \leq \|y\|/|r| = 1/|r|.$$

Thus

$$\|(t-T)^{-1}\| \leq 1/|r|.$$

3.2. REMARK. (Atkinson): The above inequality implies that for a Hermitian operator, any pole of $R(\lambda, T)$ must be simple.

3.3. THEOREM. *Let X be a reflexive Banach space and suppose that $T\in B(X)$ is hermitian. Then $X=\overline{TX}\oplus\{x \mid x\in X, Tx=0\}$ and for any real sequence $\lambda_n\to 0$, $\lambda_n\neq 0$, the sequence $\{i\lambda_n R(i\lambda_n; T)\}$ converges in the strong operator topology to the projection T_0 whose null manifold is $\overline{TX}$ and whose range is $\{x \mid Tx=0\}$.*

Proof. This result follows immediately from theorem 3.1 and [3, Cor. 5, p. 597].

3.4. THEOREM. *Let T be a hermitian operator on a Banach space. If $T=A+B$, with $AB=BA=0$ and B quasinilpotent, then $B^2=0$.*

Proof. Since $AB=BA=0$, $T^n=A^n+B^n$ for all $n\geq 1$. Thus $BT^n=B(A^n+B^n)=B^{n+1}$. Hence $B(\lambda-T)^{-1}=B(\lambda-B)^{-1}$ for $|\lambda|$ greater than the spectral radius of T,

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by the power series expansion for resolvents. Thus since the intersection of the resolvent sets of T and B is connected, we have $B(\lambda-T)^{-1}=B(\lambda-B)^{-1}$ for all $\lambda\in\rho(B)\cap\rho(T)$

By theorem 3.1,

$$\|re^{i\theta}(re^{i\theta}-T)^{-1}\| \leq 1/|\sin\theta|,$$

for $r\sin\theta\neq 0$.
Thus

$$\begin{aligned}\|re^{i\theta}B(re^{i\theta}-B)^{-1}\| &= \|re^{i\theta}B(re^{i\theta}-T)^{-1}\| \\ &\leq \|B\|\cdot(1/|\sin\theta|),\end{aligned}$$

$r\sin\theta\neq 0$.
Hence for $r>0$, we have

$$|\sin\theta|\cdot\|re^{i\theta}B(re^{i\theta}-B)^{-1}\|\leq\|B\|$$

Now the function

$$F(\lambda) = (1/\lambda)B((1/\lambda)-B^{-1}) = B(1+\lambda B+\lambda^2B^2+\cdots)$$

is entire in λ, since B is quasi-nilpotent.

Also as shown above, $|\sin\theta|\cdot\|F(re^{i\theta})\|\leq\|B\|$.
Thus by [5, Lemma 3.13.1, p. 101], $F(\lambda)$ is a constant. Since $F(\lambda)=B+\lambda B^2+\cdots$, it follows that $B^2=0$.

3.5. Theorem. *Let T be a hermitian operator of meromorphic type on a reflexive Banach space. Let $\{\lambda_n\}$ be an enumeration of the distinct poles. Let E_n be the spectral projection corresponding to λ_n. If T has minimal uniform index* 1 *relative to $\{\lambda_n\}$, then $T=\sum_{n=1}^{\infty}\lambda_nE_n$, with convergence in the uniform operator topology.*

Proof. By 3.2 each λ_n is a simple pole of $R(\lambda, T)$, so that $F_n=(T-\lambda_n)E_n=0$. Hence by [12, Theorem 8, p. 96], $T=A+B$, where $A=\sum_{n=1}^{\infty}\lambda_nE_n$ and B is quasi-nilpotent with $AB=BA=0$. We must show that $B=0$. By theorem 3.4, $B^2=0$.

By theorem 3.3, $X=\overline{TX}\oplus\{x\in X: Tx=0\}$ so that it suffices to show that $Bz=0$ for z in TX and for z in the null space of T. If z is in TX, then for some $y\in X$, we have $z=Ty$. Hence $Bz=BTy=B(A+B)y=BAy+B^2y=0$. If $Tz=0$, then since $(\lambda_n-T)E_n=0$, we have $\lambda_nE_nz=TE_nz=E_nTz=0$.
Thus

$$Az = \sum_{n=1}^{\infty}\lambda_nE_nz = 0.$$

Hence

$$Bz = Tz-Az = 0.$$

3.6. Theorem. *Let T be a herimitian operator of meromorphic type on a Banach space. Let $\{\lambda_n\}$ be an enumeration of the poles. Let E_n be the spectral projection corresponding to λ_n. If T has minimal uniform index* 1 *relative to $\{\lambda_n\}$, then $T^2=\sum_{n=1}^{\infty}\lambda_n^2E_n$, with convergence in the uniform operator topology.*

Proof. By 3.2 each λ_n is a simple pole of $R(\lambda; T)$, so it follows that $F_n=(T-\lambda_n)E_n=0$.

Hence by [12, Theorem 8, p. 96], $T=A+B$, where $A=\sum_{n=1}^{\infty}\lambda_nE_n$ and B is quasi-nilpotent, with $AB=BA=0$. Hence by theorem 3.4, $B^2=0$. Thus $(T-\sum_{n=1}^{\infty}\lambda_nE_n)^2=0$. Hence $T^2-2T\sum_{n=1}^{\infty}\lambda_nE_n+\sum_{n=1}^{\infty}\lambda_n^2E_n=0$. Thus $T^2-2\sum_{n=1}^{\infty}\lambda_n^2E_n+\sum_{n=1}^{\infty}\lambda_n^2E_n=0$, since $TE_n=\lambda_nE_n$. Hence $T^2=\sum_{n=1}^{\infty}\lambda_2^nE_n$.

3.7. Corollary. *Let T be a hermitian operator of meromorphic type on a Banach space. Let $\{\lambda_n\}$ be an enumeration of the poles of $R(\lambda; T)$. Let E_n be the spectral projection corresponding λ_n. If T has minimal uniform index* 1 *relative to $\{\lambda_n\}$, then*

$$T=\sum_{n=1}^{\infty}\lambda_nE_n-\lim_{\alpha\to 0}(i\alpha)\sum_{n=1}^{\infty}\frac{\lambda_nE_n}{(i\alpha-\lambda_n)},$$

where α is real.

Proof. By 3.2 each λ_n is a simple pole of $R(\lambda, T)$ and so $F_n=(T-\lambda_n)E_n=0$. Hence by [12, theorems 1 and 5] and theorem 3.4,

$$R(\lambda; T)=\frac{I}{\lambda}+\sum_{n=1}^{\infty}\frac{\lambda_nE_n}{\lambda(\lambda-\lambda_n)}+\frac{B}{\lambda^2},$$

where $B=T-\sum_{n=1}^{\infty}\lambda_nE_n$ and $B^2=0$. Hence $B=\lambda^2(\lambda-T)^{-1}-I.\lambda-\lambda^2\sum_{n=1}^{\infty}(\lambda_nE_n)/\lambda(\lambda-\lambda_n)$. We put $\lambda=i\alpha$ for real α, use theorem 3.1 and let $\alpha\to 0$.
We get

$$B=-\lim_{\alpha\to 0}(i\alpha)^2\sum_{n=1}^{\infty}(\lambda_nE_n)/(i\alpha)(i\alpha-\lambda_n)$$

Hence

$$T=\sum_{n=1}^{\infty}\lambda_nE_n-\lim_{\alpha\to 0}(i\alpha)\sum_{n=1}^{\infty}(\lambda_nE_n)/(i\alpha-\lambda_n).$$

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