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Discrete Sets and Associated Dynamical Systems in a Non-Commutative Setting

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Abstract. We define a uniform structure on the set of discrete sets of a locally compact topological space on which a locally compact topological group acts continuously. Then we investigate the completeness of these uniform spaces and study these spaces by means of topological dynamical systems.

1 Introduction

Since the quasicrystalline materials were discovered in 1984 by Shechtman, Blech, Gratias and Cahn ([16]), intensive studies have been done in order to analyse quasicrystals and related topics mathematically (see *e.g.,* the surveys [1, 9, 10]).

In many cases, quasicrystals or tilings are regarded as discrete sets and they are studied by using the properties of dynamical systems built from discrete subsets. Since these objects are originally in Euclidean space $\mathbb{R}^d$, the theory of discrete sets is formulated within the framework of locally compact Abelian groups. Let A be such a group (the group operation is denoted additively) and let D be the set of discrete subsets of A. Then we consider the closure of the A-orbit of $\Lambda \in D$, in a suitable topology, as dynamical system. This topology on D is usually introduced via metrics or distance functions (see *e.g.,* [8]). Thus metrizability has played an important role. An alternative approach, particularly favoured by Schlottmann [15], has been to work rather with uniformities. More precisely, let K be a compact subset of A and V be a neighbourhood of the unit element, define $U_{K,V} \subset D \times D$ by

$$U_{K,V} := \{(\Lambda_1, \Lambda_2) \mid \exists v_1, v_2 \in V \text{ such that } (v_1 + \Lambda_1) \cap K = (v_2 + \Lambda_2) \cap K\}.$$

The set $\{U_{K,V}\}$ is a fundamental system of uniformity $\mathcal{U}$ on D and it has been proved that the uniform space $(D, \mathcal{U})$ is complete. In above case (which we call the *Abelian group case*) it is by now well understood how to introduce the topology and establish the connection between finite local complexity (FLC) and compactness of the closure of A-orbit (see *e.g.,* [6, 7, 8, 15, 17]).

As the Euclidean group of plane, for example, is not Abelian, it seems interesting to consider more general cases. Here we offer some extensions of these results to the setting of discrete subsets of a locally compact Hausdorff topological space M on which a locally compact Hausdorff topological group G acts continuously.

Let D be the set of discrete subsets of M. First we show that in a way similar to that just described we can define a uniform structure $\mathcal{U}$ on D (cf. Definition 2.3).

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Although in the *group case*, where $M = G$ and G acts by left translation, we can prove the completeness of $(D, \mathcal{U})$ in the same way as the Abelian case, the general case needs another approach. Theorem 3.4 deals with (essentially) the group cases. Theorem 3.10 deals with the general case, where we now also assume that M is σ-compact. With respect to metrizability, we remark that if G and M are metrizable then the uniform space $(D, \mathcal{U})$ is metrizable as well. (*cf.* [3, Chap. IX], [4, Chap. XII].)

When $(D, \mathcal{U})$ is complete, define $X(\Lambda)$ by the closure of G-orbit of Λ in D. In §4.2 we give the definition of finite local complexity of Λ and discuss the relation between FLC and compactness of $X(\Lambda)$. Under a weak assumption, we can show that these two notions are equivalent (Theorem 4.5 and Proposition 4.8). Finally in §4.4, we briefly describe the pinwheel tiling, which offers an interesting example of above setting.

In the very recent works of Benedetti and Gambando [2] and Sadun [14], discrete subsets of a Riemannian manifold on which a finite dimensional Lie group acts, are considered. And the compactness of the hull of a discrete set and minimality of the dynamical system are discussed. But in these works metrics are used, and our approach is different from theirs.

2 Uniformity on the Space of Discrete Sets

2.1 Definition of Uniformity

Let $(X, \mathcal{U})$ be a uniform space, *i.e.,* X be a set and $\mathcal{U}$ be a uniformity on X ([3, Chap.II, §1]). The elements of $\mathcal{U}$ are called entourages. There exists a unique topology on X for which $\{V(x) \mid V \in \mathcal{U}\}$ forms the system of neighbourhoods of $x \in X$, where $V(x) := \{y \in X \mid (x, y) \in V\}$. To define a uniformity $\mathcal{U}$, it is enough to give a fundamental system $\mathcal{B}$. Then $\mathcal{U}$ is given by

$$\mathcal{U} = \{U \subset X \times X \mid U \supset V \text{ for some } V \in \mathcal{B}\}.$$

Considering the definition, we have the following proposition and its corollary.

Proposition 2.1 *A set $\mathcal{B}$ of subsets of $X \times X$ is a fundamental system of a (unique) uniformity if and only if*

(FS1) *For any $V_1, V_2 \in \mathcal{B}$, there exists $V_3 \in \mathcal{B}$ such that $V_1 \cap V_2 \supset V_3$.*
(FS2) *For any $V \in \mathcal{B}$, $V \supset \Delta := \{(x, x) \mid x \in X\}$.*
(FS3) *For any $V \in \mathcal{B}$, there exists $V' \in \mathcal{B}$ such that $V' \subset V^{-1} := \{(y, x) \mid (x, y) \in V\}$.*
(FS4) *For any $V \in \mathcal{B}$, there exists $W \in \mathcal{B}$ such that $W \circ W \subset V$.*

(For $V_1, V_2 \in X \times X$, $V_1 \circ V_2 := \{(x, z) \mid \exists y \in X \text{ such that } (x, y) \in V_1, (y, z) \in V_2\}$.)

Corollary 2.2 *Let $\mathcal{B} \subset X \times X$ be a fundamental system of a uniformity $\mathcal{U}$. Let $\mathcal{B}' \subset X \times X$ satisfy* (IN) *and* (IN$'$)*:*

(IN$'$) *For any $V \in \mathcal{B}$, there exists $V' \in \mathcal{B}'$ such that $V \supset V'$.*
(IN) *For any $V' \in \mathcal{B}'$, there exists $V \in \mathcal{B}$ such that $V' \supset V$.*

Then $\mathcal{B}'$ is also a fundamental system of the uniformity $\mathcal{U}$.

Let G be a locally compact Hausdorff topological group (the group operation is denoted multiplicatively, e being the identity element of G), and M be a locally compact Hausdorff topological space on which G acts continuously, *i.e.,* there exists a continuous mapping $G \times M \to M\colon (g, x) \mapsto gx$, such that;

(i) $(g_1g_2)x = g_1(g_2x)$, for all $g_1, g_2 \in G, x \in M$;
(ii) $ex = x$, for all $x \in M$.

Let D be the set of all discrete subsets of M:

$$D := \{\Lambda \subset M \mid \Lambda\colon \text{ discrete } \}.$$

In this note, Λ is defined to be discrete if and only if $\Lambda \cap K$ is finite for all compact subsets $K \subset M$, *i.e.,* Λ is closed and the induced topology on Λ is discrete.

For subsets $G_1 \subset G, M_1 \subset M,\ G_1M_1 := \{gx \mid g \in G_1, x \in M_1\}$. In particular, for $\Lambda \in D$ and $g \in G, g\Lambda := \{gx \mid x \in \Lambda\}$ and $g\Lambda \in D$.

Definition 2.3 For a compact subset K of M and a neighbourhood V of e in G, define $U_{K,V} \subset D \times D$ by

$$U_{K,V} := \{(\Lambda_1, \Lambda_2) \mid \exists v_1, v_2 \in V \text{ such that } (v_1\Lambda_1) \cap K = (v_2\Lambda_2) \cap K\}.$$

From the definition, the next lemma is obvious.

Lemma 2.4 *Let K_1, K_2 be compact subsets of M and V_1, V_2 be neighbourhoods of e. Then, if $K_1 \supset K_2$ and $V_1 \subset V_2$, $U_{K_1,V_1} \subset U_{K_2,V_2}$.*

Theorem 2.5

$$\mathcal{B} = \{U_{K,V} \mid K \subset M \textit{ is compact},\ V \textit{ is a neighbourhood of } e \subset G\}$$

is a fundamental system of a uniformity on D.

Proof It suffices to show the conditions (FS1) to (FS4). (FS2) and (FS3) are obvious.

(FS1): For $U_{K,V},\ U_{K',V'}$, consider $U_{K\cup K',V\cap V'}$.

(FS4): In order to find $U_{K',V'}$ such that $U_{K',V'} \circ U_{K',V'} \subset U_{K,V}$, we investigate the condition for $(\Lambda_1, \Lambda_2), (\Lambda_2, \Lambda_3) \in U_{K',V'}$ (for some K', V').

By definition, for some $v_1, v_2, v_2', v_3 \in V'$,

$$(v_1\Lambda_1) \cap K' = (v_2\Lambda_2) \cap K', \quad (v_2'\Lambda_2) \cap K' = (v_3\Lambda_3) \cap K'.$$

Thus, by translating these identities by v_2^{-1} and $v_2'^{-1}$, respectively, we have

$$((v_2^{-1}v_1)\Lambda_1) \cap (v_2^{-1}K') \cap (v_2'^{-1}K') = ((v_2'^{-1}v_3)\Lambda_3) \cap (v_2'^{-1}K') \cap (v_2^{-1}K').$$

For the original $U_{K,V}$, take a neighbourhood V' such that $\overline{V'}$ is compact and $V'^{-1}V' \subset V$ and put $K' = \overline{V'}K$. For $v \in V'$, as $vK \subset K'$, we have $K \subset v^{-1}K'$. Then the above identity shows that, for $(\Lambda_1, \Lambda_2), (\Lambda_2, \Lambda_3) \in U_{K',V'}, (\Lambda_1, \Lambda_3) \in U_{K,V}$. ■

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Lemma 2.6 *Define $U'_{K,V}$ for a compact subset K of M and a neighbourhood V of e as follows:*

$$U'_{K,V} := \{(\Lambda_1, \Lambda_2) \mid \exists v \in V \text{ such that } \Lambda_1 \cap K = (v\Lambda_2) \cap K\}.$$

Then $\mathcal{B}' := \{U'_{K,V}\}$ is also a fundamental system of the same uniformity as $\mathcal{B}$ on D. Similarly, it is true for $\mathcal{B}'^{-1} := \{U'^{-1}_{K,V}\}$.

Proof Let us use Corollary 2.2. Evidently $U'_{K,V} \subset U_{K,V}$, which shows (IN$'$).

If $(\Lambda_1, \Lambda_2) \in U_{K_1,V_1}$, we have, for some $v_1, v_2 \in V_1$,

$$\Lambda_1 \cap (v_1^{-1}K_1) = ((v_1^{-1}v_2)\Lambda_2) \cap (v_1^{-1}K_1),$$

which shows $U_{K_1,V_1} \subset U'_{K_2,V_2}$, where $K_2 = \bigcap_{v \in V_1}(v^{-1}K_1)$ as long as K_2 is non-empty and $V_1^{-1}V_1 \subset V_2$.

Thus, for given $U'_{K,V}$, take V_1 such that $\overline{V_1}$ is compact and $V_1^{-1}V_1 \subset V$, and put $K_1 = \overline{V_1}K$. Then $K_2 = \bigcap_{v \in V_1}(v^{-1}K_1) \supset K$ and $U_{K_1,V_1} \subset U'_{K_2,V} \subset U'_{K,V}$, which shows (IN). ■

$U'_{K,V}$ is more convenient to compute in some cases, and we can use both $U_{K,V}$ and $U'_{K,V}$ as entourages.

2.2 Properties of the Topological Space D

Let G, M and D be same as §2.1 and consider the topology on D given by the uniformity $\mathcal{U}$ defined in Theorem 2.5.

Proposition 2.7 *The topological space D is Hausdorff.*

Proof The topology on a uniform space $(X, \mathcal{U})$ is Hausdorff if and only if $\bigcap_{V \in \mathcal{U}} V = \Delta$ ([3, Chap. II, §1, Prop.3]). Thus, it suffices to show that, for $\Lambda_1, \Lambda_2 \in D$, $\Lambda_1 \neq \Lambda_2$, there exists $U'_{K,V} \in \mathcal{B}'$, such that $(\Lambda_1, \Lambda_2) \notin U'_{K,V}$. Take (say) $x \in \Lambda_1$, $x \notin \Lambda_2$. There exists a compact neighbourhood V of e, such that $(V^{-1}x) \cap \Lambda_2 = \varnothing$. Then, for $K = Vx, (\Lambda_1, \Lambda_2) \notin U'_{K,V}$. In fact, if not, for some $v \in V, \Lambda_1 \cap K = (v\Lambda_2) \cap K \ni x = vx'(x' \in \Lambda_2)$, which is a contradiction. ■

Proposition 2.8 *The action of G on D by left translation:*

$$G \times D \to D : (t, \Lambda) \mapsto t\Lambda$$

is continuous.

Proof We have, for $t \in G$,

$$tU_{K,V}(\Lambda) = U_{tK,V_1}(t\Lambda),$$

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where $V_1 = tVt^{-1}$. Thus it is enough to show the continuity at $t = e$. For any $U'^{-1}_{K,V} \in \mathcal{B}'^{-1}$, we will look for a compact subset K' and neighbourhoods W_0, W_1 of e satisfying,

$$W_0 U'^{-1}_{K',W_1}(\Lambda) \subset U'^{-1}_{K,V}(\Lambda).$$

If $U'^{-1}_{K_1,W_1}(\Lambda) \ni \Lambda'$, we have $\Lambda' \cap K_1 = (w_1\Lambda) \cap K_1$ for some $w_1 \in W_1$. And, for $w \in W_0$,

$$(w\Lambda') \cap (wK_1) = (ww_1\Lambda) \cap (wK_1)$$

i.e., $(w\Lambda') \in U'^{-1}_{K_2,W'}(\Lambda)$, where $K_2 = \bigcap_{w\in W_0}(wK_1)$ and $W_0 W_1 \subset W'$.

Thus, for given $U'^{-1}_{K,V}(\Lambda)$, take W such that $\overline{W}$ is compact and $WW \subset V$, and put $K' = \overline{W}^{-1}K$. Then $K_2 = \bigcap_{w\in W}(wK') \supset K$ and $WU'^{-1}_{K',W}(\Lambda) \subset U'^{-1}_{K_2,V}(\Lambda) \subset U'^{-1}_{K,V}(\Lambda)$. ■

Corollary 2.9

(i) *G acts on D as homeomorphisms.*

(ii) *If E is a G-invariant subset of D, then the closure $\overline{E}$ is also G-invariant.*

3 Sufficient Conditions for Which the Uniform Spaces $(D, \mathcal{U})$ Are Complete

3.1 Preliminaries

Let G, M and D be as §2.1. Before investigating the completeness of $(D, \mathcal{U})$, we recall several definitions and results for the convenience of readers.

Definition 3.1 Let $(X, \mathcal{U})$ be a uniform space. Let $(I, \leq)$ be a directed set.

(i) A mapping $x\colon I \to X$ (which is called a net) is a Cauchy net if and only if, for any $V \in \mathcal{U}$ there exists $k_V \in I$ such that $(x(m), x(n)) \in V$ for all $m, n \geq k_V$.

(ii) $(X, \mathcal{U})$ is complete if and only if every Cauchy net $x\colon I \to X$ converges.

Lemma 3.2 *Let $\{\Lambda_i\}_{i\in I}$ be a Cauchy net ($i \mapsto \Lambda_i$). Let J be a cofinal subset of I (*i.e.*, for any $m \in I$ there exists $n \in J$ such that $n \geq m$). If $\{\Lambda_j\}_{j\in J}$ converges, then $\{\Lambda_i\}_{i\in I}$ converges also.*

Proof The proof is similar to the case of a Cauchy sequence in a metric space. ■

Lemma 3.3 *Let $\mathcal{K}$ be a family of compact subsets of M with the property that, for every compact subset $K'(\subset M)$, there exists $K'' \in \mathcal{K}$ which contains K'. Let $\{\Lambda_i\}_{i\in I}$ be a net which satisfies the following condition:*

For any $K \in \mathcal{K}$, there exists m_K such that for all $i, j \geq m_K, \Lambda_i \cap K = \Lambda_j \cap K$.

Define $\Lambda \subset M$ by $\Lambda := \bigcup_{K\in\mathcal{K}}(K \cap \Lambda_{m_K})$. Then $\Lambda \cap K = K \cap \Lambda_{m_K}$ for all $K \in \mathcal{K}$. Moreover, $\Lambda \in D$ and $\{\Lambda_i\}_{i\in I}$ converges to Λ.

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Proof Let $x \in \Lambda \cap K, K \in \mathcal{K}$ and suppose $x \in K_1 \cap \Lambda_{m_{K_1}}$ for $K_1 \in \mathcal{K}$. Then, for $p \geq m_K, m_{K_1}, x \in \Lambda_p$ and $x \in K \cap \Lambda_p = K \cap \Lambda_{m_K}$.

For a compact subset $K' \subset M$, take $K'' \in \mathcal{K}, K'' \supset K'$. Then, $\Lambda \cap K' \subset \Lambda \cap K'' = K'' \cap \Lambda_{m_{K''}} = K'' \cap \Lambda_i (i \geq m_{K''})$, which is finite. And $\Lambda \cap K' = \Lambda_i \cap K' (i \geq m_{K''})$ shows that $(\Lambda_i, \Lambda) \in U_{K',V}$ for all V. ∎

3.2 Completeness Conditions

Theorem 3.4 *Suppose that G is transitive on M, that the stabilizer N of a point in M is a normal subgroup and that M is homeomorphic to G/N by the canonical mapping. Then the uniform space $(D, \mathcal{U})$ is complete.*

In particular, in the case where $M = G$ and G acts on M by left translation (we call it group case*), the uniform space $(D, \mathcal{U})$ is complete.*

Proof (0) We show that a Cauchy net $\{\Lambda_i\}_{i \in I}$ in D always converges. To do so, we will prove that we can reduce to the case where $\bigcap_{i \in I} \Lambda_i \neq \varnothing$ and then construct a family $\mathcal{K}$ mentioned in Lemma 3.3.

(1) At first, we can assume that there exists a compact K_0 such that, for any $m \in I$, there exists $n \geq m$ satisfying $\Lambda_n \cap K_0 \neq \varnothing$. (If not, $\{\Lambda_i\}$ converges to $\varnothing$.) Then taking a subnet if necessary, we can assume $\Lambda_i \cap K_0 \neq \varnothing$ for all i.

(2) Let $x_i \in \Lambda_i \cap K_0$. Since K_0 is compact, there exists a subnet $\{x_j\}_{j \in J}$ which converges and let x_0 be the limit of $\{x_j\}_{j \in J}$. By assumption, there exist $\tilde{g}_j \in G/N$ for $j \in J$ such that $\tilde{g}_j(x_j) = x_0$ and $\{\tilde{g}_j\}_{j \in J}$ converges to $\tilde{e} = N \in G/N$. Then, taking the subset J as I and replacing $\{\Lambda_i\}_{i \in I}$ by $\{\tilde{g}_i(\Lambda_i)\}_{i \in I}$, we can assume $\bigcap_{i \in I} \Lambda_i \ni x_0$, for $\{\tilde{g}_i(\Lambda_i)\}_{i \in I}$ is a Cauchy net, and it suffices to show that $\{\tilde{g}_i(\Lambda_i)\}_{i \in I}$ converges.

(3) We can assume that there exists a compact neighbourhood W_0 of x_0 , such that $\Lambda_i \cap W_0 = \{x_0\}$ for all i. In fact, take an open neighbourhood W of x_0 where $\overline{W}$ is compact and a neighbourhood V of e such that $Vx_0 \subset W$. Since $\{\Lambda_i\}$ is a Cauchy net, there exists m such that, for all $i, j \geq m$, $(\Lambda_i, \Lambda_j) \in U'_{\overline{W},V}$. Therefore for any $j \geq m$, there exists $v_j \in V$, satisfying

$$(v_j \Lambda_j) \cap \overline{W} = \Lambda_m \cap \overline{W}.$$

Since $\Lambda_m \cap \overline{W}$ is finite, and $v_j(x_0) \in \Lambda_m \cap \overline{W}$, by taking a subnet if necessary, we can assume that all $v_j(x_0)$ are equal, thus all $v_j^{-1}(W)$ are equal by assumption on the stabilizer, *i.e.*, $\Lambda_i \cap (v_i^{-1}(W)) = \Lambda_j \cap (v_j^{-1}(W)) \ni x_0$. ($n \in N$ fixes every point of M.) Since $\Lambda_i \cap (v_i^{-1}(W))$ is finite and $v_i^{-1}(W)$ is a neighbourhood of x_0, in $v_i^{-1}(W)$ we can find W_0 as above.

(4) Let $\mathcal{K} = \{K \subset M \mid \text{compact}, K \supset W_0\}$. We claim that $\mathcal{K}$ satisfies the conditions of Lemma 3.3. In fact, take a neighbourhood V of e such that $Vx_0 \subset W_0$ and consider the entourages $U'_{K,V}, K \in \mathcal{K}$. There exists m_K such that $(\Lambda_i, \Lambda_j) \in U'_{K,V}$ for all $i, j \geq m_K$, *i.e.*, for some $v_{ij} \in V, \Lambda_i \cap K = (v_{ij}\Lambda_j) \cap K$. Since $v_{ij}(x_0) \in W_0 \subset K$ and $v_{ij}(x_0) \in (v_{ij}\Lambda_j) \cap K = \Lambda_i \cap K, v_{ij}(x_0) \in \Lambda_i \cap W_0 = \{x_0\}$, and by assumption we have $v_{ij}\Lambda_j = \Lambda_j$. ∎

Lemma 3.5 *For a locally compact Hausdorff group G, there exists a fundamental system $\mathcal{F} = \{W_\alpha\}$ of neighbourhoods of e such that:*

(i) *There exists $W_0 \in \mathcal{F}$ such that $W_\alpha \subset W_0$ for all $W_\alpha \in \mathcal{F}$.*

(ii) *For any $W_\alpha \in \mathcal{F}$, $\overline{W_\alpha}$ is compact and ${W_\alpha}^{-1} = W_\alpha$.*

(iii) *For any $W_\alpha \in \mathcal{F}$, there exists $W_\beta \in \mathcal{F}$ such that ${W_\beta}^3 \subset W_\alpha$.*

Proof Since there exists a compact neighbourhood of e, we can easily show this lemma, considering definitions. ■

Proposition 3.6 *Let $\{\Lambda_i\}_{i\in I}$ be a Cauchy net in D.*

Then, for any compact subset K in M, there exists $\Lambda_K \in D$ satisfying that, for any neighbourhood V of e, there exists $i_V \in I$, such that, for any $i \geq i_V$, there exists $v_i \in V$ such that

$$v_i\Lambda_i \cap K = \Lambda_K \cap K.$$

In particular, if M is compact, then the uniform space $(D, \mathcal{U})$ is complete.

Proof (0) Take a fundamental system $\{W_\alpha\}$ which satisfies the conditions in Lemma 3.5. Let $\tilde{K}$ be $\overline{W_0}K$. (Then $K \subset v_0^{-1}\tilde{K}$ for any $v_0 \in \overline{W_0}$.)

(1) Consider the entourage $U'_{\tilde{K},W_\alpha}$. Since $\{\Lambda_i\}_{i\in I}$ is a Cauchy net, there exists $m_\alpha \in I$ such that, for any $i \geq m_\alpha$, there exists $v_i^{(\alpha)} \in W_\alpha$ satisfying

$$(v_i^{(\alpha)}\Lambda_i) \cap \tilde{K} = \Lambda_{m_\alpha} \cap \tilde{K}.$$

(2) Since $\overline{W_0}$ is compact, there exists a cofinal subset J of I such that $\{v_j^{(0)}\}_{j\in J}$ converges to an element $\tilde{v} \in \overline{W_0}$.

Define Λ_K by $\Lambda_K = (\tilde{v})^{-1}\Lambda_{m_0}$.

(3) For any W_α, there exists $j_\alpha \in J$ such that, for any $j \geq j_\alpha (j \in J)$, $v_j^{(0)} \in \tilde{v}W_\alpha$.

(4) Let V be a neighbourhood of e. Take W_α such that $V \supset W_\alpha$ and W_β such that ${W_\beta}^3 \subset W_\alpha$. Then take $k_\alpha \in J$ satisfying $k_\alpha \geq m_0, m_\beta, j_\beta$.

For any $p \in I$, $p \geq k_\alpha$,

$$(v_p^{(\beta)}\Lambda_p) \cap \tilde{K} = \Lambda_{m_\beta} \cap \tilde{K} = (v_{k_\alpha}^{(\beta)}\Lambda_{k_\alpha}) \cap \tilde{K}.$$

(5) By (1) for W_0, $(v_{k_\alpha}^{(0)}\Lambda_{k_\alpha}) \cap \tilde{K} = \Lambda_{m_0} \cap \tilde{K} = (\tilde{v}\Lambda_K) \cap \tilde{K}$, and by (3) $v_{k_\alpha}^{(0)} = \tilde{v}w_\beta$ for some $w_\beta \in W_\beta$. Thus

$$(w_\beta\Lambda_{k_\alpha}) \cap (\tilde{v}^{-1}\tilde{K}) = \Lambda_K \cap (\tilde{v}^{-1}\tilde{K}).$$

(6) By (4),

$$(w_\beta(v_{k_\alpha}^{(\beta)})^{-1}v_p^{(\beta)}\Lambda_p) \cap (w_\beta(v_{k_\alpha}^{(\beta)})^{-1}\tilde{K}) = (w_\beta\Lambda_{k_\alpha}) \cap (w_\beta(v_{k_\alpha}^{(\beta)})^{-1}\tilde{K}).$$

Thus by (5)

$$(\text{the left side }) \cap (\tilde{v}^{-1}\tilde{K}) = \Lambda_K \cap (\tilde{v}^{-1}\tilde{K}) \cap (w_\beta(v_{k_\alpha}^{(\beta)})^{-1}\tilde{K}).$$

Let $v_p = w_\beta (v_{k_\alpha}^{(\beta)})^{-1} v_p^{(\beta)}$. Then $v_p \in W_\beta{}^3 \subset W_\alpha \subset V$ and we have, taking the intersection with K, $(v_p \Lambda_p) \cap K = \Lambda_K \cap K$.

(7) When M is compact, $\{\Lambda_i\}_{i \in I}$ converges to Λ_M by definition. ■

Lemma 3.7 *Let $\{W_\alpha\}$ be a fundamental system as Lemma* 3.5*. Let $\{\Lambda_i\}_{i\in I}$ be a Cauchy net in D.*

Let K, K' be compact subsets of M such that $K \subset K'$. Take $\Lambda_{\tilde{K}}, \Lambda_{\tilde{K}'} \in D$ given by Proposition 3.6 *for $\tilde{K} = \overline{W_0}K$ and $\tilde{K}' = \overline{W_0}K'$ respectively. Then*

$$\Lambda_{\tilde{K}} \cap K = \Lambda_{\tilde{K}'} \cap K.$$

Proof For any W_α, take W_β such that $W_\beta{}^3 \subset W_\alpha$. Then, there exists i'' such that, for any $i \geq i''$, there exist $v_i, v_i' \in W_\beta$ such that

$$(v_i \Lambda_i) \cap \tilde{K} = \Lambda_{\tilde{K}} \cap \tilde{K}, \quad (v_i' \Lambda_i) \cap \tilde{K}' = \Lambda_{\tilde{K}'} \cap \tilde{K}'.$$

Thus we have

$$(v_i \Lambda_i) \cap (v_i v_i'^{-1} \tilde{K}') = (v_i v_i'^{-1} \Lambda_{\tilde{K}'}) \cap (v_i v_i'^{-1} \tilde{K}').$$

Since $(v_i v_i'^{-1} \tilde{K}') \supset K' \supset K$,

$$\Lambda_{\tilde{K}} \cap K = (v_i v_i'^{-1} \Lambda_{\tilde{K}'}) \cap K,$$

which shows, for all W_α, there exists $w_\alpha \in W_\alpha$ such that

$$\Lambda_{\tilde{K}} \cap K = (w_\alpha \Lambda_{\tilde{K}'}) \cap K.$$

Since $\Lambda_{\tilde{K}} \cap K$ is finite and $\Lambda_{\tilde{K}'}$ is discrete,

$$\Lambda_{\tilde{K}} \cap K \subset \Lambda_{\tilde{K}'} \cap K.$$

Similarly, for any W_α, there exists $w_\alpha{}' \in W_\alpha$ such that $\Lambda_{\tilde{K}'} \cap K = (w_\alpha{}' \Lambda_{\tilde{K}}) \cap K$. Thus we have $\Lambda_{\tilde{K}} \cap K = \Lambda_{\tilde{K}'} \cap K$ ■

Definition 3.8 A locally compact space M is said to be *σ-compact* if it is a countable union of compact subsets.

Remark 3.9 It is known that M is σ-compact if and only if there exists a sequence $\{A_n \mid n = 1, 2, \dots\}$ of open subsets of M satisfying:

(i) $\overline{A_n}$ is compact;
(ii) $\overline{A_n} \subset A_{n+1}$ for all n;
(iii) $M = \cup_n A_n$. (*cf.* [3, Chap. I, §9.9])

Theorem 3.10 *If M is σ-compact, then the uniform space $(D, \mathfrak{U})$ is complete.*

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Proof (0) As before, we show that any Cauchy net $\{\Lambda_i\}_{i\in I}$ in D converges.

Take a fundamental system $\{W_\alpha\}$ of e in G satisfying the conditions in Lemma 3.5 and a sequence $\{A_n\}$ of Remark 3.9. Let $K_n = \overline{A_n}$.

(1) Take $\tilde{\Lambda}_n = \Lambda_{\tilde{K}_n}$, $\tilde{K}_n = \overline{W_0}K_n$, as Proposition 3.6. And define Λ by

$$\Lambda := \bigcup_n (\tilde{\Lambda}_n \cap K_n).$$

Then by Lemma 3.7, $\Lambda \cap K_m = \tilde{\Lambda}_m \cap K_m$ for all m.

Since, for any compact $K \subset M$, there exists K_{n_0} such that $K \subset K_{n_0}$, $\Lambda \cap K = \Lambda \cap K_{n_0} \cap K = \tilde{\Lambda}_{n_0} \cap K_{n_0} \cap K$ is finite, *i.e.*, $\Lambda \in D$.

(2) We show that $\{\Lambda_i\}_{i\in I}$ converges to Λ.

Let V be a neighbourhood of e and K be compact (say $K \subset K_{n_0}$). By definition of $\tilde{\Lambda}_{n_0}$, there exists i_{V,n_0} such that, for any $i \geq i_{V,n_0}$, there exists $v_{i,n_0} \in V$ such that

$$v_{i,n_0}\Lambda_i \cap K_{n_0} = \tilde{\Lambda}_{n_0} \cap K_{n_0}.$$

Thus we have $v_{i,n_0}\Lambda_i \cap K = \Lambda \cap K$, which shows $(\Lambda, \Lambda_i) \in U'_{K,V}$. ∎

4 Topological Dynamical Systems and Discrete Sets

4.1 Topological Dynamical System

We call a pair (X, H) *a topological dynamical system* if X is a compact topological space and H is a locally compact topological (not necessarily Abelian) group acting on X as homeomorphisms.

Let G, M and D be same as §2.1, and we denote by $(D, \mathfrak{U})$ the uniform space defined by Theorem 2.5.

Throughout §4, we assume that $(D, \mathfrak{U})$ is *complete*.

Definition 4.1 For $\Lambda \in D$, define $X(\Lambda)$ by

$$X(\Lambda) := \overline{\{t\Lambda \mid t \in G\}}.$$

$X(\Lambda)$ is called the G-hull of Λ.

$X(\Lambda)$ is a complete Hausdorff uniform space, since it is closed in the complete space D. And G acts on $X(\Lambda)$.

4.2 FLC (Finite Local Complexity)

Definition 4.2 $\Lambda \in D$ has FLC (finite local complexity) if and only if, for any compact subset $K \subset M$ there exist, up to G-translation, only finitely many different sets of the form $\Lambda \cap tK$.

Remark 4.3 ([15])

(i) The above definition of FLC is equivalent to the following:
For any compact K, there exists a finite set $F_K \subset M$ satisfying:

$$\text{For any } t \in G, \text{ there exist } t' \in G \text{ and } F' \subset F_K \text{ such that } (t\Lambda) \cap K = t'F'.$$

(ii) In the group case, the condition that $\Lambda \in D$ has FLC is equivalent to:

$$\Lambda^{-1}\Lambda \text{ is discrete.}$$

Proof By definition, if Λ has FLC, for a compact set K there exist finitely many, finite subsets $F_i = \Lambda \cap (t_i K)$, satisfying the condition that, for $t \in G$, there exist i and $t' \in G$ such that $\Lambda \cap (tK) = t'F_i$. Then the finite set $F_K = \cup F_i$ satisfies the condition of (i).

Conversely, suppose that the condition of (i) is satisfied for F_K. For $t \in G$, there exist $t' \in G$ and $F' \subset F_K$ such that $(t^{-1}\Lambda) \cap K = t'F'$. Thus $\Lambda \cap (tK) = tt'F'$. Since F_K is finite, there exist only finitely many such F'.

Next, consider the group case and let us prove (ii). Suppose that Λ has FLC. It suffices to show that $(\Lambda^{-1}\Lambda) \cap \tilde{K}$ is finite, for any compact subset $\tilde{K} \subset G$. Adding e to $\tilde{K}$ if necessary, we can assume $\tilde{K} \ni e$. Let $k = x^{-1}y \in (\Lambda^{-1}\Lambda) \cap \tilde{K}$ ($k \in \tilde{K}, x, y \in \Lambda$). Then $y = xk \in \Lambda \cap x\tilde{K} = t'F_i$, where F_i satisfies the above condition. Since x is also in $t'F_i$, $k = {f_1}^{-1} f_2$, for some $f_1, f_2 \in F_i$. Thus $(\Lambda^{-1}\Lambda) \cap \tilde{K}$ is finite.

Conversely, suppose that $\Lambda^{-1}\Lambda$ is discrete. For a compact subset $K \subset G$, $K^{-1}K$ is compact, thus $F_K = (\Lambda^{-1}\Lambda) \cap (K^{-1}K)$ is finite. For $t \in G$, let $x_0, y \in (t\Lambda) \cap K$. Then ${x_0}^{-1} y \in F_K$. Thus $(t\Lambda) \cap K \subset (x_0 F_K)$. ∎

In the group case, we have the following beautiful result: *For $\Lambda \in D$, $X(\Lambda)$ is compact if and only if Λ has FLC.* In Abelian group case, see [15]. To prove and generalize this result, we use the following Proposition 4.4.

Proposition 4.4 *Let $(X, \mathcal{U})$ be a complete Hausdorff uniform space and $Y(\neq \varnothing)$ be a subset of X.*

Then, $\overline{Y}$ is compact if and only if Y is totally bounded, i.e., *for any $V \in \mathcal{U}$, there exists a finite set $A_V \subset Y$ such that*

$$Y \subset \bigcup_{a \in A_V} V(a).$$

(See [3, Chap. II, §4.2, Theorem 3].)

Theorem 4.5 *For $\Lambda \in D$, if $X(\Lambda)$ is compact, then Λ has FLC.*

Proof Suppose that $X(\Lambda)$ is compact. For a compact subset K, take some compact neighbourhood V of e and consider $U'^{-1}_{K,V}$. Since $\{t\Lambda \mid t \in G\}$ is totally bounded, we have a finite set $A \subset G$ such that

$$\{t\Lambda \mid t \in G\} \subset \bigcup_{a \in A} U'^{-1}_{K,V}(a\Lambda).$$

Since $(t\Lambda) \cap K = (v(a\Lambda)) \cap K = v\{(a\Lambda) \cap (v^{-1}K)\}$ for some $a \in A$ and $v \in V$,

$$F_K = \bigcup_{a \in A} \{(a\Lambda) \cap (V^{-1}K)\}$$

satisfies (i) of Remark 4.3. ∎

Definition 4.6 For a compact subset $K \subset M$, define

$$G_K := \{g \in G \mid \exists k \in K, gk \in K\}.$$

Example 4.7 In the group case, for $K \subset G$, $G_K = KK^{-1}$. In this case, the assumptions of Proposition 4.8 are satisfied.

Proposition 4.8 *Suppose that for any compact K, G_K is contained in a compact set. If Λ has FLC, then $X(\Lambda)$ is compact.*

Proof Suppose that Λ has FLC.

For a compact subset K and a neighbourhood V of e (here we can assume $\overline{V}$ compact), we will look for a finite set $A \subset G$ such that

$$\{t\Lambda \mid t \in G\} \subset \bigcup_{a \in A} U'_{K,V}(a\Lambda).$$

Define a compact set K^+ by $K^+ := \overline{V}^{-1}K (\supset K)$.

By definition, there exist finite representatives $P_1, \ldots, P_N$ of translation classes of sets of the form $(t\Lambda) \cap K^+$. Let P_i be $(t_i\Lambda) \cap K^+$, $(t_i \in G)$.

If $(t\Lambda) \cap K^+ = \varnothing$, choose one $t_0 \in G$ satisfying $(t_0\Lambda) \cap K^+ = \varnothing$, then $t\Lambda \in U'_{K,V}(t_0\Lambda)$. Thus, assume $(t\Lambda) \cap K^+ \neq \varnothing$. Suppose that there exist $t' \in G$ and i such that $(t\Lambda) \cap K^+ = t'P_i = t'\{(t_i\Lambda) \cap K^+\} \neq \varnothing$. Then $t' \in G_{K^+}$.

By assumption, G_{K^+} is contained in a compact set, we can take a finite set $\tilde{E}(\subset G)$ satisfying $G_{K^+} \subset \bigcup_{\tilde{e} \in \tilde{E}} (V^{-1}\tilde{e})$.

If $t' = v^{-1}\tilde{e}$ $(v \in V, \tilde{e} \in \tilde{E})$, $(v(t\Lambda)) \cap (vK^+) = \tilde{e}P_i$. Since $(vK^+) \supset K$, we see that, for such $t \in G$, there exist $v \in V$, $\tilde{e} \in \tilde{E}$ and i such that $(v(t\Lambda)) \cap K = (\tilde{e}P_i) \cap K$. The possibility of such $(\tilde{e}, i)$ is finite and we can choose one $a_{\tilde{e},i} \in G$ satisfying $(\tilde{e}P_i) \cap K = (a_{\tilde{e},i}\Lambda) \cap K$ for each $(\tilde{e}, i)$. And $t\Lambda \in \bigcup U'_{K,V}(a_{\tilde{e},i}\Lambda)$. ∎

4.3 Minimality

Definition 4.9 Let (X, H) be a topological dynamical system. (X, H) is minimal if the closure of every orbit coincides with X.

Proposition 4.10 *For a topological dynamical system (X, H), there exists an H-invariant closed subset $Y \neq \varnothing$ so that (Y, H) is minimal.*

For proof, consider the collection of all non-empty H-invariant closed subsets, then use Zorn's lemma and compactness of X (finite intersection property).

Definition 4.11 Let (X, H) be a topological dynamical system. Then $x \in X$ is *almost periodic* if, for any neighbourhood U of x, $P(x, U) := \{h \in H \mid hx \in U\}$ is relatively dense (*i.e.,* there exists a compact subset K_U in H such that $K_U(P(x, U)) = H$).

Proposition 4.12 ([5, Chap.1])

(i) *If (X, H) is minimal, every point of X is almost periodic.*
(ii) *If $x \in X$ is almost periodic, then $(\overline{Hx}, H)$ is minimal.*

Proof (i) For a neighbourhood U of x, $X = HU = \bigcup(h_i U)$ (finite union). Then $K_U = \{h_i\}$ satisfies $K_U(P(x, U)) = H$.

(ii) It suffices to show that, for all $y \in \overline{Hx}$, $x \in \overline{Hy}$. If not, there exists a compact neighbourhood V of x such that $V \cap \overline{Hy} = \varnothing$. Since $Hx = (K_V(P(x, V)))x \subset K_V V$ and $K_V V$ is compact, $\overline{Hx} \subset K_V V$ *i.e.,* $y = kv$ for some $k \in K_V, v \in V$, which is a contradiction. ■

We apply these to $(X(\Lambda), G)$, G, Λ being as §4.1.

Remark 4.13 By definition of $X(\Lambda)$ and $U'_{K,V}$ we have: $\Lambda' \in X(\Lambda)$ if and only if, for any compact subset K, there exists $t_K \in G$ such that $\Lambda' \cap K = (t_K \Lambda) \cap K$.

Definition 4.14 Let Λ, Λ' be in D.

(i) Λ and Λ' are *locally indistinguishable* (LI) if, for any compact subset K, there exist $t_K, t'_K \in G$ satisfying

$$\Lambda \cap K = (t'_K \Lambda') \cap K \text{ and } (t_K \Lambda) \cap K = \Lambda' \cap K.$$

(ii) Λ is *repetitive* if, for any compact subset K, $P_K := \{t \in G \mid (t\Lambda) \cap K = \Lambda \cap K\}$ is relatively dense.

Remark 4.15 From the definition, Λ and Λ' are LI if and only if $X(\Lambda) = X(\Lambda')$.

Proposition 4.16 *Let $\Lambda \in D$. Suppose that $X(\Lambda)$ is compact. Then the followings are equivalent:*

(i) *For all $\Lambda' \in X(\Lambda)$, Λ' and Λ are* LI.
(ii) *$(X(\Lambda), G)$ is minimal.*
(iii) *Λ is almost periodic,* i.e., *$P(\Lambda, U) = \{g \in G \mid g\Lambda \in U\}$ is relatively dense for all neighbourhoods U of Λ.*
(iv) *Λ is repetitive.*

Proof The properties mentioned above show easily the equivalence of (i), (ii) and (iii).

Let K be a compact subset. For all V, $P_K \subset P(\Lambda, U'_{K,V}(\Lambda))$, we have (iv)$\Rightarrow$ (iii). Conversely, fix a compact neighbourhood V and take a compact subset C such that $C(P(\Lambda, U'_{K,V}(\Lambda))) = G$. Put $C_1 = CV^{-1}$, which is compact. Then $C_1 P_K = G$. In fact, write $g = ct, c \in C, t \in P(\Lambda, U'_{K,V}(\Lambda))$. For t, there exists $v \in V$ such that $(v(t\Lambda)) \cap K = \Lambda \cap K$. Then $vt \in P_K$ and $g = (cv^{-1})(vt)$. ■

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Proposition 4.17 (Point-set version of Radin-Wolff [13]) *If $\Lambda \in D$ has FLC and the assumptions of Proposition* 4.8 *are satisfied, then there exists $\Lambda' \in D$ satisfying:*

(i) *Λ' is repetitive and has FLC.*
(ii) *For any compact K, there exists $t_K \in G$ such that $\Lambda' \cap K = (t_K\Lambda) \cap K$.*

For proof, consider in $X(\Lambda)$ a G-invariant minimal closed subset which is of the form $X(\Lambda')$. For (ii), use Remark 4.13.

4.4 An Example–Pinwheel Tiling

The pinwheel tiling (or the Radin aperiodic plane set) is an aperiodic tiling of plane with a single prototile consisting of a right-angled triangle with sides of length 1, 2, $\sqrt{5}$. (In this section, "triangle" means a triangle congruent to this one.) For the construction of this tiling, see *e.g.*, [11, p. 474] and [12]. A triangle in the plane is described by its location (in the plane), orientation and chiral type. The location is indicated by the coordinates of one point fixed to each triangle all at once, *e.g.*, the vertex with a right angle. The orientation is indicated by an element of a circle S, and chiral type is indicated by + or $-$. Thus there is a one-to-one correspondence between the set of triangles in the plane and the direct product set $S \times \{\pm 1\} \times \mathbb{R}^2$ or $M = O(2) \times \mathbb{R}^2$ (as a set), where $O(2)$ denotes the real orthogonal group of degree 2.

The Euclidean group $E(2)$ of plane is isomorphic to the semi-direct product $O(2) \cdot \mathbb{R}^2$ and the group operation is given by

$$(\sigma, a)(\tau, b) = (\sigma\tau, \sigma b + a) \quad (\text{ for } \sigma, \tau \in O(2), a, b \in \mathbb{R}^2)$$

The group $E(2)$ and the subgroup G_0 of $E(2)$ consisting of the transformations which keep the orientation of the plane act naturally on the plane, triangles and the tiling. The action of $E(2)$ on M is described by

$$(\sigma, a)(\rho, c) = (\sigma\rho, \sigma c + a) \,(\text{ for } (\sigma, a) \in E(2), (\rho, c) \in M)$$

Namely, if we consider the set M as the underlying set of the group $E(2)$, the action is left translation of group elements. Since they satisfy the condition in Theorem 3.10, the uniform space $(D, \mathcal{U})$, where D is the set of discrete subsets of M and $\mathcal{U}$ is given by Theorem 2.5 for the group G_0 or $E(2)$, is complete. And we can apply the results in §4.

Let Λ be the tiling set of the pinwheel tiling, and consider the hull $X(\Lambda)$. The following geometric observation shows that Λ has FLC for the group G_0 or $E(2)$. Thus by Proposition 4.8 we know that $X(\Lambda)$ is compact. To show that Λ has FLC, we use Remark 4.3 (i) and consider the case of G_0. It is enough to consider as K the set $O(2) \times K_0$ where K_0 is the closed disc with radius R ($R \geq \sqrt{5}$) and centre $O = (0, 0)$. Let $t \in G_0$ and $K_1 = t^{-1}K_0$ be the closed disc with centre $P_1 = t^{-1}O$. Let T_1 be a triangle in $K_1 \cap \Lambda$ such that $P_1 \in T_1$. Take $s \in G_0$ which moves T_1 to T_0 or $T_0{}'$, where T_0 is the triangle with vertices $(0, 0), (2, 0), (2, 1)$ and $T_0{}'$ is the triangle with vertices $(0, 0), (2, 0), (2, -1)$. The triangles in Λ have the following property (PT):

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(PT) Any vertex of a triangle is a vertex or the middle point of a side of triangles which contain that vertex.

The set $s(\Lambda\cap(t^{-1}K))$ is a subset of tiling $s\Lambda$ and can be considered as a set of triangles which have common points with the disc sK_1, the centre of which is $st^{-1}O(\in T_0$ or $T_0{}')$. The number of different coverings by triangles of the disc with radius $(R+2\sqrt{5})$ and centre O, which have the above property (PT) and contain T_0 or $T_0{}'$, is finite. Thus the number of triangles which appear in these coverings, is also finite. Since this disc contains K_0 and sK_1, we know that there exists a finite set $F_K(\subset M)$ which satisfies the condition of Remark 4.3 (i).

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