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FIXED POINTS OF ASYMPTOTICALLY REGULAR MULTIVALUED MAPPINGS

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Abstract

Some results on fixed point of asymptotically regular multivalued mapping are obtained in metric spaces. The structure of common fixed points and coincidence points of a pair of compatible multivalued mappings is also discussed. Our work generalizes known results of Aubin and Siegel, Dube, Dube and Singh, Hardy and Rogers, Hu, Iseki, Jungck, Kaneko, Nadler, Ray and Shiau, Tan and Wong.

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1. Introduction

Let T be a single valued self mapping on a metric space X. A sequence $\{x_n\}$ in X is said to be asymptotically T-regular if $d(x_n, Tx_n) \to 0$. The presence of a sequence $\{x_n\}$ for which $d(x_n, Tx_n) \to 0$ is related to some property of T (see [3], [7], [8], [24], [25], and [27]) and hence is exploited to obtain fixed points of T. The aim of the present paper is to bring out the thrust of a similar assumption for multivalued mappings. The weakly dissipative multivalued mappings recently introduced by Aubin and Siegel in [3] satisfy such an assumption. As stated in Aubin and Siegel [3], such fixed point theorems have application to control theory, system theory and optimization problems. Moreover, such a sequence (for multivalued mappings) has been used by Itoh and Takahashi [12] and Rhoades, Singh and Kulshrestha [26].

Rhoades [24], [25] had compared these contractive conditions. Most of the contractive conditions used imply the asymptotic regularity of the mappings under consideration, so the study of such mappings play an important role in fixed point theory.

In Section 2, some notation, definitions and facts, used in subsequent sections are listed. In Section 3 we prve the existence of a common fixed point of two multivalued mappings satisfying a contractive type condition in a metric space. In Section 4 a class of multivalued mappings is introduced which is larger (even in the case of single valued mappings) than those that Wong [32] refers to as Kannan mappings. The fixed point theorems therein are proved under less restrictive hypotheses and for wider classes than the results of Shiau, Tan and Wong [30]. In Section 5 we extend the idea of Jungck [13] to multivalued mappings and obtain a coincidence theorem for a pair of compatible multivalued mappings. The structure of common fixed points of these mappings is also studied.

2. Preliminaries

Let (X, d) be a metric space and let $CB(X)$ denote the family of all nonempty bounded closed subsets of X. For $A, B \in CB(X)$, let $H(A, B)$ denote the distance between A and B in Hausdorff metric, that is

$$H(A, B) = \begin{cases} \inf E_{A,B} & \text{if } E_{A,B} \neq \varnothing, \\ +\infty & \text{if } E_{A,B} = \varnothing, \end{cases}$$

where $N(\varepsilon, A) = \{x \in X : d(x, A) < \varepsilon\}$ and

$$E_{A,B} = \{\varepsilon > 0 : A \subseteq N(\varepsilon, B),\ B \subseteq N(\varepsilon, A)\}.$$

Let $T: X \to CB(X)$ be a mapping and $\{x_n\}$ a sequence in X. Then $\{x_n\}$ is said to be *asymptotically T-regular* if $d(x_n, Tx_n) \to 0$. Let $f: X \to X$ be a mapping such that $TX \subseteq fX$. Then $\{x_n\}$ is called *asymptotically T-regular with respect to f* if $d(fx_n, Tx_n) \to 0$. (cf. [27]). A point x is said to be a *fixed point* of a single valued mapping f (multivalued mapping T) provided $x = fx$ $(x \in Tx)$. The point x is called a *coincidence point* of f and T if $fx \in Tx$. We shall require the following well-known facts (cf. [21]).

LEMMA 2.1. *If $A, B \in CB(X)$ with $H(A, B) < \varepsilon$, then for each $a \in A$, there exists an element $b \in B$ such that $d(a, b) < \varepsilon$.*

LEMMA 2.2. *Let $\{A_n\}$ be a sequence in $CB(X)$ and $\lim_{n\to\infty} H(A_n, A) = 0$ for $A \in CB(X)$. If $x_n \in A_n$ and $\lim_{n\to\infty} d(x_n, x) = 0$, then $x \in A$.*

3. Common fixed point of multivalued generalized contractions

Wong [31] extended the result of Hardy and Rogers [9] by showing that two self mappings S and T on a complete metric space satisfying a contractive type condition have a common fixed point. In this section we extend this result of Wong to the case when S and T are multivalued and satisfy a more general contractive type condition.

THEOREM 3.1. *Let X be a complete metric space, $S\colon X \to CB(X)$ and $T\colon X \to CB(X)$. If there exists a constant α, $0 \le \alpha < 1$, such that for each $x, y \in X$,*

$$(1)\quad H(Tx, Sy) \le \alpha \max\{d(x, y), d(x, Sx), d(y, Ty), (d(x, Ty) + d(y, Sx))/2\}$$

then there exists a common fixed point of S and T.

PROOF. Assume that $\beta = \sqrt{\alpha}$.

Let x_0 be an arbitrary but fixed element of X and choose $x_1 \in Sx_0$. Then

$$H(Sx_0, Tx_1) < \beta \max\{d(x_0, x_1), d(x_0, Sx_0), d(x_1, Tx_1), (d(x_0, Tx_1) + d(x_1, Sx_0))/2\}.$$

Lemma 2.1 implies that there exists a point $x_2 \in Tx_1$ such that

$$\begin{aligned} d(x_1, x_2) &< \beta \max\{d(x_0, x_1), d(x_0, Sx_0), d(x_1, Tx_1), (d(x_0, Tx_1) + d(x_1, Sx_0))/2\} \\ &< \beta \max\{d(x_0, x_1), d(x_1, x_2), (d(x_0, x_1) + d(x_1, x_2))/2\}. \end{aligned}$$

If $d(x_1, x_2) > d(x_0, x_1)$, then $d(x_1, x_2) \le \beta d(x_1, x_2)$, a contradiction. Thus $d(x_1, x_2) < \beta d(x_0, x_1)$. Now

$$H(Tx_1, Sx_2) < \beta \max\{d(x_1, x_2), d(x_1, Tx_1), d(x_2, Sx_2), (d(x_1, Sx_2) + d(x_2, Tx_1))/2\}.$$

Again using Lemma 2.1, we obtain a point $x_3 \in Sx_2$ such that

$$\begin{aligned} d(x_2, x_3) &< \beta \max\{d(x_1, x_2), d(x_1, Tx_1), d(x_2, Sx_2), (d(x_1, Sx_2) + d(x_2, Tx_1))/2\} \\ &< \beta d(x_1, x_2). \end{aligned}$$

By induction we produce a sequence $\{x_n\}$ of points of X, such that, for $k \geq 0$,

$$x_{2k+1} \in Sx_{2k}, \qquad x_{2k+2} \in Tx_{2k+1}$$

and

$$\begin{aligned} d(x_n, x_{n+1}) &\leq \beta d(x_{n-1}, x_n) \\ &\leq \beta^n d(x_0, x_1). \end{aligned}$$

Furthermore, for $m > n$,

$$\begin{aligned} d(x_n, x_m) &\leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \cdots + d(x_{m-1}, x_m) \\ &\leq \{\beta^n + \beta^{n+1} + \cdots + \beta^{m-1}\} d(x_0, x_1). \end{aligned}$$

It follows that $\{x_n\}$ is a Cauchy sequence and there exists a point $t \in X$ such that $x_n \to t$. It further implies that $x_{2k+1} \to t$, and $x_{2k+2} \to t$. Thus we have,

$$\begin{aligned} d(t, St) &\leq d(t, x_{2k+2}) + d(x_{2k+2}, St) \\ &\leq d(t, x_{2k+2}) + H(Tx_{2k+1}, St) \\ &\leq d(t, x_{2k+2}) + \beta \max\{d(x_{2k+1}, t), d(t, St), \\ &\qquad d(x_{2k+1}, x_{2k+2}), (d(t, x_{2k+2}) + d(x_{2k+1}, St))/2\}. \end{aligned}$$

Letting $k \to \infty$, we have $d(t, St) \leq \beta d(t, St)$. Hence $t \in St$. Similarly,

$$d(t, Tt) \leq d(t, x_{2k+1}) + H(Sx_{2k}, Tt) \leq \beta d(t, Tt).$$

Therefore $t \in Tt$.

Corollary 3.2. *Let X be a complete metric space and $T: X \to CB(X)$. If there exists a constant α, $0 \leq \alpha < 1$, such that for each $x, y \in X$,*

$$\begin{aligned} H(Tx, Ty) \leq \alpha \max\{d(x, y), d(x, Tx), d(y, Ty), \\ (d(x, Ty) + d(y, Tx))/2\} \end{aligned}$$

then there exists a sequence $\{x_n\}$ which is asymptotically T-regular and converges to a fixed point of T.

Remark 3.3. Theorem 3.1 improved the results of Kaneko [20], which considered the mapping T of a reflexive Banach space X into the family of weakly compact subsets of X. The proximinality of the set Tx is a consequence of his assumption and it is used in his proof. No such assumption is required in Theorem 3.1.

Remark 3.4. In [22], Ray proved a fixed point theorem for a multivalued mapping $T: X \to CB(X)$ satisfying

$$\begin{aligned} H(Tx, Ty) \leq ad(x, y) + b(d(x, Tx) + d(y, Ty)) \\ + c(d(x, Ty) + d(y, Tx)), \end{aligned}$$

where a, b and c are non-negative real numbers and $0 < a + 2b + c < 1$. Theorem 3.1 (even in the particular case for $S = T$) is not a special case of the theorem of Ray [22] since T is not assumed to have closed graph. It also illustrates that the compactness of Tx is not necessary for the theorem of Aubin and Siegel [3].

Several other results may also be seen to follow as immediate corollaries to Theorem 3.1. Included among these are Dube [5, Theorem 1], Dube and Singh [6, Theorem 1], Iseki [11], Nadler [21, Theorem 5], Hardy and Rogers [9] and Wong [31].

4. Fixed point of Kannan type multivalued mappings

In this section we consider the mapping $T: X \to CB(X)$ satisfying the condition

$$H(Tx, Ty) \leq \alpha_1(d(x, Tx))d(x, Tx) + \alpha_2(d(y, Ty))d(y, Ty),$$

where $\alpha_i: \mathbb{R} \to [0, 1)$ $(i = 1, 2)$. Such a mapping T is not a special case of the mapping considered in Section 1. In 1968 Kannan [17] had established a fixed point theorem for a single valued mapping T defined on a complete metric space X satisfying

$$d(Tx, Ty) \leq \alpha(d(x, Tx) + d(y, Ty)),$$

where $0 < \alpha < \frac{1}{2}$ and $x, y \in X$. Within the context of a complete metric space the assumption $0 < \alpha < \frac{1}{2}$ is crucial even to the existence part of this result, but within a more restrictive yet quite natural setting, an elaborate fixed point theory exists for the case $\alpha = \frac{1}{2}$. Mappings of this wider class were studied by Kannan in [18]. In recent years, Beg and Azam [4], Shiau Tan and Wong [29] and Wong [32] have also studied such mappings.

THEOREM 4.1. *Let X be a complete metric space and $T: X \to CB(X)$ a mapping satisfying*

$$H(Tx, Ty) \leq \alpha_1(d(x, Tx))d(x, Tx) + \alpha_2(d(y, Ty))d(y, Ty), \tag{2}$$

for all $x, y \in X$, where $\alpha_i: \mathbb{R} \to [0, 1)$ $(i = 1, 2)$. If there exists an asymptotically T-regular sequence $\{x_n\}$ in X, then T has a fixed point x^* in X. Moreover $Tx_n \to Tx^*$.

PROOF. By hypothesis, we have

$$H(Tx_n, Tx_m) \leq \alpha_1(d(x_n, Tx_n))d(x_n, Tx_n) + \alpha_2(d(x_m, Tx_m))d(x_m, Tx_m).$$

Thus $\{Tx_n\}$ is a Cauchy sequence. Since $(CB(X), H)$ is complete (see [2]), there exists a $K^* \in CB(X)$, such that $H(Tx_n, K^*) \to 0$. Let $x^* \in K^*$. Then

$$\begin{aligned} d(x^*, Tx^*) &\le H(K^*, Tx^*) \\ &= \lim_{n\to\infty} H(Tx_n, Tx^*) \\ &\le \lim_{n\to\infty} (\alpha_1(d(x_n, Tx_n))d(x_n, Tx_n) + \alpha_2(d(x^*, Tx^*))d(x^*, Tx^*)) \\ &\le \alpha_2(d(x^*, Tx^*))d(x^*, Tx^*). \end{aligned}$$

It further implies that

$$(1 - \alpha_2(d(x^*, Tx^*)))d(x^*, Tx^*) \le 0.$$

Therefore $d(x^*, Tx^*) = 0$. Thus $x^* \in Tx^*$. Now,

$$\begin{aligned} H(K^*, Tx^*) &= \lim_{n\to\infty} H(Tx_n, Tx^*) \\ &\le \alpha_2(d(x^*, Tx^*))d(x^*, Tx^*) \\ &\le d(x^*, Tx^*) = 0. \end{aligned}$$

It follows that

$$Tx^* = K^* = \lim_{n\to\infty} Tx_n.$$

THEOREM 4.2. *Let X be a complete metric space and $T: X \to CB(X)$ a mapping satisfying (2). If there exists an asymptotically T-regular sequence $\{x_n\}$ in X and Tx_n is compact for each n, then each cluster point of $\{x_n\}$ is a fixed point of T.*

PROOF. Let $y_n \in Tx_n$ be such that $d(x_n, y_n) = d(x_n, Tx_n)$. Obviously, a cluster point of $\{x_n\}$ is a cluster point of $\{y_n\}$. If y^* is such a cluster point of $\{x_n\}$ and $\{y_n\}$, then with x^* (as in Theorem 4.1),

$$\begin{aligned} d(y_n, Tx^*) &\le H(Tx_n, Tx^*) \\ &\le \alpha_1(d(x_n, Tx_n))d(x_n, Tx_n) + \alpha_2(d(x^*, Tx^*))d(x^*, Tx^*) \\ &\le \alpha_1(d(x_n, Tx_n))d(x_n, Tx_n). \end{aligned}$$

Therefore $y^* \in Tx^*$. Now

$$\begin{aligned} d(y^*, Ty^*) &\le H(Tx^*, Ty^*) \\ &\le \alpha_1(d(x^*, Tx^*))d(x^*, Tx^*) + \alpha_2(d(y^*, Ty^*))d(y^*, Ty^*). \end{aligned}$$

It follows that $\{1 - \alpha_2(d(y^*, Ty^*))\}d(y^*, Ty^*) \le 0$. Hence $y^* \in Ty^*$.

THEOREM 4.3. *Let* X *be a complete metric space and* $T\colon X \to CB(X)$ *a mapping satisfying* (2) *with* $\alpha_1(d(x, Tx)) + \alpha_2(d(y, Ty)) \le 1$. *If* $\inf\{d(x, Tx)\colon x \in X\} = 0$ *then* T *has a fixed point.*

PROOF. It is sufficient to show that there exists an asymptotically T-regular sequence $\{x_n\}$ in X.

Let x_0 be an arbitrary but fixed element of X. Consider the sequence $\{x_n\}$, $x_n \in Tx_{n-1}$. The inequality (2) implies that

$$\begin{aligned} d(x_n, Tx_n) &\le H(Tx_{n-1}, Tx_n) \\ &\le \alpha_1(d(x_{n-1}, Tx_{n-1}))d(x_{n-1}, Tx_{n-1}) \\ &\qquad + \alpha_2(d(x_n, Tx_n))d(x_n, Tx_n) \\ &\le \frac{\alpha_1(d(x_{n-1}, Tx_{n-1}))}{1-\alpha_2(d(x_n, Tx_n))} d(x_{n-1}, Tx_{n-1}) \\ &\le d(x_{n-1}, Tx_{n-1}). \end{aligned}$$

It follows that the sequence $\{d(x_n, Tx_n)\}$ is decreasing. Therefore

$$d(x_n, Tx_n) \to \inf\{d(x_n, Tx_n)\colon n \in N\} \quad \text{and} \quad d(x_n, Tx_n) \to 0.$$

Hence $\{x_n\}$ is asymptotically T-regular.

Theorems 4.1, 4.2 and 4.3 generalize results of Shiau, Tan and Wong [30]. Here we desire to emphasize not only that our T belongs to a wider class of mappings but also that the hypothesis of compactness of Tx (in [30, Theorem 1]) is dropped.

5. Coincidence point of compatible multivalued mappings

Jungck [14] introduced a contraction condition for single valued compatible mappings on a metric space. He also pointed out in [15] and [16] the potential of compatible mappings for generalized fixed point theorems. Subsequently a variety of extensions, generalizations and applications of this followed; for example, see [1], [28] and [29]. This section is a continuation of these investigations for multivalued compatible mappings.

DEFINITION. Let X be a metric space. Mappings $T\colon X \to CB(X)$, $f\colon X \to X$ are *compatible* if, whenever there is a sequence $\{x_n\} \subset X$ satisfying $\lim fx_n \in \lim Tx_n$ (provided $\lim fx_n$ exists in X and $\lim Tx_n$ exists in $CB(X)$), then $\lim H(fTx_n, Tfx_n) = 0$.

If T is a single valued self mapping on X, this definition of compatibility becomes that of Jungck [14]. Let $X = \mathbb{R}$, with Euclidean metric, $Tx =$

$[x^2/4, x^2/2]$, $fx = x^2/8$. Then f and T are compatible but they do not commute.

Let $\varphi: (0, \infty) \to [0, 1)$ be a function having the following property (cf. [10], [23]):
(P) for $t > 0$, there exists $\delta(t) > 0$, $s(t) < 1$, such that

$$0 \leq r - t < \delta(t) \text{ implies } \varphi(r) \leq s(t).$$

The following Theorem is a generalization of Hu [10, Theorem 2], Jungck [13], Kaneko [19] and Nadler [21, Theorem 5].

THEOREM 5.1. *Let T be a mapping from a complete metric space X into $CB(X)$. Let $f: X \to X$ be a continuous mapping such that $TX \subseteq fX$. If f and T are compatible and for all $x, y \in X$,*

$$H(Tx, Ty) < \varphi(d(fx, fy))d(fx, fy), \tag{3}$$

then there exists a sequence $\{x_n\}$ which is asymptotically T-regular with respect to f, and fx_n converges to a coincidence point of f and T.

PROOF. Let x_0 be an arbitrary, but fixed element of X. We shall construct two sequences $\{x_n\}$ and $\{y_n\}$ of points of X as follow. Let $y_0 = fx_0$ and $x_1 \in X$ be such that $y_1 = fx_1 \in Tx_0$. Then inequality (3) implies that

$$H(Tx_0, Tx_1) < \varphi(d(fx_0, fx_1))d(fx_0, fx_1).$$

Using Lemma 2.1 and the fact that $TX \subseteq fX$, we may choose $x_2 \in X$ such that $y_2 = fx_2 \in Tx_1$ and

$$\begin{aligned} d(y_1, y_2) &= d(fx_1, fx_2) \\ &< \varphi(d(fx_0, fx_1))d(fx_0, fx_1) \\ &< d(fx_0, fx_1). \end{aligned}$$

By induction we produce two sequences of points of X such that $y_n = fx_n \in Tx_{n-1}$, $n \geq 0$. Furthermore,

$$\begin{aligned} d(y_{n+1}, y_{n+2}) &= d(fx_{n+1}, fx_{n+2}) \\ &< \varphi(d(fx_n, fx_{n+1}))d(fx_n, fx_{n+1}) \\ &< d(fx_n, fx_{n+1}) = d(y_n, y_{n+1}). \end{aligned}$$

It follows that the sequence $\{d(y_n, y_{n+1})\}$ is decreasing and converges to its greatest lower bound which we denote by t. Now $t \geq 0$; in fact $t = 0$. Otherwise by property (P) of φ, there exists $\delta(t) > 0$, $s(t) < 1$, such that,

$$0 \leq r - t < \delta(t) \quad \text{implies} \quad \varphi(r) \leq s(t).$$

For this $\delta(t) > 0$, there exists a natural number N such that,

$$0 \leq d(y_n, y_{n+1})) - t < \delta(t), \text{ whenever } n \geq N.$$

Hence

$$\varphi(d(y_n, y_{n+1})) \leq s(t), \text{ whenever } n \geq N.$$

Let $K = \max\{\varphi(d(y_0, y_1)), \varphi(d(y_1, y_2)), \dots, \varphi(d(y_{N-1}, y_N)), s(t)\}$. Then, for $n = 1, 2, 3, \dots$,

$$\begin{aligned} d(y_n, y_{n+1}) &< \varphi(d(y_{n-1}, y_n))d(y_{n-1}, y_n) \\ &\leq Kd(y_{n-1}, y_n) \\ &\leq K^n d(y_0, y_1) \to 0 \quad \text{as} \quad n \to \infty, \end{aligned}$$

which contradicts the assumption that $t > 0$. Consequently,

$$\lim_{n\to\infty} d(y_n, y_{n+1}) = 0,$$

which implies that $d(fx_n, Tx_n) \to 0$. Hence the sequence $\{x_n\}$ is asymptotically T-regular with respect to f.

Assume that $\{fx_n\}$ is not a Cauchy sequence. Then there exists a positive number t^* and subsequences $\{n(i)\}$, $\{m(i)\}$ of the natural numbers with $n(i) < m(i)$ and such that $d(y_{n(i)}, y_{m(i)}) \geq t^*$, $d(y_{n(i)}, y_{m(i)-1}) < t^*$ for $i = 1, 2, 3 \dots$. Then

$$\begin{aligned} t^* &\leq d(y_{n(i)}, y_{m(i)}) \\ &\leq d(y_{n(i)}, y_{m(i)-1}) + d(y_{m(i)-1}, y_{m(i)}). \end{aligned}$$

Letting $i \to \infty$ and using the fact that $d(y_{n(i)}, y_{m(i)-1}) < t^*$, we obtain $\lim_{i\to\infty} d(y_{n(i)}, y_{m(i)}) = t^*$. For this $t^* > 0$, there exists $\delta(t^*) > 0$, $s(t^*) < 1$, such that

$$0 \leq r - t^* < \delta(t^*) \quad \text{implies} \quad \varphi(r) \leq s(t^*).$$

For this $\delta(t^*) > 0$, there exists a natural number N_0 such that,

$$i \geq N_0 \text{ implies } 0 \leq d(y_{n(i)}, y_{m(i)}) - t^* < \delta(t^*).$$

Hence $\varphi(d(y_{n(i)}, y_{m(i)})) \leq s(t^*)$ for $i \geq N_0$. Thus

$$\begin{aligned} d(y_{n(i)}, y_{m(i)}) &\leq d(y_{n(i)}, y_{n(i)+1}) + d(y_{n(i)+1}, y_{m(i)+1}) + d(y_{m(i)+1}, y_{m(i)}) \\ &\leq d(y_{n(i)}, y_{n(i)+1}) + \varphi(d(y_{n(i)}, y_{m(i)}))d(y_{n(i)}, y_{m(i)}) \\ &\qquad + d(y_{m(i)+1}, y_{m(i)}) \\ &\leq d(y_{n(i)}, y_{n(i)+1}) + s(t^*)d(y_{n(i)}, y_{m(i)}) + d(y_{m(i)+1}, y_{m(i)}). \end{aligned}$$

Letting $i \to \infty$, we get $t^* \leq s(t^*)t^* < t^*$, a contradiction. Hence $\{fx_n\}$ is a Cauchy sequence. By completeness of the space, there exists an element $p \in$

X such that $d(y_n, p) \to 0$. Continuity of f implies that $d(fy_n, fp) \to 0$. Hence

$$\begin{aligned} H(Ty_n, Tp) &< \varphi(d(fy_n, fp))d(fy_n, fp) \\ &< d(fy_n, fp) \to 0. \end{aligned}$$

Inequality (3) and the fact that $\{fx_n\}$ is a Cauchy sequence imply that there exists $A \in CB(X)$ such that $Tx_n \to A$. Furthermore,

$$d(p, A) \le \lim_{n\to\infty} H(Tx_{n-1}, Tx_n) = 0.$$

Now

$$d(fy_{n+1}, Ty_n) \le H(fTx_n, Tfx_n).$$

Letting $n \to \infty$, we obtain $d(fp, Tp) = 0$. Hence $fp \in Tp$.

EXAMPLE 5.2. Let $X = [0, \infty)$ with the Euclidean metric $Tx = [0, x]$ and $fx = 10^4 x$. Then f and T do not satisfy the condition of the theorems in [10], [13] and [21]. Considering the function $\varphi(x) = c$, where $10^{-4} < c < 1$, it is easily seen that all the hypotheses of Theorem 5.1 are valid. Thus f and T have a coincidence point.

COROLLARY 5.3. *If, in addition to the hypotheses of Theorem* 5.1 *the mapping* f *satisfies, for all* $x, y \in X$,

(4)

$$d(fx, fy) \le \gamma \max\{d(x, y), d(x, fx), d(y, fy), (d(x, fy)+d(y, fx))/2\}$$

where $0 \le \gamma < 1$, *then there exists a common fixed point of* f *and* T.

PROOF. Let $\beta = \sqrt{\gamma}$. As in the proof of Theorem 5.1 there is a coincidence point p of f and T. Define the iterative sequence $\{t_n\}$ as follows: $t_0 = p$ and $t_n = ft_{n-1} = f^n t_0$, $n = 0, 1, 2\dots$. Now inequality (4) implies that

$$\begin{aligned} d(t_n, t_{n+1}) &= d(ft_{n-1}, ft_n) \\ &< \beta \max\{d(t_{n-1}, t_n), d(t_n, t_{n+1}), d(t_{n-1}, t_{n+1})/2\} \\ &< \beta d(t_{n-1}, t_n) \\ &< \beta^n d(t_0, t_1). \end{aligned}$$

It further implies that $\{t_n\}$ is a Cauchy sequence. By the completeness of X, we have $f^n t_0 \to x^* \in X$.

Now consider a constant sequence $\{u_n\} \subset X$ as follows: $u_n = t_0$. Then

$$\lim_{n\to\infty} fu_n = ft_0 \in Tt_0 = \lim_{n\to\infty} Tu_n.$$

Thus by the compatibility of f and T,

$$H(fTt_0, Tft_0) = \lim_{n\to\infty} H(fTu_n, Tfu_n) = 0.$$

Hence $f^2t_0 = fft_0 \in fTt_0 = Tft_0$. Choose another constant sequence, $v_n = ft_0$. Then

$$\lim_{n\to\infty} fv_n = f^2t_0 \in Tft_0 = \lim_{n\to\infty} Tv_n,$$

and

$$H(fTft_0, Tf^2t_0) = \lim_{n\to\infty} H(fTv_n, Tfv_n) = 0.$$

Thus $f^3t_0 = ff^2t_0 \in fTft_0 = Tf^2t_0$. Consequently, we have $f^{n+1}t_0 \in Tf^nt_0$. Using (3), we get $\lim_{n\to\infty} Tx_n = Tx^*$. Hence by Lemma 2.2, we obtain, $x^* \in Tx^*$. Moreover,

$$fx^* = f\lim_{n\to\infty} f^nt_0 = \lim_{n\to\infty} f^{n+1}t_0 = x^*.$$

Hence x^* is a common fixed point of f and T.

In Theorem 5.1 our hypothesis that f is continuous implies that T is continuous. And we use the continuity of f and T in our proof. In the next theorem we show that if fX is complete then the continuity and compatibility of f and T are not required.

THEOREM 5.4. *Let T be a mapping of a metric space X into $CB(X)$. Let $f: X \to X$ be a mapping such that $TX \subseteq fX$, fX is complete and the condition* (3) *is satisfied. Then*

(i) *there exists a sequence $\{x_n\}$ which is asymptotically T-regular with respect to f and*

(ii) *f and T have a coincidence point.*

PROOF. Examining the proof of Theorem 5.1, we see that the only change is that the completeness of fX allows us to obtain $z \in X$ such that $fx_n \to p = fz$. Then

$$\begin{aligned} d(fz, Tz) &\le d(fz, fx_{n+1}) + d(fx_{n+1}, Tz) \\ &\le d(fz, fx_{n+1}) + H(Tx_n, Tz) \\ &\le d(fz, fx_{n+1}) + \varphi(d(fx_n, fz))d(fx_n, fz) \\ &\le d(fz, fx_{n+1}) + d(fx_n, fz). \end{aligned}$$

Letting $n \to \infty$, we obtain

$$d(fz, Tz) \le d(fz, p) + d(p, fz) = 0.$$

Hence $fz \in Tz$.

COROLLARY 5.5. *Suppose that, in addition to the hypotheses of Theorem 5.4, f satisfies* (4) *and f and T are compatible. Then $\{fx_n\}$ converges to a coincidence point (say p) of f and T, and $\{f^n p\}$ converges to a common fixed point of f and T.*

PROOF. By Theorem 5.4, there exists $z \in X$ such that $fz \in Tz$. As in Corollary 5.3, compatibility of f and T implies that $ffz = ftz = Tfz$.

Since $fx_n \to fz$ (see Theorem 5.4), fx_n converges to a coincidence point of f and T.

Now, inequality (4) implies that $\{f^n z\}$ is a Cauchy sequence. Let $f^n z \to x^*$. Since (as in Corollary 5.3) $f^{n+1}z \in Tf^n z$, we have

$$\begin{aligned} d(x^*, Tx^*) &\le d(x^*, f^{n+1}z) + H(Tf^n z, Tx^*) \\ &\le d(x^*, f^{n+1}z) + \varphi(d(f^n z, x^*))d(f^n z, x^*) \\ &\le d(x^*, f^{n+1}z) + d(f^n z, x^*). \end{aligned}$$

Letting $n \to \infty$, we obtain $d(x^*, Tx^*) = 0$, i.e., $x^* \in Tx^*$. Moreover,

$$\begin{aligned} d(x^*, fx^*) &\le d(x^*, f^{n+1}z) + d(f^{n+1}z, fx^*) \\ &\le d(x^*, f^{n+1}z) \\ &\qquad + \gamma \max\{d(f^n z, x^*), d(f^n z, f^{n+1}z), d(x^*, fx^*), \\ &\qquad\qquad (d(f^n z, fx^*) + d(f^{n+1}z, x^*))/2\}. \end{aligned}$$

Letting $n \to \infty$, we have $d(x^*, fx^*) \le \gamma d(x^*, fx^*)$. Hence $x^* = fx^*$.

We show that the assumption of $TX \subseteq fX$ (Theorem 5.4) and compatibility of f and T (Corollary 5.5) cannot be dropped.

EXAMPLE 5.6. Let $X = \mathbb{R}$ with the Euclidean metric $Tx = [0, |x|/3]$, $fx = (x+3)/2$ and $\varphi(x) = 2/3$. Then all the hypotheses of Theorem 5.4 are satisfied and $f(-2) \in T(-2)$. Moreover f and T are not compatible, but the other assumptions of Corollary 5.5 are satisfied, $f^n(-2) \to 3$ and 3 is not a common fixed point of f and T.

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References

[1] A. Asad and S. Sessa, 'Common fixed points for nonself compatible maps on compacta' (to appear). Presented during the International Conference on Fixed Point Theory and its Application, Marseille (France), June 5–10, 1989.
[2] J. P. Aubin, *Applied abstract analysis* (John Wiley & Sons, New York, 1977).
[3] J. P. Aubin and J. Siegel, 'Fixed points and stationary points of dissipative multivalued maps', *Proc. Amer. Math. Soc.* **78** (1980), 391–398.
[4] I. Beg and A. Azam, 'Fixed point theorems for Kannan mappings', *Indian J. Pure and App. Math.* **17** (11) (1986), 1270–1275.
[5] L. S. Dube, 'A theorem on common fixed points of multivalued mappings', *Annal. Soc. Sci. Bruxells* **84** (4) (1975), 463–468.
[6] L. S. Dube and S. P. Singh, 'On multivalued contraction mapping', *Bull. Math. de'la Soc. Sci. Math. de la R. S. de Roumanie* **14**(62) (1970), 307–310.
[7] H. W. Engl, 'Weak convergence of asymptotically regular sequences for nonexpansive mappings and connections with certain Chebyshef-centers', *Nonlinear Anal.* **1**(5) (1977), 495–501.
[8] M. D. Guay and K. L. Singh, 'Fixed points of asymptotically regular mappings', *Math. Vesnik* **35** (1983), 101–106.
[9] G. E. Hardy and T. D. Rogers, 'A generalization of a fixed point theorem of Reich', *Canad. Math. Bull.* **16** (1973), 201–206.
[10] T. Hu, 'Fixed point theorems for multivalued mappings', *Canad. Math. Bull.* **23** (1980), 193–197.
[11] K. Iseki, 'Multivalued contraction mappings in complete metric spaces', *Rend. Sem. Math. Univ. Padova* **53** (1975), 15–19.
[12] S. Itoh and W. Takahashi, 'Single valued mappings, multivalued mappings and fixed point theorems', *J. Math. Anal Appl.*, **59** (1977), 514–521.
[13] G. Jungck, 'Commuting mappings and fixed points', *Amer. Math. Monthly* **83** (1976), 261–263.
[14] G. Jungck, 'Compatible mappings and common fixed points', *Internat. J. Math. and Math. Sci.* **9** (1986), 771–779.
[15] G. Jungck, 'Compatible mappings and common fixed points (2)', *Internat. J. Math. and Math. Sci.* **9** (1986), 285–288.
[16] G. Jungck, 'Common fixed points for commuting and compatible maps on compacta', *Proc. Amer. Math. Soc.* **103** (3) (1988), 977–983.
[17] R. Kannan, 'Some results on fixed points', *Bull. Calcutta Math. Soc.* **60** (1968), 71–76.
[18] R. Kannan, 'Fixed point theorems in reflexive Banach space', *Proc. Amer. Math. Soc.* **38** (1973), 111–118.
[19] H. Kaneko, 'Single valued and multivalued f-contraction', *Boll. U.M.I.* **44** (1985), 29–33.
[20] H. Kaneko, 'A comparison of contractive conditions for multivalued mappings', *Kobe J. Math.* **3** (1986), 37–45.
[21] S. B. Nadler, Jr., 'Multivalued contraction mappings', *Pacific J. Math.* **30** (1969), 475–480.
[22] B. K. Ray, 'On Ciric's fixed point theorem', *Fund. Math.* **94** (1977), 221–229.
[23] S. Reich, 'Fixed points of contractive functions', *Boll. U.M.I.* (4) A **5** (1972), 26–42.
[24] B. E. Rhoades, 'A comparison of various definitions of contractive mappings', *Trans. Amer. Math. Soc.* **226** (1977), 257–290.
[25] B. E. Rhoades, 'Contractive definitions revisited, Topological methods in nonlinear functional analysis', *Contemporary Math., Amer. Math. Soc.* **21** (1983), 189–205.
[26] B. E. Rhoades, S. L. Singh and C. Kulshrestha, 'Coincidence theorem for some multivalued mappings', *Internat. J. Math. and Math. Sci.* **7** (3) (1984), 429–434.

[27] B. E. Rhoades, S. Sessa, M. S. Khan and M. Swaleh, 'On fixed points of asymptotically regular mappings', *J. Austral. Math. Soc.* (*Series* A) **43** (1987), 328–346.
[28] B. E. Rhoades, S. Park and K. B. Moon, 'On generalization of the Meir-Keeler type contraction maps', *J. Math. Anal. Appl.* **146** (1990), 482–494.
[29] S. Sessa, B. E. Rhoades and M. S. Khan, 'On common fixed points of compatible mappings in metric and Banach spaces', *Internat. J. Math. and Math. Sci.* **11** (2) (1988), 375–392.
[30] C. Shiau, K. K. Tan and C. S. Wong, 'A class of quasi-nonexpansive multivalued maps', *Canad. Math. Bull.* **18** (1975), 709–714.
[31] C. S. Wong, 'Common fixed points of two mappings', *Pacific J. Math.* **48** (1973), 299–312.
[32] C. S. Wong, 'On Kannan maps', *Proc. Amer. Math. Soc.* **47** (1975), 105–111.

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