SUBdireCtLy IrRdeCuBLE SEMIRINGS AND SEMIGRouPS wItHouT nOnZerO NIlpOTENTS

BY
WILLIAM H. CORNISH

1. Introduction. It follows from [1, p. 377, Lemma 1] that a noncommutative subdirectly irreducible ring, with no nonzero nilpotent elements, cannot possess any proper zero-divisors. From [2, p. 193, Corollary 1] a subdirectly irreducible distributive lattice, with more than one element, is isomorphic to the chain with two elements. Hence we can say that a subdirectly irreducible distributive lattice with 0 possesses no proper zero-divisors.

In this paper we consider two generalizations of these results. Firstly, we show that there exists a commutative semiring with 0 and 1 having no nonzero nilpotents which is subdirectly irreducible and yet has proper zero-divisors. Secondly, it is proved that each subdirectly irreducible semigroup with 0 and no nonzero nilpotents cannot contain proper zero-divisors.

2. Semirings. A semiring is an algebra \((S, +, \cdot, 0)\) such that \((S, +)\) is a commutative semigroup, \((S, \cdot)\) is a semigroup, 0 is the zero, i.e. \(x + 0 = x\) and \(x \cdot 0 = 0 = 0 \cdot x\) for every \(x \in S\), and \(\cdot\) distributes over \(+\) from the left and the right. The rest of the terminology is used as in ring theory and universal algebra. In particular \(\omega\) and \(\iota\) respectively denote the smallest and largest congruences and a semiring is called simple if these are its only congruences.

Theorem 2.1. There exists a subdirectly irreducible commutative semiring which has no nonzero nilpotents and yet contains proper divisors of zero.

Proof. Let \(S = \{0, a, b, c, 1\}\). Define addition as the supremum in the lattice of Figure 1 and multiplication as the infimum in the lattice of Figure 2.

The distributive law holds so that \(S\) is a commutative semiring with 1 as the identity element. It has no nonzero nilpotents and \(a, b \neq 0\) while \(a \cdot b = 0\).

Besides \(\omega\) and \(\iota\) a routine computation shows that the only other congruences together with their associated partitions of \(S\) are: \(\Theta\) with partition \(\{0, a\}, \{b, c, 1\}\), \(\Phi\) with partition \(\{0, b\}, \{a, c, 1\}\), and \(\Theta \land \Phi\) with partition \(\{0\}, \{a\}, \{b\}, \{c, 1\}\).

Thus \(S\) is subdirectly irreducible since \(\Theta \land \Phi\) is the smallest congruence not equal to \(\omega\).
As a contrast we have the following positive result.

**Theorem 2.2.** A simple semiring with no nonzero nilpotents contains no proper divisors of zero.

**Proof.** Let $x$ be an arbitrary element of the semiring $S$. Define $\Phi_x$ by $y \equiv z (\Phi_x)$ if and only if $y + v = z + w$ for some $v, w \in J_x = \{ s \in S : sx = 0 \}$. As $S$ has no nonzero nilpotents $J_x = \{ s \in S : xs = 0 \}$, $\Phi_x$ is a congruence and $J_x = \{ s \in S : s \equiv 0 (\Phi_x) \}$. Since $S$ is simple, $\Phi_x$ is either $\omega$ or $\iota$. In the first case $J_x = \{ 0 \}$ so $x$ is a non-divisor of zero. In the second case $J_x = S$ whence $x^2 = 0$ so $x = 0$. Whence every nonzero element is a non-divisor of zero.

3. Semigroups period

**Theorem 3.1.** A subdirectly irreducible semigroup, with $0$ and no nonzero nilpotents, contains no proper zero-divisors.

**Proof.** Let $S$ be any semigroup with $0$ and no nonzero nilpotents. Though [1, p. 377, Lemma 1] is stated for rings it clearly applies to semigroups. Hence $S$ possesses a set of ideals, $\{ P_a : a \in I \}$, such that $xy \in P_a$ implies $x \in P_a$ or $y \in P_a$ for
For each $a \in I$ define $\Theta_a$ by $x \equiv y(\Theta_a)$ iff $x, y \in P_a$ or $x = y$. The following are easily verified: (i) each $\Theta_a$ is a congruence on $S$, (ii) for each $a$, the factor semigroup $S_a = S/\Theta_a$ is a semigroup with 0, and no proper zero divisors, (iii) $\wedge \Theta_a = \omega$ in the lattice of congruences. Hence, $S$ is a subdirect product of semigroups $S_a$ with 0 and no proper zero-divisors. If $S$ is subdirectly irreducible then $S$ must be isomorphic to some $S_a$.

**REFERENCES**


THE FLINDERS UNIVERSITY OF SOUTH AUSTRALIA,
BEDFORD PARK, AUSTRALIA