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## Non-contractible periodic orbits in Hamiltonian dynamics on closed symplectic manifolds

Viktor L. Ginzburg and Başak Z. Gürel

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# Non-contractible periodic orbits in Hamiltonian dynamics on closed symplectic manifolds

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## ABSTRACT

We study Hamiltonian diffeomorphisms of closed symplectic manifolds with non-contractible periodic orbits. In a variety of settings, we show that the presence of one non-contractible periodic orbit of a Hamiltonian diffeomorphism of a closed toroidally monotone or toroidally negative monotone symplectic manifold implies the existence of infinitely many non-contractible periodic orbits in a specific collection of free homotopy classes. The main new ingredient in the proofs of these results is a filtration of Floer homology by the so-called augmented action. This action is independent of capping and, under favorable conditions, the augmented action filtration for toroidally (negative) monotone manifolds can play the same role as the ordinary action filtration for atoroidal manifolds.

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## 1. Introduction

In this paper, focusing on closed symplectic manifolds, we study Hamiltonian diffeomorphisms with non-contractible periodic orbits. We show that in a variety of settings the presence of one non-contractible one-periodic orbit of a Hamiltonian diffeomorphism of a closed symplectic manifold guarantees the existence of infinitely many non-contractible periodic orbits.

More specifically, we concentrate on closed symplectic manifolds  $(M^{2n}, \omega)$  which are toroidally monotone or toroidally negative monotone and Hamiltonians with at least one non-contractible one-periodic orbit  $x$ . Then we prove that under minor assumptions on the free homotopy class  $\mathfrak{f}$  of  $x$  and under certain dynamical and Floer theoretic conditions on  $x$  and  $M$ , the Hamiltonian has infinitely many simple periodic orbits in the collection of free homotopy classes  $\mathfrak{f}^{\mathbb{N}} := \{\mathfrak{f}^k \mid k \in \mathbb{N}\}$ ; see Theorems 2.1 and 2.2. The conditions on  $\mathfrak{f}$  are automatically satisfied when the homology class  $[\mathfrak{f}]$  is non-zero in  $H_1(M; \mathbb{Z})/\text{Tor}$  or when  $\mathfrak{f} \neq 1$  and  $\pi_1(M)$

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is hyperbolic and torsion free. These theorems partially extend the main result of [Gü13] from atoroidal symplectic manifolds to toroidally monotone or negative monotone manifolds. (We show in the next section that there are numerous manifolds and Hamiltonians meeting the requirements of the theorems.) The phenomenon we consider here is  $C^\infty$ -generic. To be more precise, we show in Theorem 2.7 that the presence of one non-contractible periodic orbit  $x$  in a class  $\mathfrak{f}$  such that  $1 \notin \mathfrak{f}^{\mathbb{N}}$  implies  $C^\infty$ -generically the existence of infinitely many non-contractible periodic orbits in  $\mathfrak{f}^{\mathbb{N}}$ . Finally, we also refine the results from [Gü13] for atoroidal symplectic manifolds; see Theorem 2.4.

All these results are manifestations of the same underlying phenomenon that the presence of a periodic orbit which is geometrically or homologically unnecessary forces a Hamiltonian system to have infinitely many periodic orbits. (On a closed symplectic manifold non-contractible periodic orbits are clearly unnecessary. For example, a  $C^2$ -small autonomous Hamiltonian has only constant one-periodic orbits.) This phenomenon of ‘forced existence’ of infinitely many periodic orbits is very general and has been observed in a variety of other settings. For instance, the celebrated theorem of Franks [Fra92, Fra96], asserting that a Hamiltonian diffeomorphism (or, even, an area-preserving homeomorphism) of  $S^2$  with at least three fixed points must have infinitely many periodic orbits, is a prototypical result along these lines; see also [LC06] for further refinements and [CKRTZ12, Ker12] for a symplectic topological proof. Another instance is a theorem from [GG14] that for a certain class of closed monotone symplectic manifolds including  $\mathbb{C}\mathbb{P}^n$  any Hamiltonian diffeomorphism with a hyperbolic fixed point must necessarily have infinitely many periodic orbits. Yet, the specific question of forced existence for non-contractible periodic orbits considered here is largely unexplored except for [Gü13] focusing on symplectically atoroidal manifolds. We refer the reader to, e.g., [Gü14] for some other related results and to [GG15] for a detailed discussion of the phenomenon and further references.

The proofs of our main theorems rely on the machinery of Floer homology for non-contractible periodic orbits. In the course of the last two decades, this version of Floer homology has been studied and used in a number of papers, but usually in a more topological context and focusing on Hamiltonians on open manifolds such as twisted or ordinary cotangent bundles; see, e.g., [BPS03, BH01, GL00, Lee03, Nic06, SW12, Web06, Xue14]. (The recent works [Bat15b, PS14] are closer to the setting considered in this paper, which on the conceptual level can be thought of as a continuation of [Gü13].) The main difficulty in applying the technique to closed manifolds is that the global Floer homology for non-contractible orbits vanishes and, moreover, already for closed surfaces, a Hamiltonian may have no non-contractible periodic orbits of any period. (As a consequence, the Floer complex can be trivial for all iterations.) Thus, to infer that a Hamiltonian has a number of periodic orbits, e.g., infinitely many, an additional input is required. In our case, this is one non-contractible periodic orbit of the Hamiltonian, which serves as a seed generating infinitely many offsprings. The new periodic orbits are detected by analyzing the change in certain filtered Floer homology groups under the iteration of the Hamiltonian and using the ‘stability of the filtered homology’. This argument shares many common elements with the reasoning in, e.g., [GG10, Gü13, Gü14]. We feel that it can also be cast in the framework of the barcode and persistent homology theory for Floer homology (cf. [PS14, UZ15]) and it would be interesting to see if a systematic use of this theory would lead to new results in this class of questions.

There are three key ingredients to the proofs in this paper.

The main new component is the observation, perhaps of independent interest, that under favorable circumstances the Floer homology for a toroidally monotone or toroidally negative monotone manifold is filtered by the so-called augmented action. The augmented action  $\tilde{\mathcal{A}}_H$  is the difference  $\mathcal{A}_H - (\lambda/2)\Delta_H$  between the standard symplectic action  $\mathcal{A}_H$  and the (renormalized)

mean index  $\Delta_H$  of an orbit, where  $\lambda$  is the monotonicity constant; cf. [GG09a, § 1.4]. The key feature of the augmented action is that it is independent of capping and hence is assigned to the orbit itself. When the augmented action gap is sufficiently large, the Floer differential does not increase the augmented action, and the augmented action filtration is defined. In this case, the filtration behaves similarly to the ordinary action filtration in the aspherical or atoroidal case and can be used in the same way. (One essential difference, which is a source of several complications, is that the augmented action filtration is not strict: in general, the Floer differential can connect orbits with equal augmented action even if the gap is large.)

The other two ingredients are the stability of the filtered homology already mentioned above and the ball-crossing theorem [GG14, Theorem 3.1] used in one of the proofs. This theorem gives an iteration-independent lower bound on the energy of a Floer trajectory asymptotic to a hyperbolic orbit.

The paper is organized as follows. In § 2, we set our conventions and notation, introduce the necessary notions, state the main results of the paper, and discuss in detail the classes of manifolds and Hamiltonians these results apply to. In § 3, we define the augmented action filtration and establish its key properties. Finally, in § 4, we prove the main theorems of the paper.

## 2. Main results

### 2.1 Conventions and notation

To state the main results of the paper, let us first introduce some relevant definitions and set our conventions and notation.

Throughout the paper, we assume that  $(M^{2n}, \omega)$  is a closed *toroidally monotone* or *toroidally negative monotone* symplectic manifold unless explicitly stated otherwise. To be more specific, recall that a cohomology class  $w$  is *atoroidal* if for every map  $v : \mathbb{T}^2 \rightarrow M$  the integral of  $w$  over  $v$  vanishes:  $\langle w, [v] \rangle = 0$ . A symplectic manifold  $(M, \omega)$  is said to be *toroidally monotone* (respectively, *toroidally negative monotone*) when for some constant  $\lambda \geq 0$  (respectively,  $\lambda < 0$ ) the class  $w = [\omega] - \lambda c_1(TM)$  is atoroidal. The constant  $\lambda$  is referred to as the *toroidal monotonicity constant*. Note that the case  $\lambda = 0$  corresponds to an atoroidal class  $[\omega]$ . Toroidally (negative) monotone manifolds are automatically spherically (negative) monotone. We refer the reader to § 2.2 for examples of toroidally monotone or negative monotone manifolds. We call the positive generator  $N_T$  of the group generated by the integrals  $\langle c_1(TM), [v] \rangle$  for all tori the *minimal toroidal first Chern number* of  $M$ . We set  $N_T = \infty$  when this group is  $\{0\}$ , i.e.,  $c_1(TM)$  is atoroidal. For a toroidally monotone or negative monotone manifold, this implies that  $[\omega]$  is also atoroidal.

We denote by  $\tilde{\pi}_1(M)$  the set of homotopy classes of free loops in  $M$ . The free homotopy class of a loop  $x$  and its integer homology class (modulo torsion) are denoted by  $[[x]]$  and, respectively, by  $[x]$ . Likewise, we write  $[f] \in H_1(M; \mathbb{Z})/\text{Tor}$  for the homology class modulo torsion of a free homotopy class  $f \in \tilde{\pi}_1(M)$ .

All Hamiltonians  $H$  are assumed to be one-periodic in time, i.e.,  $H : S^1 \times M \rightarrow \mathbb{R}$ , and we set  $H_t = H(t, \cdot)$  for  $t \in S^1 = \mathbb{R}/\mathbb{Z}$ . The Hamiltonian vector field  $X_H$  of  $H$  is defined by  $i_{X_H}\omega = -dH$ . The (time-dependent) flow of  $X_H$  is denoted by  $\varphi_H^t$  and its time-one map by  $\varphi_H$ . Such time-one maps are called *Hamiltonian diffeomorphisms*. For the sake of brevity, we will refer to the periodic orbits of  $\varphi_H$  or, equivalently, the periodic orbits of  $\varphi_H^t$  with integer period as the periodic orbits of  $H$ . For a Hamiltonian  $H$  and a collection of free homotopy classes

$\mathfrak{c} \subset \tilde{\pi}_1(M)$ , we set  $\mathcal{P}_k(H, \mathfrak{c})$  to be the set of  $k$ -periodic orbits of  $H$  in  $\mathfrak{c}$ . For instance,  $\mathcal{P}_k(H, \gamma)$ , where  $\gamma \in H_1(M; \mathbb{Z})/\text{Tor}$ , comprises the  $k$ -periodic orbits of  $H$  in the homology class  $\gamma$ .

For a class  $\mathfrak{f} \in \tilde{\pi}_1(M)$ , let us fix a reference loop  $z \in \mathfrak{f}$ . A capping of  $x : S^1 \rightarrow M$  with free homotopy class  $\mathfrak{f}$  is a cylinder (i.e., a homotopy)  $\Pi : [0, 1] \times S^1 \rightarrow M$  connecting  $x$  and  $z$  taken up to a certain equivalence relation. Namely, two cappings  $\Pi$  and  $\Pi'$  are equivalent if the integral of  $c_1(TM)$ , and hence of  $\omega$ , over the torus obtained by attaching  $\Pi'$  to  $\Pi$  is equal to zero.

The action of  $H$  on a capped loop  $\bar{x} = (x, \Pi)$  is

$$\mathcal{A}_H(\bar{x}) = - \int_{\Pi} \omega + \int_{S^1} H_t(x(t)) dt.$$

Clearly,  $\mathcal{A}_H(\bar{x})$  is well defined. Moreover, the critical points of  $\mathcal{A}_H$  are exactly the capped one-periodic orbits of  $H$  in the homotopy class  $\mathfrak{f}$ . The action spectrum  $\mathcal{S}(H, \mathfrak{f})$  is the set of critical values of  $\mathcal{A}_H$ . It has zero measure; see, e.g., [HZ94].

Furthermore, let us fix a trivialization of  $TM$  along the reference loop  $z$ . Then, to a capped one-periodic orbit  $\bar{x}$  with  $x \in \mathfrak{f}$ , one can associate the mean index  $\Delta_H(\bar{x})$  in a standard way. Namely, we extend the trivialization of  $TM|_z$  to the capping of  $x$  and then use the resulting trivialization of  $TM|_x$  to turn the linearized flow  $d\varphi_H^t|_x$  along  $x$  into a path in the group  $\text{Sp}(2n)$ . The mean index  $\Delta_H(\bar{x})$  is by definition the mean index of the resulting path; see, e.g., [Lon02, SZ92]. The mean index measures, roughly speaking, the total rotation number of certain unit eigenvalues of the linearized flow along  $x$ . The mean index  $\Delta_H(x)$  of a non-capped orbit  $x$  is well defined as an element of  $\mathbb{R}/2N_T\mathbb{Z}$ .

By analogy with the case of contractible orbits (see [GG09a]), we define the augmented action of a one-periodic orbit  $x$  to be

$$\tilde{\mathcal{A}}_H(x) = \mathcal{A}_H(\bar{x}) - \frac{\lambda}{2} \Delta_H(\bar{x}).$$

The action and the mean index change under recapping in the same way, up to the factor  $\lambda/2$ , and hence the augmented action of  $x$  is well defined, i.e., independent of the capping. Note, however, that the augmented action depends on the choices of the reference loop  $z$  and the trivialization. When  $\lambda = 0$ , i.e.,  $[\omega]$  is atoroidal, the augmented action turns into the ordinary action.

The augmented action spectrum  $\tilde{\mathcal{S}}(H, \mathfrak{f})$  is the collection of the augmented action values for all one-periodic orbits of  $H$  in the class  $\mathfrak{f}$ , i.e.,

$$\tilde{\mathcal{S}}(H, \mathfrak{f}) = \{\tilde{\mathcal{A}}_H(x) \mid x \in \mathcal{P}_1(H, \mathfrak{f})\}.$$

In contrast with the ordinary action spectrum,  $\tilde{\mathcal{S}}(H, \mathfrak{f})$  need not have zero measure; in fact, it can contain whole intervals, unless, of course,  $H$  has finitely many periodic orbits in  $\mathfrak{f}$ . However,  $\tilde{\mathcal{S}}(H, \mathfrak{f})$  is a compact set, which depends continuously on  $H$ . To be more precise, as is easy to see, for any open set  $U \supset \tilde{\mathcal{S}}(H, \mathfrak{f})$ , we have  $U \supset \tilde{\mathcal{S}}(K, \mathfrak{f})$  when  $K$  is sufficiently  $C^1$ -close to  $H$ .

Assume now that all one-periodic orbits of  $H$  in the class  $\mathfrak{f}$  with augmented action in an open interval  $I$  are isolated. Then we set  $\chi(H, I, \mathfrak{f})$  to be the sum of the Poincaré–Hopf indices of their return maps. This definition extends by continuity to all Hamiltonians  $H$  with possibly non-isolated orbits as long as the end points of  $I$  are outside  $\tilde{\mathcal{S}}(H, \mathfrak{f})$ . For instance,  $\chi(H, I, \mathfrak{f}) \neq 0$  when  $I \cap \tilde{\mathcal{S}}(H, \mathfrak{f})$  contains only one point  $a$  and  $H$  is non-degenerate and has an odd number of one-periodic orbits (e.g., one) in the class  $\mathfrak{f}$  with augmented action  $a$ .

The above definitions generalize in an obvious way to the setting where one-periodic orbits are replaced by  $k$ -periodic orbits or where one class  $\mathfrak{f} \in \tilde{\pi}_1(M)$  is replaced by a collection of classes  $\mathfrak{c} \subset \tilde{\pi}_1(M)$ . In the latter case, a reference loop  $z$  and a trivialization of  $TM|_z$  must be fixed for every  $\mathfrak{f} \in \mathfrak{c}$ .

**2.2 Results**

Now we are in a position to state our main results. We recall that *all manifolds considered in this paper are assumed to be toroidally monotone or toroidally negative monotone unless specifically stated otherwise*. Furthermore, since the proofs heavily depend on Hamiltonian Floer homology, we need to either assume throughout the paper that  $M$  is weakly monotone or rely on the machinery of virtual cycles. To be more precise, recall that a manifold  $M^{2n}$  is said to be weakly monotone if one of the following conditions is satisfied:  $M$  is (not necessarily strictly) monotone, i.e.,  $[\omega]|_{\pi_2(M)} = \lambda c_1(TM)|_{\pi_2(M)}$  with  $\lambda \geq 0$ , or  $c_1(TM)|_{\pi_2(M)} = 0$ , or  $N \geq n - 2$ , where  $N$  is the minimal Chern number. Under any of these conditions, the Floer homology is defined and has standard properties; see [HS95, MS12, Ono95] and references therein. We refer the reader to, e.g., [FO99, HWZ11, LT98, Par16] for various incarnations of the technique of virtual cycles.

Our first result asserts that under certain additional assumptions on  $M$  the presence of one non-contractible hyperbolic one-periodic orbit implies the existence of infinitely many non-contractible periodic orbits; cf. [GG14].

**THEOREM 2.1.** *Assume that  $N_T \geq n/2 + 1$ , where  $2n = \dim M$ , and that a Hamiltonian  $H$  on  $M$  has a hyperbolic one-periodic orbit  $x$  such that all homotopy classes in the set  $\llbracket x \rrbracket^{\mathbb{N}} = \{\llbracket x \rrbracket^k \mid k \in \mathbb{N}\}$  are distinct and non-trivial. (This is the case, e.g., when  $[x] \neq 0$  in  $H_1(M; \mathbb{Z})/\text{Tor}$ .) Then  $H$  has infinitely many simple periodic orbits with homotopy class in  $\llbracket x \rrbracket^{\mathbb{N}}$ . In particular, if all such orbits are isolated, there are simple non-contractible periodic orbits of arbitrarily large period.*

The condition that  $N_T \geq n/2 + 1$  appears to be purely technical even though it plays an essential role in the proof. As we will show below, there are numerous symplectic manifolds  $M$  and Hamiltonians  $H$  meeting the requirements of the theorem. For instance, the theorem applies to  $M = \Sigma_g \times \mathbb{C}P^m$ , where  $\Sigma_g$  is a closed surface of genus  $g \geq 2$ .

The second result is more accurate and covers a broader range of manifolds and maps, although it still relies on some additional topological assumptions about the flow of  $H$ ; cf. [Gü13].

**THEOREM 2.2.** *Assume that  $\mathcal{P}_1(H, \mathfrak{f})$  is finite and  $\chi(H, I, \mathfrak{f}) \neq 0$  for some interval  $I$  with end points outside  $\tilde{\mathcal{S}}(H, \mathfrak{f})$ , where  $\mathfrak{f} \in \tilde{\pi}_1(M)$  and  $[\mathfrak{f}] \neq 0$  in  $H_1(M; \mathbb{Z})/\text{Tor}$ . Then, for every sufficiently large prime  $p$ , the Hamiltonian  $H$  has a simple periodic orbit in the homotopy class  $\mathfrak{f}^p$  and with period either  $p$  or  $p'$ , where  $p'$  is the first prime greater than  $p$ . Moreover, when  $\pi_1(M)$  is hyperbolic and torsion free, the condition  $[\mathfrak{f}] \neq 0$  can be replaced by  $\mathfrak{f} \neq 1$  and no finiteness requirement is needed.*

*Remark 2.3.* We emphasize that in these theorems we impose no non-degeneracy conditions on  $H$ . An interesting new point in the second part of Theorem 2.2 is that, in contrast with many other results of this type, we do not need to require  $\mathcal{P}_1(H, \mathfrak{f})$  to be finite to have simple periodic orbits with arbitrarily large period. It is immediate to see that, if  $H$  is non-degenerate,  $\chi(H, I, \mathfrak{f}) \neq 0$  for a short interval  $I$  centered at  $a \in \tilde{\mathcal{S}}(H, \mathfrak{f})$  when the one-periodic orbits of  $H$  in the class  $\mathfrak{f}$  have different augmented action values.

Also note that by passing to an iteration one can replace in both of the theorems one-periodic orbits by  $k$ -periodic ones. Then the first theorem remains correct as stated. In the second theorem, after replacing  $H$  by the iterated Hamiltonian  $H^{\natural k}$  (see § 3.1), we can only conclude that  $H$  has infinitely many simple periodic orbits in  $\mathfrak{f}^{\mathbb{N}}$ . (Of course, the theorem still applies literally to  $H^{\natural k}$  in place of  $H$ , but simple periodic orbits of  $H^{\natural k}$  are not necessarily simple as periodic orbits of  $H$ .)

Let us now further discuss the conditions of Theorems 2.1 and 2.2, beginning with those concerning the manifold and then moving on to the Hamiltonian.

The manifolds  $M$  meeting the requirements of these theorems exist in abundance. To construct specific examples, let us start with a symplectically atoroidal manifold  $(M_1, \omega_1)$ , i.e., a closed symplectic manifold such that  $[\omega_1]$  and  $c_1(TM_1)$  are atoroidal. Among these are, for instance, surfaces of genus  $g \geq 2$  and, more generally, all Kähler manifolds with negative sectional curvature. (See, e.g., [Gü13] for further references and a discussion of such manifolds.) Next, let  $(M_0, \omega_0)$  be a closed spherically monotone or negative monotone symplectic manifold. There are numerous examples of such manifolds including, in the negative monotone case, those with arbitrarily large minimal (spherical) Chern number  $N_S$ . For instance, let  $M_0$  be a smooth complete intersection in  $\mathbb{C}\mathbb{P}^{m+k}$  given by  $k$  homogeneous polynomials of degrees  $d_1, \dots, d_k$ . Then  $M_0$  is monotone or negative monotone with  $N_S = |m + k - d|$ , where  $d = d_1 + \dots + d_k$ , unless  $N_S = 0$  (see, e.g., [LM89, p. 88]). To be more precise,  $M_0$  is monotone when  $m + k - d > 0$ , negative monotone when  $m + k - d < 0$ , and  $c_1(TM_0) = 0$  when  $m + k - d = 0$ . Now it is easy to see that  $M = M_0 \times M_1$  is toroidally monotone or toroidally negative monotone (when  $N_S \neq 0$ ) with  $N_T = N_S$ .

Furthermore,  $\pi_1(M) = \pi_1(M_0) \times \pi_1(M_1)$ . In particular,  $H_1(M; \mathbb{Z})/\text{Tor} \neq 0$  when  $H_1(M_1; \mathbb{Z})/\text{Tor} \neq 0$ . Moreover, when  $M_0$  is a complete intersection and  $M_1$  is a Kähler manifold with negative sectional curvature,  $\pi_1(M)$  is hyperbolic and torsion free. Indeed, in this case,  $\pi_1(M) = \pi_1(M_1)$  since complete intersections are simply connected (see, e.g., [Sha94, ch. IX, § 4.1]).

Next, note that for any symplectic manifold  $M$  and  $\mathfrak{f} \in \tilde{\pi}_1(M)$  there exists a Hamiltonian  $H : S^1 \times M \rightarrow \mathbb{R}$  with a hyperbolic one-periodic orbit in the class  $\mathfrak{f}$ ; see, e.g., [Bat15a, Proposition 1.3]. (In fact, one can prescribe arbitrarily a periodic orbit and the linearization of the flow along it.) Thus, whenever  $M$  satisfies the conditions of Theorem 2.1 and  $H_1(M; \mathbb{Z})/\text{Tor} \neq 0$  or  $\pi_1(M)$  is hyperbolic and torsion free, there exists a  $C^\infty$ -open, non-empty set of Hamiltonians this theorem applies to. Furthermore, in the setting of Theorem 2.2, the collection of Hamiltonians with one-periodic orbits in  $\mathfrak{f}$  is non-empty and, as one can easily see, has a locally non-empty interior, i.e., the intersection of this collection with a neighborhood of any of its points has a non-empty interior. It is also not hard to show that the Hamiltonians meeting the requirements of the theorem form a  $C^\infty$ -open and dense subset in this collection. (Indeed, non-degenerate Hamiltonians form a  $C^\infty$ -open and dense subset, and one can further ensure by a  $C^\infty$ -small perturbation of  $H$  that all one-periodic orbits have distinct augmented actions.)

It is worth pointing out that it is not clear how large this open subset is, i.e., how common the Hamiltonians with at least one non-contractible periodic orbit are. This is an interesting question and to the best of our understanding very little is known about this problem. Consider, for instance, time-dependent Hamiltonians on a closed surface  $M$  of positive genus. Then a bump function with small support or, as Paul Seidel pointed out to us, a self-indexing Morse function are examples of Hamiltonians without non-contractible periodic orbits of any period. Furthermore, a simple KAM argument shows that for a  $C^\infty$ -generic autonomous Hamiltonian  $H$  on  $M = \mathbb{T}^2$  such that one of the components of the level  $\{H = 0\}$  is a parallel, no Hamiltonian sufficiently  $C^\infty$ -close to  $H$  has periodic orbits in the free homotopy class collinear to the class of the meridian. (This observation is due to Leonid Polterovich.) However, as far as we know there are no counterexamples to the conjecture that a  $C^1$ -generic (or even  $C^\infty$ -generic) Hamiltonian on  $M$  has a non-contractible periodic orbit. In fact, as has been pointed out to us by Patrice Le Calvez and Andrés Koropecki, the conjecture holds for  $M = \mathbb{T}^2$  for  $C^\infty$ -generic Hamiltonians [LCT15, Proposition J]. (See also [Tal15] for some possibly relevant results.)

Our next theorem is a minor refinement of [Gü13, Theorem 1.1]. This is a result stronger than Theorems 2.1 and 2.2, but applicable to a much more narrow class of manifolds.

**THEOREM 2.4.** *Assume that the class  $[\omega]$  is atoroidal and let  $H$  be a Hamiltonian having a non-degenerate one-periodic orbit  $x$  with homotopy class  $\mathfrak{f}$  such that  $[\mathfrak{f}] \neq 0$  in  $H_1(M; \mathbb{Z})/\text{Tor}$  and  $\mathcal{P}_1(H, [\mathfrak{f}])$  is finite. Then, for every sufficiently large prime  $p$ , the Hamiltonian  $H$  has a simple periodic orbit in the homotopy class  $\mathfrak{f}^p$  and with period either  $p$  or  $p'$ , where  $p'$  is the first prime greater than  $p$ . Moreover, when  $\pi_1(M)$  is hyperbolic and torsion free, the condition  $[\mathfrak{f}] \neq 0$  can be replaced by  $\mathfrak{f} \neq 1$ .*

*Remark 2.5.* Here, as in [Gü13, Theorem 3.1], the requirement that  $x$  is non-degenerate can be replaced by a much less restrictive condition that  $x$  is isolated and has non-trivial local Floer homology; see Theorem 4.1. The key difference between Theorems 2.4 and 2.2 is that a non-degenerate Hamiltonian  $H$  can, at least hypothetically, have several one-periodic orbits in the class  $\mathfrak{f}$ , yet  $\chi(H, I, \mathfrak{f}) = 0$  for any interval  $I$ . (The reason why in Theorem 2.2, in contrast with Theorem 2.4 or [Gü13], it is not sufficient to assume that  $H$  has a non-contractible periodic orbit is that, as has already been mentioned in the introduction, the augmented action filtration is not strict. We will come back to this issue in § 3.3.) The new points of Theorem 2.4 as compared with the results from [Gü13] are the ‘moreover’ part of the theorem and the control of the homotopy classes of the orbits rather than the homology classes.

Among symplectic manifolds with atoroidal class  $[\omega]$  are the Kähler manifolds with negative sectional curvature mentioned above and also some other classes of symplectic manifolds; see, e.g., [BK08, Keç09].

*Remark 2.6 (Growth).* In the settings of Theorems 2.2 and 2.4, the number of simple non-contractible periodic orbits with period less than or equal to  $k$ , or the number of distinct homotopy classes represented by such orbits, is bounded from below by  $\text{const} \cdot k/\ln k$ . An immediate consequence of the theorems is that  $H$  has infinitely many simple periodic orbits with homology class in  $\mathbb{N}[\mathfrak{f}]$  regardless of whether or not the set of one-periodic orbits (in the class  $[\mathfrak{f}]$ ) is finite.

The simplest manifold the above theorems do not apply to is the standard symplectic torus  $\mathbb{T}^{2n}$  with  $2n \geq 4$ . In dimension two, it is easy to see that for strongly non-degenerate Hamiltonian diffeomorphisms the presence of one non-contractible orbit in a homotopy class  $\mathfrak{f}$  implies the existence of infinitely many simple periodic orbits with homotopy class in  $\mathfrak{f}^{\mathbb{N}}$ . (Recall that a diffeomorphism is said to be strongly non-degenerate if all its periodic orbits are non-degenerate.) The proof of this fact amounts to the observation that in dimension two the mean index determines the Conley–Zehnder index and is similar to the proofs of [Abb01, Theorem 5.1.9] or [GG09b, Theorem 1.7]. However, somewhat surprisingly, it is not clear at all whether the non-degeneracy condition here can be replaced as in, e.g., [Gü13, Gü14] by the requirement that the orbit has non-vanishing local Floer homology. When  $2n \geq 4$ , no results along these lines have been established for  $\mathbb{T}^{2n}$ .

It is interesting to contrast the previous theorems with the following, admittedly superficial,  $C^\infty$ -generic existence result. Namely,  $C^\infty$ -generically, the existence of one non-contractible one-periodic orbit is sufficient to infer the existence of infinitely many simple non-contractible periodic orbits under no conditions on  $M$  and with only very minor assumptions about  $\mathfrak{f}$ ; cf. [GG09b]. To be more precise, denote by  $\mathcal{F}_{\mathfrak{f}}$  the collection of strongly non-degenerate



Hamiltonian diffeomorphisms with a one-periodic orbit in  $\mathfrak{f}$ . Clearly,  $\mathcal{F}_{\mathfrak{f}}$  is  $C^\infty$ -open in the group of Hamiltonian diffeomorphisms. As has been pointed out above, this set is non-empty by, e.g., [Bat15a, Proposition 1.3]. Recall also that a subset is called residual, or second Baire category, when it is the intersection of a countable collection of open and dense subsets.

**THEOREM 2.7.** *Assume that  $\mathfrak{f}^k \neq 1$  for all  $k \in \mathbb{N}$ . Then the subset  $\mathcal{F}_{\mathfrak{f}}^\infty$  of  $\mathcal{F}_{\mathfrak{f}}$  formed by Hamiltonian diffeomorphisms with infinitely many simple periodic orbits in  $\mathfrak{f}^\mathbb{N}$  is  $C^\infty$ -residual.*

It is essential that in this theorem the ambient manifold  $M$  is not required to be toroidally monotone or negative monotone. In fact, no condition on  $M$ , other than compactness, is necessary. However, as in the case of other results of this paper, the proof makes use of the Hamiltonian Floer homology, and we need to either assume that  $M$  is weakly monotone or rely on the machinery of virtual cycles.

A consequence of Theorem 2.7 in the spirit of an observation in [PS14] is that non-autonomous Hamiltonian diffeomorphisms (i.e., Hamiltonian diffeomorphisms that cannot be generated by autonomous Hamiltonians) form a  $C^\infty$ -residual subset of  $\mathcal{F}_{\mathfrak{f}}$ . In fact, every  $\varphi \in \mathcal{F}_{\mathfrak{f}}^\infty$  is necessarily non-autonomous. Indeed, when  $k > 1$ , simple  $k$ -periodic orbits of an autonomous Hamiltonian diffeomorphism are never isolated and, hence, in particular, never non-degenerate.

### 3. Augmented action filtration

Our goal in this section is to show that when the augmented action gap is sufficiently large the Floer homology for non-contractible periodic orbits is filtered by the augmented action, and to analyze the behavior of this filtration under continuation maps. As in the rest of the paper, we assume that  $(M^{2n}, \omega)$  is a closed, toroidally monotone or toroidally negative monotone symplectic manifold. However, the construction of the Floer homology (but not of the augmented action filtration) goes through in general for any compact manifold  $M$ , at least when  $M$  is weakly monotone or via the technique of virtual cycles.

#### 3.1 Preliminaries: iterated Hamiltonians

Let  $H : S^1 \times M \rightarrow \mathbb{R}$  be a one-periodic in time Hamiltonian on  $M$ . The augmented action of  $H$  is homogeneous under the iterations of  $\varphi_H$ . To make this more precise, let us recall a few standard definitions.

Let  $K$  and  $H$  be two one-periodic Hamiltonians. The ‘composition’  $K \natural H$  is, by definition, the Hamiltonian

$$(K \natural H)_t = K_t + H_t \circ (\varphi_K^t)^{-1},$$

and the flow of  $K \natural H$  is  $\varphi_K^t \circ \varphi_H^t$ . We set  $H^{\natural k} = H \natural \dots \natural H$  ( $k$  times). Abusing terminology, we will refer to  $H^{\natural k}$  as the  $k$ th iteration of  $H$ . (Note that the flow  $\varphi_{H^{\natural k}}^t = (\varphi_H^t)^k$ ,  $t \in [0, 1]$ , is homotopic with fixed end points to the flow  $\varphi_H^t$ ,  $t \in [0, k]$ .)

In general,  $H^{\natural k}$  is not one-periodic, even when  $H$  is. However,  $H^{\natural k}$  becomes one-periodic when, for example,  $H_0 \equiv 0 \equiv H_1$ . The latter condition can always be met by reparametrizing the Hamiltonian as a function of time without changing the time-one map. This procedure does not affect the Hofer norm, and actions and indices of the periodic orbits. Thus, in what follows, we usually treat  $H^{\natural k}$  as a one-periodic Hamiltonian. Alternatively, the Hamiltonian diffeomorphism  $\varphi_H^k$  can be obtained as the time- $k$  flow of  $H$ . Thus, in some instances such as the proof of Lemma 4.2, it is more convenient to treat  $H^{\natural k}$  as the  $k$ -periodic Hamiltonian  $H_t$  with  $t \in \mathbb{R}/k\mathbb{Z}$ .

We will always state specifically when this is the case. Clearly, these two Hamiltonians, both denoted by  $H^{\natural k}$ , have canonically isomorphic filtered Floer homology.

The  $k$ th iteration of a one-periodic orbit  $x$  of  $H$  is denoted by  $x^k$ . More specifically,  $x^k$  is the  $k$ -periodic orbit  $x(t)$ ,  $t \in [0, k]$ , of  $H$ . There is an action- and mean-index-preserving one-to-one correspondence between the one-periodic orbits of  $H^{\natural k}$  and the  $k$ -periodic orbits of  $H$ . Thus, we can also think of  $x^k$  as the one-periodic orbit  $x^k(t) = \varphi_{H^{\natural k}}^t(x(0))$  of  $H^{\natural k}$ .

Assume now that all iterated homotopy classes  $\mathfrak{f}^k$ ,  $k \in \mathbb{N}$ , are distinct and non-trivial. As above, we have a reference loop  $z \in \mathfrak{f}$  fixed together with a trivialization of  $TM|_z$ . Let us choose the iterated loop  $z^k$  with the ‘iterated trivialization’ as the reference loop for  $\mathfrak{f}^k$ . Then the action and the mean index are both homogeneous with respect to the iteration and, as a consequence,

$$\tilde{\mathcal{A}}_{H^{\natural k}}(x^k) = k\tilde{\mathcal{A}}_H(x).$$

### 3.2 Floer homology for non-contractible periodic orbits

The key tool used in the proofs of Theorems 2.1 and 2.2 is the Floer homology for non-contractible periodic orbits of Hamiltonian diffeomorphisms. Various flavors of Floer homology in this case for both open and closed manifolds have been considered in several other works; see, e.g., [BPS03, GL00, GG14, Gü13, Lee03, Nic06, Web06]. Below we assume that  $M^{2n}$  is closed and toroidally monotone or toroidally negative monotone. In the latter case, to have the Floer homology defined, one must either rely on the machinery of multivalued perturbations (and set the coefficient ring to be  $\mathbb{Q}$ ) or require in addition that  $N_S \geq n$  to ensure that  $M$  is weakly monotone, where  $N_S$  is the minimal spherical Chern number; cf. [LO96].

Let us now briefly describe the elements of the construction of the Floer homology relevant to our argument. Fix  $\mathfrak{f} \in \tilde{\pi}_1(M)$ . Let  $H$  be a Hamiltonian such that all one-periodic orbits of  $H$  in  $\mathfrak{f}$  are non-degenerate. (Here  $x$  is said to be non-degenerate if the linearized return map  $d\varphi_H : T_{x(0)}M \rightarrow T_{x(0)}M$  does not have one as an eigenvalue.) The Floer complex  $\text{CF}(H, \mathfrak{f})$  is generated, over some fixed coefficient ring, by these orbits. The Floer differential is defined in the standard way. With this definition, the complex  $\text{CF}(H, \mathfrak{f})$  is neither graded nor does it carry an action filtration. The homology  $\text{HF}(H, \mathfrak{f})$  of  $\text{CF}(H, \mathfrak{f})$  is equal to zero when  $\mathfrak{f} \neq 1$ . Indeed, by the standard continuation argument,  $\text{HF}(H, \mathfrak{f})$  is independent of  $H$  (cf. § 3.4) and, since all one-periodic orbits of a  $C^2$ -small autonomous Hamiltonian  $H$  are contractible, we have  $\text{HF}(H, \mathfrak{f}) = 0$ . As is well known,  $\text{HF}(H, 1) = H_*(M)$  at least over  $\mathbb{Q}$ ; see, e.g., [MS12] for further references.

To give the complex  $\text{CF}(H, \mathfrak{f})$  some more structure, let us fix a reference loop  $z \in \mathfrak{f}$  and a trivialization of  $TM|_z$ . Using this trivialization, we can define the Conley–Zehnder index  $\mu_{\text{CZ}}(H, \bar{x}) \in \mathbb{Z}$  of a capped non-degenerate orbit  $\bar{x}$  as in, e.g., [MS12, SZ92]. For future reference, note that

$$|\Delta_H(\bar{x}) - \mu_{\text{CZ}}(\bar{x})| \leq n. \tag{3.1}$$

Similarly to the contractible case, the Conley–Zehnder index  $\mu_{\text{CZ}}(H, x)$  of an orbit without capping is defined only modulo  $2N_T$ . As a result, we obtain a  $\mathbb{Z}_{2N_T}$ -grading of the complex  $\text{CF}(H, \mathfrak{f})$  and of the homology  $\text{HF}(H, \mathfrak{f})$  and, in particular, a  $\mathbb{Z}_2$ -grading. Replacing the one-periodic orbits of  $H$  by the capped one-periodic orbits, one could define the Floer complex and the homology of  $H$  as a module over a suitably chosen Novikov ring and, as in the contractible case, this complex and the homology would be  $\mathbb{Z}$ -graded and filtered by the action. However, for our purposes it is more convenient to work with the homology  $\text{HF}(H, \mathfrak{f})$  and the complex  $\text{CF}(H, \mathfrak{f})$  generated by the non-capped orbits and defined as above.

The constructions from this section readily carry over to the case where a single free homotopy class  $f$  is replaced by a collection of free homotopy classes. For instance, one can specify the collection of free homotopy classes of loops by prescribing a homology class.

**3.3 Filtration**

Let, as above,  $M^{2n}$  be a toroidally monotone or toroidally negative monotone closed symplectic manifold with monotonicity constant  $\lambda$ . In what follows, we have a free homotopy class  $f$  or a collection of such classes together with the reference loops and trivializations fixed and suppressed in the notation. Thus, we write  $CF(H)$  for  $CF(H, f)$ , etc. With these auxiliary data fixed, the augmented action spectrum  $\tilde{\mathcal{S}}(H) := \tilde{\mathcal{S}}(H, f)$  is defined for any Hamiltonian  $H$  on  $M$ .

The *augmented action gap* is the infimum of the distance between two distinct points in the augmented action spectrum  $\tilde{\mathcal{S}}(H)$ , i.e.,

$$\text{gap}(H) = \inf |s - s'| \in [0, \infty], \quad \text{where } s \text{ and } s' \neq s \text{ are in } \tilde{\mathcal{S}}(H).$$

We emphasize that  $\text{gap}(H)$  is defined even when  $H$  is degenerate. It is also worth pointing out that  $\text{gap}(H)$  is neither upper nor lower semicontinuous in  $H$ .

Set

$$c_0(M) = |\lambda| \frac{2n \pm 1}{2}, \tag{3.2}$$

where the sign  $\pm$  is  $\text{sign}(\lambda)$ . We say that the *gap condition* is satisfied whenever

$$\text{gap}(H) > c_0(M). \tag{3.3}$$

**PROPOSITION 3.1.** *Assume that all one-periodic orbits of  $H$  in  $f$  are non-degenerate and (3.3) holds. Then the complex  $CF(H)$ , and hence the homology  $HF(H)$ , is filtered by the augmented action. In other words,*

$$\tilde{\mathcal{A}}_H(y) \leq \tilde{\mathcal{A}}_H(x) \tag{3.4}$$

whenever  $y$  enters  $\partial x = \sum a_y y$  with non-zero coefficient.

*Remark 3.2.* In contrast with the standard action filtration, the augmented action filtration is not necessarily strict, i.e., equality in (3.4) can occur. Note also that in this proposition it suffices to have a non-strict inequality in the gap condition (3.3).

*Proof.* Throughout the proof, let us assume that  $\lambda \geq 0$ , i.e.,  $M$  is toroidally monotone. The negative monotone case is dealt with by a similar (up to some signs) calculation.

To establish (3.4), let us fix a capping of  $x$ . Then an orbit  $y$  entering  $\partial x$  with non-zero coefficient inherits a capping from  $\bar{x}$ . We have

$$\begin{aligned} \tilde{\mathcal{A}}_H(y) &= \mathcal{A}_H(\bar{y}) - \frac{\lambda}{2} \Delta_H(\bar{y}) \\ &< \mathcal{A}_H(\bar{x}) - \frac{\lambda}{2} (\mu_{CZ}(\bar{y}) - n) \\ &= \mathcal{A}_H(\bar{x}) - \frac{\lambda}{2} (\mu_{CZ}(\bar{x}) - n - 1) \\ &\leq \mathcal{A}_H(\bar{x}) - \frac{\lambda}{2} (\Delta_H(\bar{x}) - 2n - 1) \\ &= \tilde{\mathcal{A}}_H(x) + c_0(M). \end{aligned}$$

Here we used (3.1) and the facts that  $\mu_{CZ}(\bar{y}) = \mu_{CZ}(\bar{x}) - 1$  and that  $\partial$  is action decreasing. Thus, we have shown that  $\partial$  does not increase the augmented action by more than  $c_0(M)$ . Now the required inequality (3.4) follows once the augmented action gap is greater than  $c_0(M)$ , i.e., when the gap condition (3.3) holds.  $\square$

With Proposition 3.1 in mind, we can define the augmented action filtration on the homology exactly in the same way as for the ordinary action. Thus, assume that (3.3) is satisfied and  $a \notin \tilde{\mathcal{S}}(H)$  and denote by  $\widetilde{CF}^{(-\infty, a)}(H)$  the subcomplex of  $CF(H)$  generated by the orbits with augmented action below  $a$ . Let  $\widetilde{HF}^{(-\infty, a)}(H)$  be the homology of this subcomplex. Furthermore, when  $I = (a, b)$  is an interval with end points outside  $\tilde{\mathcal{S}}(H)$ , we set

$$\widetilde{CF}^I(H) = \widetilde{CF}^{(-\infty, b)}(H) / \widetilde{CF}^{(-\infty, a)}(H).$$

In other words,  $\widetilde{CF}^I(H)$  is the complex generated by the orbits with augmented action in  $I$ , equipped with the naturally defined differential. We denote the homology of this complex by  $\widetilde{HF}^I(H)$ . (The role of the tilde here is to emphasize that we use the augmented action rather than the ordinary action and that  $I$  is an augmented action range.) We have the long exact sequence

$$\dots \rightarrow \widetilde{HF}^{(-\infty, a)}(H) \rightarrow \widetilde{HF}^{(-\infty, b)}(H) \rightarrow \widetilde{HF}^I(H) \rightarrow \dots$$

and a similar exact sequence for three intervals

$$\dots \rightarrow \widetilde{HF}^{(c, a)}(H) \rightarrow \widetilde{HF}^{(c, b)}(H) \rightarrow \widetilde{HF}^{(a, b)}(H) \rightarrow \dots \tag{3.5}$$

Our next goal is to show that the construction of the augmented action filtered Floer homology extends by continuity to all, not necessarily non-degenerate, Hamiltonians.

**PROPOSITION 3.3.** *Let  $H$  be a Hamiltonian on  $M$ , not necessarily non-degenerate, such that the gap condition (3.3) is satisfied and let  $a \notin \tilde{\mathcal{S}}(H)$ . Then, for any non-degenerate Hamiltonian  $K$  sufficiently  $C^1$ -close to  $H$ , the subspace  $\widetilde{CF}^{(-\infty, a)}(K) \subset CF(K)$  is a subcomplex.*

This result is not an immediate consequence of Proposition 3.1. Since the augmented action gap is not lower semicontinuous in the Hamiltonian, we cannot guarantee that (3.3) holds for  $K$  if it holds for  $H$ , and thus *a priori* Proposition 3.1 need not apply to  $K$ .

*Proof.* Let  $x$  be a one-periodic orbit of  $K$  with  $\tilde{\mathcal{A}}_K(x) < a$  and let  $y$  be an orbit entering  $\partial x$  with non-zero coefficient. We need to show that  $\tilde{\mathcal{A}}_K(y) < a$ .

The orbits  $x$  and  $y$  are  $C^1$ -small perturbations of one-periodic orbits  $x'$  and  $y'$  of  $H$  with augmented actions close to those of  $x$  and  $y$ . By continuity of the augmented action spectrum, we necessarily have  $\tilde{\mathcal{A}}_H(x') < a$  when  $K$  is  $C^1$ -close to  $H$ .

If  $\tilde{\mathcal{A}}_H(y') > \tilde{\mathcal{A}}_H(x')$ , we have  $\tilde{\mathcal{A}}_H(y') - \tilde{\mathcal{A}}_H(x') > c_0(M)$  by (3.3), and therefore  $\tilde{\mathcal{A}}_K(y) - \tilde{\mathcal{A}}_K(x) > c_0(M)$ . This is impossible because, as we have seen from the proof of Proposition 3.1, the differential cannot increase the augmented action by more than  $c_0(M)$ . Thus,  $\tilde{\mathcal{A}}_H(y') \leq \tilde{\mathcal{A}}_H(x')$ . Then

$$\tilde{\mathcal{A}}_K(y) \approx \tilde{\mathcal{A}}_H(y') \leq \tilde{\mathcal{A}}_H(x') < a,$$

and hence  $\tilde{\mathcal{A}}_K(y) < a$  when  $K$  is  $C^1$ -close to  $H$ .  $\square$

Now, for any Hamiltonian  $H$ , when the end points of an interval  $I$  are outside  $\tilde{\mathcal{S}}(H)$  and (3.3) holds, we can, utilizing Proposition 3.3, set  $\widetilde{\text{HF}}^I(H) := \widetilde{\text{HF}}^I(K)$ , where  $K$  is a  $C^1$ -small non-degenerate perturbation of  $H$ . Using standard continuation arguments (cf. § 3.4), it is easy to see that the resulting homology is well defined, i.e., independent of  $K$ .

*Example 3.4.* The setting we are interested in where the gap condition (3.3) is satisfied is when  $H$  is a high prime order iteration of some Hamiltonian  $F$ , i.e.,  $H = F^{\flat k}$  and  $k$  is a large prime. In this case, either  $F$  has simple  $k$ -periodic orbits or  $\text{gap}(H) = k \text{gap}(F)$ . Thus, either new periodic orbits are created or the gap grows linearly under the iterations of  $F$  and eventually becomes greater than  $c_0(M)$ .

It is clear that all these constructions respect the  $\mathbb{Z}_{2N_T}$ -grading (and hence the  $\mathbb{Z}_2$ -grading) of the complexes and the homology. Thus, for instance, (3.5) is an exact sequence of graded complexes and the connecting map has degree  $-1$ .

For degenerate Hamiltonians with isolated one-periodic orbits, one can, similarly to the case of the standard action filtration, view the local Floer homology as building blocks for the Floer homology filtered by the augmented action. For instance, assume that  $\tilde{\mathcal{S}}(H) \cap I = \{c\}$  and all one-periodic orbits  $x$  with augmented action  $c$  are isolated. Then it is not hard to see that there exists a spectral sequence with  $E^1 = \bigoplus_x \text{HF}(x)$  converging to  $\widetilde{\text{HF}}^I(H)$ , where  $\text{HF}(x)$  stands for the local Floer homology of  $x$ . (We refer the reader to, e.g., [Gin10, GG10, McL12] for the definition and a discussion of the local Floer homology.) In contrast with the case of the ordinary action filtration, we do not necessarily have  $E^1 = \widetilde{\text{HF}}^I(H)$  even when  $H$  is non-degenerate. The reason is that the augmented action filtration is not strict and the Floer differential, or more generally Floer trajectories, can connect orbits with equal augmented action.

However, as is easy to see, for any interval  $I$  with end points outside  $\tilde{\mathcal{S}}(H)$ , we have

$$\chi(H, I) = (-1)^n [\dim \widetilde{\text{HF}}^I_{\text{even}}(H) - \dim \widetilde{\text{HF}}^I_{\text{odd}}(H)]. \tag{3.6}$$

In particular,  $\widetilde{\text{HF}}^I(H) \neq 0$  if  $\chi(H, I) \neq 0$ . (Here, as everywhere in this section, we have suppressed the class  $\mathfrak{f}$  in the notation.)

### 3.4 Homotopy and continuation

The behavior of the augmented action under homotopies is similar to that of the ordinary action. Namely, recall that a continuation map shifts the action filtration upward by a certain constant; see, e.g., [Gin07, § 3.2.2]. This is still true for the augmented action, although the size of the shift is slightly different. Furthermore, when the homotopy is monotone decreasing, the action shift is zero, and the induced map in homology preserves the action filtration. This fact does not have a direct analogue for the augmented action, but the augmented action filtration is preserved when the augmented action gaps for the Hamiltonians are large enough.

To be more precise, consider a homotopy  $H^s$  from a Hamiltonian  $H^0$  to a Hamiltonian  $H^1$  on  $M$ , and set

$$c_a(H^s) = \int_{-\infty}^{\infty} \int_{S^1} \max_M \partial_s H^s dt ds.$$

For instance,

$$c_a(H^s) = \int_{S^1} \max_M (H^1 - H^0) dt$$

when  $H^s$  is a linear homotopy from  $H^0$  to  $H^1$ . The augmented action shift is governed by the constant

$$c_h(H^s) = \max\{0, c_a(H^s)\} + |\lambda|n \geq 0. \tag{3.7}$$

PROPOSITION 3.5. Assume that both Hamiltonians  $H^0$  and  $H^1$  satisfy (3.3), i.e.,

$$\text{gap}(H^0) > c_0(M) \quad \text{and} \quad \text{gap}(H^1) > c_0(M). \tag{3.8}$$

Then a homotopy  $H^s$  from  $H^0$  to  $H^1$  induces a map in the Floer homology shifting the action filtration upward by  $c_h(H^s)$ :

$$\widetilde{\text{HF}}^I(H^0) \rightarrow \widetilde{\text{HF}}^{c_h(H^s)+I}(H^1),$$

where  $c_h(H^s) + I$  stands for the interval  $I$  moved to the right by  $c_h(H^s)$ . Furthermore, if

$$\text{gap}(H^0) > c_h(H^s) \quad \text{and} \quad \text{gap}(H^1) > c_h(H^s)$$

in addition to (3.8), the map induced by the homotopy preserves the augmented action filtration, i.e., we have  $\widetilde{\text{HF}}^I(H^0) \rightarrow \widetilde{\text{HF}}^I(H^1)$ .

Note that here the shift  $c_h(H^s)$  can be replaced by any constant  $a > c_h(H^s)$ . The proof of the proposition is standard and we omit it. Here we only mention that the first term in (3.7) is the maximal action shift induced by the homotopy (see, e.g., again [Gin07, §3.2.2]) and the second term is the maximal mean index shift, as can be seen from an argument similar to the proof of Proposition 3.1.

*Remark 3.6.* The arguments from this section carry over to contractible periodic orbits, i.e., to the case where  $\mathfrak{f} = 1$ , with some simplifications and straightforward modifications. Namely, in this case, it is enough to assume that  $M$  is monotone or negative monotone to have the augmented action defined; see [GG09a]. An analogue of (3.3) is still sufficient to ensure that the Floer complex and the homology are filtered by the augmented action and Propositions 3.1, 3.3, and 3.5 still hold.

### 4. Proofs

With the action filtration introduced, we are now in a position to prove the main results of the paper.

#### 4.1 Proofs of Theorems 2.2 and 2.4

*Proof of Theorem 2.2: the case  $[\mathfrak{f}] \neq 0$ .* Since  $\mathcal{P}_1(H, [\mathfrak{f}])$  is finite, only finitely many distinct free homotopy classes  $\mathfrak{f}_i \in \tilde{\pi}_1(M)$  occur as the free homotopy classes of one-periodic orbits of  $H$  in the homology class  $[\mathfrak{f}]$ . We claim that then, for a sufficiently large prime  $p$ , the classes  $\mathfrak{f}_i^p$  are also distinct.

To see this, first note that for any two elements  $g \neq h$  in any group there is at most one prime  $p$  such that  $g^p = h^p$ . Indeed, assume that there are two such distinct primes  $p$  and  $q$ . Then, since  $p$  and  $q$  are relatively prime,  $ap + bq = 1$  for some integers  $a$  and  $b$ . Hence,

$$g = (g^p)^a (g^q)^b = (h^p)^a (h^q)^b = h,$$

which is impossible since  $g \neq h$ . Clearly, the same is true for conjugacy classes. As a consequence, for any finite collection of distinct conjugacy classes, their large prime powers are also distinct.

Throughout the proof we will always require  $p$  to be a sufficiently large prime to satisfy the above condition for the collection  $\mathfrak{f}_i$ . (Later we will need to impose additional lower bounds on  $p$ .)

Let us assume that  $H$  has no simple  $p$ -periodic orbits in the class  $\mathfrak{f}^p$ . Our goal is to show that it has a simple  $p'$ -periodic orbit, where  $p'$  is the first prime greater than  $p$ , in the homotopy class  $\mathfrak{f}^p$ .

Then all  $p$ -periodic orbits in  $\mathfrak{f}^p$  are the  $p$ th iterations of one-periodic orbits, since  $p$  is prime. Furthermore, by the above requirement on  $p$ , these one-periodic orbits are necessarily in the free homotopy class  $\mathfrak{f}$ . Thus, we have

$$\tilde{\mathcal{S}}(H^{\natural p}, \mathfrak{f}^p) = p \tilde{\mathcal{S}}(H, \mathfrak{f}) \tag{4.1}$$

with respect to the  $p$ th iteration of the reference loop  $z \in \mathfrak{f}$  and of the trivialization of  $TM|_z$ . As a consequence,

$$\text{gap}(H^{\natural p}, \mathfrak{f}^p) = p \text{gap}(H, \mathfrak{f}), \tag{4.2}$$

and the augmented action filtration on the Floer homology  $\widetilde{\text{HF}}(H^{\natural p}, \mathfrak{f}^p)$  is defined once  $p$  is so large that  $p \text{gap}(H, \mathfrak{f}) > c_0(M)$ ; see (3.2) and § 3.3. We also have

$$\tilde{\mathcal{S}}(H^{\natural p}, \mathfrak{f}^p) \cap pI = p(\tilde{\mathcal{S}}(H, \mathfrak{f}) \cap I). \tag{4.3}$$

Here  $pI = (pa, pb)$  for  $I = (a, b)$ .

Next we claim that, when  $p$  is sufficiently large,

$$\chi(H^{\natural p}, pI, \mathfrak{f}^p) = \chi(H, I, \mathfrak{f}). \tag{4.4}$$

To see this, denote by  $x_i$  the one-periodic orbits of  $H$  in the class  $\mathfrak{f}$  and with augmented action in  $I$ . This is a finite collection of orbits since  $\mathcal{P}_1(H, [\mathfrak{f}])$  is finite. Then all sufficiently large primes  $p$  are admissible in the sense of [GG10] for all orbits  $x_i$ , i.e., 1 has the same multiplicity as a generalized eigenvalue of the linearized return maps  $d\varphi_H$  and  $d\varphi_H^p$  at  $x_i$  and the two maps have the same number of eigenvalues in  $(-1, 0)$ . (Indeed, it suffices to require  $p$  to be larger than 2 and larger than the degree of any root of unity among the eigenvalues of  $d\varphi_H$  at  $x_i$ .) By the Shub–Sullivan theorem (see [CMPY83, SS74]), the orbits  $x_i$  and  $x_i^p$  have the same Poincaré–Hopf index. The orbits  $x_i^p$  are the only  $p$ -periodic orbits of  $H$  in  $\mathfrak{f}^p$  with augmented action in  $pI$ , and (4.4) follows. Alternatively, one can argue as in the proof of the case  $\mathfrak{f} \neq 1$  of the theorem; see below.

By (3.6) and since  $\chi(H, I, \mathfrak{f}) \neq 0$ , we conclude that

$$\widetilde{\text{HF}}^{pI}(H^{\natural p}, \mathfrak{f}^p) \neq 0.$$

Now we are in a position to show that  $H$  must have at least one  $p'$ -periodic orbit in the class  $\mathfrak{f}^p$ , where  $p'$  is the first prime greater than  $p$ , provided again that  $p$  is sufficiently large. Then, as the last step of the proof, we will show that this  $p'$ -periodic orbit is necessarily simple.

Arguing by contradiction, assume that there are no such orbits. Then

$$\text{gap}(H^{\natural p'}, \mathfrak{f}^p) = \infty$$

and, obviously, the augmented action filtration is defined on the Floer homology for  $H^{\natural p'}$  and  $\mathfrak{f}^p$ . (Of course, the resulting complex and the homology are zero for any augmented action interval, but this is not essential at this point.) By roughly following the line of reasoning from [Gü13, Gü14] and relying on the fact that the filtered homology is defined, we will show that the homology is non-trivial for a certain augmented action interval and thus arrive at a contradiction with the assumption that  $H$  has no  $p'$ -periodic orbits in the class  $\mathfrak{f}^p$ .

Set

$$e_+ = \max \left\{ \int_{S^1} \max_M H_t dt, 0 \right\}$$

and

$$e_- = \max \left\{ - \int_{S^1} \min_M H_t dt, 0 \right\}.$$

Then

$$a_{\pm} := (p' - p)e_{\pm} + |\lambda|n \geq c_{\pm},$$

where the constants  $c_{\pm} = c_h(H^s)$  are defined by (3.7) for the linear homotopies from  $H^{\natural p}$  to  $H^{\natural p'}$  and from  $H^{\natural p'}$  to  $H^{\natural p}$ .

Furthermore, recall that  $p' - p = o(p)$  as  $p \rightarrow \infty$ ; see, e.g., [BHP01]. Thus, when  $p$  is sufficiently large, we have

$$\text{gap}(H^{\natural p}, \mathfrak{f}^p) = p \text{gap}(H, \mathfrak{f}) > a_{\pm} \geq c_{\pm}.$$

Hence, the conditions of Proposition 3.5 are satisfied, and the continuation maps

$$\widetilde{\text{HF}}^{pI}(H^{\natural p}, \mathfrak{f}^p) \rightarrow \widetilde{\text{HF}}^{pI+a_+}(H^{\natural p'}, \mathfrak{f}^p)$$

and

$$\widetilde{\text{HF}}^{pI+a_+}(H^{\natural p'}, \mathfrak{f}^p) \rightarrow \widetilde{\text{HF}}^{pI+a_++a_-}(H^{\natural p}, \mathfrak{f}^p)$$

are defined.

Now consider the following commutative diagram.

$$\begin{array}{ccc} \widetilde{\text{HF}}^{pI}(H^{\natural p}, \mathfrak{f}^p) & & \\ \downarrow & \searrow \cong & \\ \widetilde{\text{HF}}^{pI+a_+}(H^{\natural p'}, \mathfrak{f}^p) & \longrightarrow & \widetilde{\text{HF}}^{pI+a_++a_-}(H^{\natural p}, \mathfrak{f}^p) \end{array}$$

Here the diagonal map is an isomorphism. To see this, denote by  $\delta > 0$  the distance from the end points of  $I$  to  $\tilde{\mathcal{S}}(H, \mathfrak{f})$ . Then the distance from the end points of  $pI$  to  $\tilde{\mathcal{S}}(H^{\natural p}, \mathfrak{f}^p)$  is  $p\delta$  and, when  $p$  is large,

$$p\delta > a_+ + a_-$$

again because  $p' - p = o(p)$ . Hence, the intervals  $(pI + a_+ + a_-) \setminus pI$  and  $pI \setminus (pI + a_+ + a_-)$  contain no points of  $\tilde{\mathcal{S}}(H^{\natural p}, \mathfrak{f}^p)$ , and the diagonal map is indeed an isomorphism. (In fact, this argument shows that, as in the second part of Proposition 3.5, one can eliminate the shifts  $a_{\pm}$  and  $a_+ + a_-$  in the continuation maps when  $p$  is sufficiently large.)

Moreover, as we have shown above,  $\widetilde{\text{HF}}^{pI}(H^{\natural p}, \mathfrak{f}^p) \neq 0$ . Therefore, the middle group  $\widetilde{\text{HF}}^{pI+a_+}(H^{\natural p'}, \mathfrak{f}^p)$  in the diagram is also non-trivial, and thus  $H$  must have a  $p'$ -periodic orbit in the homotopy class  $\mathfrak{f}^p$ .

It remains to show that this orbit is necessarily simple. However, otherwise, it would be the  $p'$ th iteration of a one-periodic orbit in the homology class  $p[\mathfrak{f}]/p'$ . This is impossible because  $p[\mathfrak{f}]/p'$  is not an integer homology class when  $p, p'$  are large since  $[\mathfrak{f}] \neq 0$ . This completes the proof of the case  $[\mathfrak{f}] \neq 0$  of the theorem.  $\square$



*Proof of Theorem 2.2: the case  $f \neq 1$ .* The proof follows the same path as in the case where  $[f] \neq 0$ , and here we only indicate the necessary changes in the argument.

The key to the proof is the fact that when  $\pi_1(M)$  is a torsion-free hyperbolic group and  $f \neq 1$  in  $\tilde{\pi}_1(M)$ , there exists a constant  $r(f) \in \mathbb{N}$  such that the equation

$$f^p = h^q$$

in  $\tilde{\pi}_1(M)$ , where  $p$  and  $q$  are primes greater than  $r(f)$  and  $h \in \tilde{\pi}_1(M)$ , is satisfied only when  $h = f$  and  $p = q$ .

To see this, we first note that it is sufficient to prove this fact for  $\pi_1(M)$  rather than  $\tilde{\pi}_1(M)$ . In other words, given  $f \in \pi_1(M)$ ,  $f \neq 1$ , we need to show that  $f^p = h^q$  for sufficiently large primes  $p$  and  $q$  (depending on  $f$ ) only when  $h = f$  and  $p = q$ . To this end, recall that for any hyperbolic group  $G$ , every element  $f \in G$  of infinite order is contained in a unique maximal virtually cyclic subgroup  $E(f)$  and

$$E(f) = \{g \in G \mid g^{-1}f^l g = f^{\pm l} \text{ for some } l \in \mathbb{N}\};$$

see [Ols93] (and also [Gro87]). Applying this to  $f \neq 1$  in  $G = \pi_1(M)$ , which is also assumed to be torsion free, and using the fact that a torsion-free and virtually cyclic group is cyclic, we conclude that  $E(f)$  is infinite cyclic, i.e.,  $\mathbb{Z}$ . Furthermore,  $h \in E(f)$  as a consequence of the condition  $f^p = h^q$ . Indeed,

$$h^{-1}f^p h = h^{-1}h^q h = h^q = f^p.$$

This reduces the question to the case where  $f$  and  $h$  belong to the infinite cyclic group  $E(f)$ , and in this case the result is obvious.<sup>1</sup>

From now on, we require that  $p \geq r(f)$ . (Later we will need to introduce additional lower bounds for  $p$ .) As in the proof of the first case of the theorem, assume that  $H$  has no simple  $p$ -periodic orbits in the class  $f^p$ . Then every  $p$ -periodic orbit is the  $p$ th iteration of a one-periodic orbit and, by the above observation (with  $p = q$ ), this one-periodic orbit must also be in the class  $f$ . Clearly, (4.1), (4.2), and (4.3) still hold.

Furthermore, (4.4) also holds, i.e.,  $\chi(H^{bp}, pI, f^p) = \chi(H, I, f)$ , although now the reason is slightly different. Consider the set  $F$  of the initial conditions  $x(0)$  for all one-periodic orbits of  $H$  in the class  $f$  and with augmented action in  $I$ . Since the end points of  $I$  are outside  $\tilde{\mathcal{S}}(H, f)$ , the set  $F$  is closed. Under a small non-degenerate perturbation  $\tilde{H}$  of  $H$ , the set  $F$  splits into a finite collection of the initial conditions of the orbits  $\tilde{x}_i$  of  $\tilde{H}$  in  $f$  with augmented action in  $I$ , and  $\chi(H, I, f)$  is the sum of the Poincaré–Hopf indices of the orbits  $\tilde{x}_i$ . We can furthermore ensure that there are no roots of unity among the Floquet multipliers of these orbits. By our assumptions,  $F$  is also the set of the initial conditions for all  $p$ -periodic orbits of  $H$  in the class  $f^p$  with augmented action in  $pI$ . If  $\tilde{H}$  is sufficiently close to  $H$ , the only  $p$ -periodic orbits of  $\tilde{H}$  in  $f^p$  with augmented action in  $pI$  are  $\tilde{x}_i^p$ . Hence,  $\chi(H^{bp}, pI, f^p)$  is the sum of the Poincaré–Hopf indices of the orbits  $\tilde{x}_i^p$ . When  $p > 2$ , the orbits  $\tilde{x}_i$  and  $\tilde{x}_i^p$  have the same Poincaré–Hopf index due to the assumption that none of the Floquet multipliers is a root of unity. As a consequence, we have (4.4).

The rest of the proof is identical to the argument in the case where  $[f] \neq 0$  except for the very last step. Thus, we have proved the existence of a  $p'$ -periodic orbit  $x$  in the class  $f^p$  and

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<sup>1</sup> The authors are grateful to Denis Osin for this argument.

now need to show that this orbit is simple. Assume the contrary. Then, since  $p'$  is prime,  $x$  is necessarily the  $p'$ th iteration of a one-periodic orbit in some class  $\mathfrak{h}$ . We have

$$\mathfrak{f}^p = \mathfrak{h}^{p'}$$

and hence  $p' = p$  when  $p > r(\mathfrak{f})$ . This is impossible since  $p'$  is the first prime greater than  $p$ .  $\square$

Turning to Theorem 2.4, note that, as in [Gü13], a more general result holds. Namely, recall that the local Floer homology  $\text{HF}(x)$  is associated to an isolated periodic orbit  $x$  of  $H$ . The group  $\text{HF}(x)$ , already mentioned in § 3.3, is roughly speaking the homology of the Floer complex generated by the orbits which  $x$  splits into under a non-degenerate perturbation; see, e.g., [GG10] for more details. In particular, when  $x$  is non-degenerate,  $\text{HF}(x)$  is equal to the ground ring and concentrated in degree  $\mu_{\text{CZ}}(x)$ . We have the following generalization of Theorem 2.4.

**THEOREM 4.1.** *Assume that the class  $[\omega]$  is atoroidal and let  $H$  be a Hamiltonian having an isolated one-periodic orbit  $x$  with homotopy class  $\mathfrak{f}$  such that  $\text{HF}(x) \neq 0$  and that  $[\mathfrak{f}] \neq 0$  in  $\text{H}_1(M; \mathbb{Z})/\text{Tor}$  and  $\mathcal{P}_1(H, [\mathfrak{f}])$  is finite. Then, for every sufficiently large prime  $p$ , the Hamiltonian  $H$  has a simple periodic orbit in the homotopy class  $\mathfrak{f}^p$  and with period either  $p$  or  $p'$ , where  $p'$  is the first prime greater than  $p$ . Moreover, when  $\pi_1(M)$  is hyperbolic and torsion free, the condition  $[\mathfrak{f}] \neq 0$  can be replaced by  $\mathfrak{f} \neq 1$ .*

The main new point here is the ‘moreover’ part of the theorem. The case of the theorem where  $[\mathfrak{f}] \neq 0$ , proved in [Gü13], is included for the sake of completeness. The proof of the ‘moreover’ part is a combination of the proof of [Gü13, Theorem 3.1] and the proof of the case  $\mathfrak{f} \neq 1$  of Theorem 2.2. Here we only briefly touch upon this argument.

*On the proof of Theorem 4.1.* The key difference between the settings of Theorems 2.4 and 4.1 and that of Theorem 2.2 is that now the class  $[\omega]$  is atoroidal and thus we have the standard action filtration  $\text{HF}^I(H; \mathfrak{f})$  on the Floer homology of  $H$  rather than the augmented action filtration. On the level of complexes, the action filtration is strictly monotone, i.e., the differential is strictly action decreasing. (In contrast, the augmented action is only non-increasing; see the discussion in § 3.3.) As a consequence,  $\text{HF}^I(H; \mathfrak{f}) \neq 0$  when  $I$  is a small interval centered at the action  $\mathcal{A}_H(x)$  whenever  $x$  and  $\mathcal{A}_H(x)$  are isolated and  $\text{HF}(x) \neq 0$ . Furthermore, when  $H$  has no simple  $p$ -periodic orbits in the class  $\mathfrak{f}^p$ , we have  $\text{HF}^{pI}(H^{\sharp p}; \mathfrak{f}^p) \neq 0$  by the persistence of the local Floer homology results from [GG09a]. With this in mind, one argues essentially word-for-word as in the proof of Theorem 2.2, with some straightforward simplifications. We omit the details.  $\square$

### 4.2 Proof of Theorem 2.1

Arguing by contradiction, assume that  $H$  has only finitely many simple periodic orbits with homotopy class in  $\mathfrak{f}^{\mathbb{N}} = \{\mathfrak{f}^k \mid k \in \mathbb{N}\}$ , where  $\mathfrak{f} = \llbracket x \rrbracket$ . We denote these orbits by  $x_j$  and let  $k_j$  be the period of  $x_j$ . Set  $F = H^{\sharp 2k_0}$  and  $y_j = x_j^{2k_0/k_j}$ , where  $k_0$  is the least common multiple of the periods  $k_j$ . The orbits  $y_j$  are the one-periodic orbits of  $F$ .

We claim that for all  $k \in \mathbb{N}$  every  $k$ -periodic orbit  $z$  of  $F$  in the collection of the homotopy classes  $\mathfrak{f}^{\mathbb{N}}$  is the  $k$ th iteration of one of the orbits  $y_j$ . Indeed, then  $z$  is also a  $2kk_0$ -periodic orbit of  $H$  in  $\mathfrak{f}^{\mathbb{N}}$ . Hence,  $z = x_j^{2kk_0/k_j} = y_j^k$  for some  $j$ .

Thus, we have a collection of free homotopy classes  $\mathfrak{f}^{\mathbb{N}}$  generated by  $\mathfrak{f} \in \tilde{\pi}_1(M)$ , and a Hamiltonian  $F$  with finitely many one-periodic orbits  $y_j$  in  $\mathfrak{f}^{\mathbb{N}}$  and no other simple periodic

orbits in  $\mathfrak{f}^{\mathbb{N}}$ . One of these orbits,  $h$ , is an even iteration of the original hyperbolic orbit. Hence,  $h$  is hyperbolic with an even number of eigenvalues in  $(-1, 0)$ .

From now on we focus on the Hamiltonian  $F$  and its periodic orbits. Set  $\mathfrak{h} = \llbracket h \rrbracket$ ; clearly,  $\mathfrak{h}^{\mathbb{N}} \subset \mathfrak{f}^{\mathbb{N}}$ .

Let us fix reference curves and trivializations for the collection  $\mathfrak{h}^{\mathbb{N}}$ . Namely, it is convenient to take  $h$  as a reference curve for  $\mathfrak{h}$ . Then the reference trivialization is fixed by the condition that  $\Delta_F(\bar{h}) = 0$ , where  $\bar{h}$  stands for  $h$  equipped with the ‘identity’ capping. (Such a trivialization exists since  $h$  is hyperbolic and has an even number of eigenvalues in  $(-1, 0)$ , and hence the mean index of  $h$  with respect to any trivialization is an even integer.) The class  $\mathfrak{h}^k$  is then given the iterated reference curve  $h^k$  and the ‘iterated’ trivialization. We fix cappings of the orbits  $y_j$ , suppressed in the notation, and equip the iterated orbits with ‘iterated’ cappings. As a consequence, the action, the mean index, and the augmented action are homogeneous under iterations for periodic orbits in  $\mathfrak{h}^{\mathbb{N}}$ . (It is essential here that all free homotopy classes in  $\mathfrak{h}^{\mathbb{N}}$  are distinct and non-trivial, and hence the reference curve and the trivialization are well defined.)

Without loss of generality, by adding if necessary a constant to  $F$ , we can ensure that  $\tilde{\mathcal{A}}_F(h) = 0$ .

By our assumptions, we have

$$\tilde{\mathcal{S}}(F^{\natural k}, \mathfrak{h}^k) = k\tilde{\mathcal{S}}(F, \mathfrak{h})$$

and

$$\text{gap}(F^{\natural k}, \mathfrak{h}^k) = k \text{gap}(F, \mathfrak{h}).$$

It follows that the augmented action filtered Floer homology of  $F^{\natural k}$  is defined when  $k$  is large enough.

Furthermore, let  $I$  be an interval such that  $0 = \tilde{\mathcal{A}}_F(h)$  is the only point in  $\tilde{\mathcal{S}}(F, \mathfrak{h}) \cap I$ , and the end points of  $I$  are not in  $\tilde{\mathcal{S}}(F, \mathfrak{h})$ . Then we also have

$$\tilde{\mathcal{S}}(F^{\natural k}, \mathfrak{h}^k) \cap kI = \{0\} \tag{4.5}$$

and the end points of  $kI$  are outside  $\tilde{\mathcal{S}}(F^{\natural k}, \mathfrak{h}^k)$ .

LEMMA 4.2.  $\widetilde{\text{HF}}^{k_i I}(F^{\natural k_i}, \mathfrak{h}^{k_i}) \neq 0$  for some sequence  $k_i \rightarrow \infty$ .

Assuming the lemma, let us finish the proof of the theorem. Similarly to the proof of Theorem 2.2, set

$$a_+ = \max \left\{ \int_{S^1} \max_M F_t dt, 0 \right\} + |\lambda|n$$

and

$$a_- = \max \left\{ -\int_{S^1} \min_M F_t dt, 0 \right\} + |\lambda|n.$$

These constants are greater than or equal to the constants  $c_h$  given by (3.7) for the linear homotopies from  $F^{\natural k}$  to  $F^{\natural(k+1)}$  and from  $F^{\natural(k+1)}$  to  $F^{\natural k}$ , and denoted again by  $c_{\pm}$ . Thus, when  $k$  is sufficiently large,

$$\text{gap}(F^{\natural k}, \mathfrak{h}^k) = k \text{gap}(F, \mathfrak{h}) > a_{\pm} \geq c_{\pm}.$$

Hence, by Proposition 3.5, the continuation maps

$$\widetilde{\text{HF}}^{kI}(F^{\natural k}, \mathfrak{h}^k) \rightarrow \widetilde{\text{HF}}^{kI+a_+}(F^{\natural(k+1)}, \mathfrak{h}^k)$$

and

$$\widetilde{\text{HF}}^{kI+a_+}(F^{\natural(k+1)}, \mathfrak{h}^k) \rightarrow \widetilde{\text{HF}}^{kI+a_++a_-}(F^{\natural k}, \mathfrak{h}^k)$$

are defined.

Let  $\delta > 0$  be the distance from the end points of  $I$  to  $\tilde{\mathcal{S}}(F, \mathfrak{h})$ . Then the distance from the end points of  $kI$  to  $\tilde{\mathcal{S}}(H^{\natural k}, \mathfrak{h}^k)$  is  $k\delta$ . When  $k$  is large,  $k\delta > a_++a_-$ , and the intervals  $(kI+a_++a_-)\setminus kI$  and  $kI\setminus(kI+a_++a_-)$  contain no points of  $\tilde{\mathcal{S}}(F^{\natural k}, \mathfrak{h}^k)$ . Thus, the natural quotient-inclusion map

$$\widetilde{\text{HF}}^{kI}(F^{\natural k}, \mathfrak{h}^k) \rightarrow \widetilde{\text{HF}}^{kI+a_++a_-}(F^{\natural k}, \mathfrak{h}^k)$$

is an isomorphism.

Let now  $k$  be one of the sufficiently large entries in the sequence  $k_i$  from Lemma 4.2. Consider the following commutative diagram.

$$\begin{array}{ccc} \widetilde{\text{HF}}^{kI}(F^{\natural k}, \mathfrak{h}^k) & & \\ \downarrow & \searrow \cong & \\ \widetilde{\text{HF}}^{kI+a_+}(F^{\natural(k+1)}, \mathfrak{h}^k) & \longrightarrow & \widetilde{\text{HF}}^{kI+a_++a_-}(F^{\natural k}, \mathfrak{h}^k) \end{array}$$

Here the diagonal map is an isomorphism and, by Lemma 4.2,  $\widetilde{\text{HF}}^{kI}(F^{\natural k}, \mathfrak{h}^k) \neq 0$ . Therefore, the middle group  $\widetilde{\text{HF}}^{kI+a_+}(F^{\natural(k+1)}, \mathfrak{h}^k)$  in the diagram is also non-trivial, and  $F$  has a  $(k+1)$ -periodic orbit  $z$  in the homotopy class  $\mathfrak{h}^k$ .

We have  $z = y_j^{k+1}$  for some  $j$ . Furthermore,  $\mathfrak{h} = \mathfrak{f}^a$  and  $\llbracket y_j \rrbracket = \mathfrak{f}^b$  for some  $a$  and  $b$  in  $\mathbb{N}$ . Thus,  $\mathfrak{f}^{ak} = \mathfrak{f}^{b(k+1)}$ . Since all homotopy classes in  $\mathfrak{f}^{\mathbb{N}}$  are distinct, we infer that  $ak = b(k+1)$ , where  $a$  is independent of  $k$ . This is clearly impossible for  $k \geq a$  since  $k$  and  $k+1$  are relatively prime, and we have arrived at a contradiction. To finish the proof of the theorem, it remains to prove the lemma.

*Proof of Lemma 4.2.* Throughout the proof, it is convenient to interpret the iterated Hamiltonian  $F^{\natural k}$  as the  $k$ -periodic Hamiltonian  $F_t$  with  $t \in \mathbb{R}/k\mathbb{Z}$ . Furthermore, without loss of generality, we can ensure that  $\lambda/2 = 1$  by rescaling  $\omega$ , and thus  $\tilde{\mathcal{A}}_F = \mathcal{A}_F - \Delta_F$ . Since  $\tilde{\mathcal{A}}_F(\bar{h}) = 0$  and  $\Delta_F(\bar{h}) = 0$ , we also have

$$\mathcal{A}_F(\bar{h}) = 0 = \Delta_F(\bar{h}). \tag{4.6}$$

Recall that, by our assumptions,  $0$  is the only point of the action spectrum  $\tilde{\mathcal{S}}(F^{\natural k}, \mathfrak{h}^k)$  in the interval  $kI$  (see (4.5)) and that  $y_j^k$  are the only  $k$ -periodic orbits of  $F$ .

Fix a neighborhood  $U$  of  $h$  which does not intersect any of the other orbits  $y_j$  and a small parameter  $\epsilon > 0$ , depending on  $U$ , to be specified later. There exists a sequence  $k_i \rightarrow \infty$  such that for all  $j$  and  $k_i$ , we have

$$\|\Delta_{F^{\natural k_i}}(y_j^{k_i})\|_{2N_T} < \epsilon, \tag{4.7}$$

where  $\|a\|_{2N_T}$  stands for the distance from  $a \in S_{2N_T}^1 = \mathbb{R}/2N_T\mathbb{Z}$  to  $0$  or, equivalently, from  $a \in \mathbb{R}$  to the nearest point in the lattice  $2N_T\mathbb{Z} \subset \mathbb{R}$ . Here we can treat the mean index

$$\Delta_{F^{\natural k_i}}(y_j^{k_i}) = k_i \Delta_F(y_j)$$

as a real number when  $y_j$  is capped or as a point in  $S_{2N_T}^1$  when the capping is discarded.

To prove (4.7), consider the torus  $\mathbb{T}^m = (S_{2N_T}^1)^m$ , where  $m$  is the number of the orbits  $y_j$ , and set

$$\Delta = (\Delta_F(y_1), \dots, \Delta_F(y_m)) \in \mathbb{T}^m.$$

The closure  $\Gamma$  of the orbit  $\{k\Delta \mid k \in \mathbb{Z}\}$  is a subgroup in  $\mathbb{T}^m$  and, for every  $k_0$ , the set  $\{k\Delta \mid k > k_0\}$  is dense in  $\Gamma$ . Hence, the point  $k\Delta$  is within the  $\epsilon$ -neighborhood of  $0 \in \Gamma$  for infinitely many values of  $k$ .

From now on  $k$  will stand for one of the entries in the sequence  $k_i$ .

Let  $G$  be a  $C^\infty$ -small, non-degenerate perturbation of  $F^{\natural k}$  equal to  $F^{\natural k}$  on the neighborhood  $U$ . The orbits  $y_j^k$ , other than  $h^k$ , split into a finite collection of non-degenerate orbits of  $G$  in the class  $\mathfrak{h}^k$ . Among these we are interested exclusively in the orbits with augmented action in  $kI$ . These orbits can only come from the orbits  $y_j^k$  with action in  $kI$ , i.e., by (4.5), from the orbits  $y_j$  with  $\tilde{A}_F(y_j) = 0$ . We denote the resulting orbits of  $G$  by  $z_j$ . (The number of the orbits  $z_j$  may be different from the number of the orbits  $y_j$ .) It is clear that the orbits  $z_j$  do not enter  $U$  and that

$$|\tilde{A}_G(z_j)| \leq \eta, \tag{4.8}$$

where  $\eta = O(\|F^{\natural k} - G\|_{C^1})$ .

It suffices now to show that when  $\epsilon > 0$  is sufficiently small the orbit  $h^k$  of  $G$  is closed (i.e., a cycle), but not exact, in the Floer complex  $\widetilde{CF}^{kI}(G, \mathfrak{h}^k)$ . To this end, we will prove that  $h^k$  cannot be connected to any of the orbits  $z_j$  by a Floer trajectory of relative index  $\pm 1$ .

By (4.6), we have

$$\Delta_G(\bar{h}^k) = k\Delta_F(\bar{h}) = 0 \quad \text{and} \quad \mathcal{A}_G(\bar{h}^k) = k\mathcal{A}_F(\bar{h}) = 0.$$

In particular,  $\mu_{CZ}(h^k) = \Delta_G(h^k) = 0$ .

Let now  $\bar{z}$  be one of the capped orbits  $\bar{z}_j$ . Our goal is to show that every Floer trajectory  $u$  connecting the capped orbits  $\bar{z}$  and  $\bar{h}^k$  has relative index different from  $\pm 1$ . Since  $\mu_{CZ}(\bar{h}^k) = 0$  and by (3.1), it is enough to prove that

$$|\Delta_G(\bar{z})| > n + 1. \tag{4.9}$$

The orbit  $z$  does not enter  $U$ . Thus, by [GG14, Theorem 3.1], there exists a constant  $e > 0$ , depending on  $U$ , but not on  $k$ , such that the energy of  $u$  is bounded from below by  $e$ . In other words, using the fact that  $\mathcal{A}_G(\bar{h}^k) = 0$ , we have

$$|\mathcal{A}_G(\bar{z})| > e.$$

Set  $\epsilon < e$ .

By (4.7),  $\Delta_G(\bar{z}) \in (\ell - \epsilon, \ell + \epsilon)$  for some  $\ell \in \mathbb{Z}$ . If  $\ell = 0$ , and hence  $|\Delta_G(\bar{z})| < \epsilon$ , we also have

$$|\mathcal{A}_G(\bar{z})| < \epsilon + \eta$$

by (4.8). This is impossible when  $\eta > 0$  is smaller than  $|e - \epsilon|$ , i.e., when  $G$  is sufficiently  $C^1$ -close to  $F^{\natural k}$ , since  $\epsilon < e$ . Thus,  $\ell \neq 0$ , and therefore

$$|\Delta_G(\bar{z})| > 2N_T - \epsilon.$$

Recall that  $N_T \geq n/2 + 1$  by the assumptions of the theorem. Hence, when  $\epsilon < 1$ , we have

$$|\Delta_G(\bar{z})| \geq n + 2 - \epsilon > n + 1.$$

This proves (4.9), completing the proofs of the lemma and of the theorem. □

### 4.3 Proof of Theorem 2.7

The argument is quite standard; it follows the same line of reasoning as the proof of [GG09b, Proposition 1.6], which, in turn, has a lot of similarities with, e.g., an argument in [Hin84].

*Proof.* It suffices to show that a Hamiltonian diffeomorphism  $\varphi_H \in \mathcal{F}_f$  has a non-hyperbolic periodic orbit in some homotopy class  $f^k$  or  $\varphi_H$  has infinitely many periodic orbits in  $f^{\mathbb{N}}$ . Indeed, the presence of a non-hyperbolic periodic orbit  $x$  implies, by the Birkhoff–Lewis–Moser fixed-point theorem (see [Mos77]), the existence of infinitely many periodic orbits in a tubular neighborhood of  $x$   $C^\infty$ -generically for Hamiltonian diffeomorphisms close to  $\varphi_H$ .

To this end, observe that since  $\text{HF}(H, f) = 0$  due to the condition  $f \neq 1$ , the Hamiltonian  $H$  necessarily has one-periodic orbits in the class  $f$  with odd and with even Conley–Zehnder indices. As a consequence, it has either a non-hyperbolic one-periodic orbit or a hyperbolic orbit with an odd number of real Floquet multipliers in the interval  $(-1, 0)$ . In the former case, the proof is finished.

In the latter case, let us apply this argument to  $\varphi_H^2$ . By the assumptions of the theorem,  $f^2 \neq 1$  and thus  $\text{HF}(H^{i2}, f^2) = 0$ . Hence,  $H$  has two-periodic orbits in the class  $f^2$  with odd and with even Conley–Zehnder indices. The second iterations of one-periodic orbits from the class  $f$  are necessarily positive hyperbolic. Therefore, there exists a simple two-periodic orbit in  $f^2$ , which is either non-hyperbolic or hyperbolic with an odd number of real Floquet multipliers in the interval  $(-1, 0)$ . In the former case, the proof is finished, and in the latter we repeat this process for  $\varphi^4$  and so forth.

As a result, we will either find a non-hyperbolic orbit in some class  $f^k$  or construct a sequence of simple periodic orbits in  $f^{\mathbb{N}}$ .  $\square$

*Remark 4.3.* As is clear from the proof, it is sufficient to assume only that  $f^k \neq 1$  when  $k$  is a power of 2. In this case, the result asserts the generic existence of infinitely many periodic orbits in the set of homotopy classes  $f^{\mathbb{N}}$ , although these orbits may now be contractible.

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Viktor L. Ginzburg [ginzburg@ucsc.edu](mailto:ginzburg@ucsc.edu)  
 Department of Mathematics, UC Santa Cruz,  
 Santa Cruz, CA 95064, USA

Başak Z. Gürel [basak.gurel@ucf.edu](mailto:basak.gurel@ucf.edu)  
 Department of Mathematics, University of Central Florida,  
 Orlando, FL 32816, USA