**4. The similar problem for cosine series.** Here we wish to determine the coefficients $b_n$ in the dual series

$$\left.\begin{aligned} \sum_{n=1}^{\infty} n^p b_n \cos nx &= P(x) \quad (0 < x < c), \\ \sum_{n=1}^{\infty} b_n \cos nx &= Q(x) \quad (c < x < \pi), \end{aligned}\right\} \qquad (9)$$

where $p = \pm 1$ and $P(x)$, $Q(x)$ are now prescribed. Writing

$$a_n = b_n/n, \quad F(x) = \int_0^x P(t)\, dt, \quad G(x) = -\int_x^\pi Q(t)\, dt, \qquad (10)$$

we easily see, by integrating with respect to $x$ between $0, x$ and $x, \pi$ respectively, that equations (9) reduce to equations (1) and this problem can be solved as before.

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# SOME TRIPLE INTEGRAL EQUATIONS

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**1. Introduction.** Potential problems in which different conditions hold over *two* different parts of the same boundary can often be conveniently reduced to the solution of a pair of dual integral equations. In some problems, however, the boundary condition is such that different conditions hold over *three* different parts of the boundary and, in such cases, the integral equations involved are frequently of the form

$$\left.\begin{aligned} \int_0^\infty \phi(u) J_\nu(ru)\, du &= f(r) \quad (0 < r < a), \\ \int_0^\infty u^{2p} \phi(u) J_\nu(ru)\, du &= g(r) \quad (a < r < b), \\ \int_0^\infty \phi(u) J_\nu(ru)\, du &= 0 \quad (b < r < \infty), \end{aligned}\right\} \qquad (1)$$

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where $f(r)$, $g(r)$ are specified functions of $r$, $p = \pm\frac{1}{2}$ and $\phi(u)$ is to be found. Such equations might well be called *triple integral equations* and, in this note, I point out certain special cases which I have found to be capable of solution in closed form.

**2. The reduction to dual series.** By taking

$$\phi(u) = u^{-p}\sum_{n=1}^{\infty} C_n J_{\nu+2n-1+p}(bu), \quad \ldots\ldots(2)$$

and using the result (Watson [**1** (*a*)]),

$$\int_0^\infty u^{-p} J_{\nu+2n-1+p}(bu) J_\nu(ru)\, du = 0 \quad (b < r), \quad \ldots\ldots(3)$$

the third of equations (1) is automatically satisfied. Also, when $0 < r < b$,

$$\int_0^\infty u^{p} J_{\nu+2n-1+p}(bu) J_\nu(ru)\, du = \frac{2^p r^\nu \Gamma(n+\nu+p)}{b^{\nu+1+p}\Gamma(\nu+1)\Gamma(n)} F\left(n+\nu+p,\ -n+1\,;\ \nu+1\,;\ \frac{r^2}{b^2}\right), \quad \ldots\ldots(4)$$

$$\int_0^\infty u^{-p} J_{\nu+2n-1+p}(bu) J_\nu(ru)\, du = \frac{2^{-p} r^\nu \Gamma(n+\nu)}{b^{\nu+1-p}\Gamma(\nu+1)\Gamma(n+p)} F\left(n+\nu,\ -n+1-p\,;\ \nu+1\,;\ \frac{r^2}{b^2}\right)$$

$$= \frac{2^{-p} r^\nu \Gamma(n+\nu)}{b^{\nu+1-p}\Gamma(\nu+1)\Gamma(n+p)} \left(1-\frac{r^2}{b^2}\right)^p F\left(-n+1,\, n+\nu+p\,;\ \nu+1\,;\ \frac{r^2}{b^2}\right), \quad \ldots\ldots(5)$$

the last step resulting from the use of the transformation formula for the hypergeometric function. The convergence of the integrals in (3), (4), (5) requires that $\mathrm{Re}(\nu) > -1$ when $p = \frac{1}{2}$, $\mathrm{Re}(\nu) > -\frac{1}{2}$ when $p = -\frac{1}{2}$ and I shall assume that these conditions apply in what follows.

Substituting from (4) and (5) in the first two of equations (1) and interchanging the order of integration and summation

$$\left.\begin{aligned} &\sum_{n=1}^{\infty} \frac{\Gamma(n+\nu)}{\Gamma(n+p)} C_n F\left(-n+1,\, n+\nu+p\,;\ \nu+1\,;\ \frac{r^2}{b^2}\right) = 2^p \Gamma(\nu+1) b^{\nu+1-p} r^{-\nu} \left(1-\frac{r^2}{b^2}\right)^{-p} f(r) \\ &\qquad\qquad (0 < r < a), \\ &\sum_{n=1}^{\infty} \frac{\Gamma(n+\nu+p)}{\Gamma(n)} C_n F\left(-n+1,\, n+\nu+p\,;\ \nu+1\,;\ \frac{r^2}{b^2}\right) = 2^{-p} \Gamma(\nu+1) b^{\nu+1+p} r^{-\nu} g(r) \\ &\qquad\qquad (a < r < b). \end{aligned}\right\} \quad \ldots\ldots(6)$$

These are a pair of dual series and the determination of the coefficients $C_n$ from these series would enable the solution of the triple integral equations (1) to be completed.

I have been unable to find closed expressions for the coefficients $C_n$ except in the special cases $\nu = \pm\frac{1}{2}$. In these cases, the dual series (6) reduce to dual trigonometrical series and the coefficients can then be found by the method I have recently given [**2**]. These special cases are of some importance in practice as they arise when solving boundary value problems involving long strips and an illustration is given below.

**3. An example.** As an example, I consider the electrostatic potential due to the two parallel and coplanar infinite strips $a < x < b, y = 0$ and $-b < x < -a, y = 0$ charged respectively to potentials $\pm 1$.

The potential $V$ has to satisfy

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0, \qquad \text{.......(7)}$$

and the boundary conditions

$$V = 0 \quad \text{when} \quad x = 0, \qquad \text{.......(8)}$$

$$\left.\begin{array}{l} V = 1 \quad (a < x < b), \\ \dfrac{\partial V}{\partial y} = 0 \quad (0 < x < a \quad \text{and} \quad b < x < \infty), \end{array}\right\} \text{when } y = 0. \qquad \text{.......(9)}$$

Equation (7) and the boundary condition (8) are satisfied by taking

$$V = \int_0^\infty \xi(u) e^{-|y|u} \sin xu \, du, \qquad \text{.......(10)}$$

and substitution in (9) then gives the triple integral equations

$$\left.\begin{array}{ll} \displaystyle\int_0^\infty u\xi(u) \sin xu \, du = 0 & (0 < x < a), \\ \displaystyle\int_0^\infty \xi(u) \sin xu \, du = 1 & (a < x < b), \\ \displaystyle\int_0^\infty u\xi(u) \sin xu \, du = 0 & (b < x < \infty), \end{array}\right\} \qquad \text{.......(11)}$$

for the determination of the unknown function $\xi(u)$.

Since $\sin xu = (\frac{1}{2}\pi xu)^{\frac{1}{2}} J_{\frac{1}{2}}(xu)$, equations (11) can be identified with equations (1) with $\nu = \frac{1}{2}, p = -\frac{1}{2}, r = x, f(x) = 0, g(x) = (2/\pi x)^{\frac{1}{2}}$, if we write

$$u^{\frac{3}{2}}\xi(u) = \phi(u). \qquad \text{.......(12)}$$

Hence, from equations (2) and (6),

$$\xi(u) = u^{-1} \sum_{n=1}^\infty C_n J_{2n-1}(bu), \qquad \text{.......(13)}$$

where the coefficients $C_n$ are given by the dual series

$$\left.\begin{array}{ll} \displaystyle\sum_{n=1}^\infty (n-\tfrac{1}{2}) C_n F\left(-n+1, n; \frac{3}{2}; \frac{x^2}{b^2}\right) = 0 & (0 < x < a), \\ \displaystyle\sum_{n=1}^\infty C_n F\left(-n+1, n; \frac{3}{2}; \frac{x^2}{b^2}\right) = \frac{b}{x} & (a < x < b). \end{array}\right\} \qquad \text{.......(14)}$$

Writing $x = b\cos\frac{1}{2}\theta$, $a = b\cos\frac{1}{2}c$ and expressing the hypergeometric functions as trigonometrical functions [3(*a*)], equations (14) become

$$\left.\begin{array}{ll} \displaystyle\sum_{n=1}^\infty (-1)^{n-1} C_n \cos\tfrac{1}{2}(2n-1)\theta = 0 & (c < \theta < \pi), \\ \displaystyle\sum_{n=1}^\infty \frac{(-1)^{n-1} C_n}{2n-1} \cos\tfrac{1}{2}(2n-1)\theta = 1 & (0 < \theta < c). \end{array}\right\} \qquad \text{.......(15)}$$

I have already shown [2] that the coefficients $C_n$ in these dual series are given by

$$(-1)^{n-1}C_n = 2P_{n-1}(\cos c)/K(\cos \tfrac{1}{2}c)$$

in the usual notation for Legendre polynomials and elliptic integrals. Hence the potential is given by equation (10) where, from equation (13),

$$\xi(u) = \frac{2}{uK(a/b)} \sum_{n=1}^{\infty} (-1)^{n-1} P_{n-1}\left(2\frac{a^2}{b^2}-1\right) J_{2n-1}(bu). \qquad \text{(16)}$$

The surface density $\sigma$ of charge on the strip $a < x < b$, $y = 0$ is given by

$$4\pi\sigma = -\left(\frac{\partial V}{\partial y}\right)_{y=0} = \int_0^\infty u\xi(u) \sin xu \, du \quad (a < x < b). \qquad \text{(17)}$$

Substitution from (16) and interchange of the order of integration and summation then yields

$$2\pi K\left(\frac{a}{b}\right)\sigma = \sum_{n=1}^{\infty} (-1)^{n-1} P_{n-1}\left(2\frac{a^2}{b^2}-1\right) \int_0^\infty J_{2n-1}(bu) \sin xu \, du \quad (a < x < b)$$

$$= \sum_{n=1}^{\infty} (-1)^{n-1} P_{n-1}\left(2\frac{a^2}{b^2}-1\right) \frac{\sin\{(2n-1)\sin^{-1}(x/b)\}}{(b^2-x^2)^{\frac{1}{2}}}$$

$$= (b^2-x^2)^{-\frac{1}{2}} \sum_{n=1}^{\infty} P_{n-1}\left(2\frac{a^2}{b^2}-1\right) \cos\{(2n-1)\cos^{-1}(x/b)\},$$

the value of the definite integral being given by Watson [**1**(*b*)]. The sum of the series on the right [**3**(*b*)] is known to be $\frac{1}{2}b(x^2-a^2)^{-\frac{1}{2}}$ and the surface density is therefore given by

$$\sigma = \frac{b}{4\pi K(a/b)} \cdot \frac{1}{(x^2-a^2)^{\frac{1}{2}}(b^2-x^2)^{\frac{1}{2}}}.$$

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