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INVARIANCE OF INFINITE-DIMENSIONAL CLASSES OF SPACES

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Abstract

The central area of investigation is in the isolation of conditions on mappings which leave invariant the classes of locally finite-dimensional metric spaces and strongly countable-dimensional metric spaces. Examples of such properties are open and closed with discrete point-inverses, open and finite-to-one, or open, closed, and countable-to-one.

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We shall be concerned principally with conditions on mappings which leave invariant two classes of infinite-dimensional metric spaces. The first of these classes is that of strongly countable-dimensional spaces; J. Nagata [see 6] defined a space to be *strongly countable-dimensional* (abbreviated *scd*) if and only if it is the countable union of finite-dimensional closed subspaces. The other class is that of locally finite-dimensional spaces [see 9]; a space is *locally finite-dimensional* (abbreviated *lfd*) if and only if each point has a finite-dimensional neighborhood. Every finite-dimensional space is lfd, and every lfd space is scd, but the converse of neither is true [see 9].

All spaces will be understood to be metric, and all functions or mappings are continuous surjections. The term dimension (or the abbreviation dim) will refer to the covering dimension of Lebesgue [see 7, p. 9]. All neighborhoods are assumed to be open.

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The following known results are stated here for later reference:

THEOREM 1. *X is locally finite-dimensional if and only if X has a locally finite closed cover consisting of locally finite-dimensional (or finite-dimensional) subsets* [9, *Theorem* 5].

THEOREM 2. *X is strongly countable-dimensional if and only if X has a σ-locally finite closed cover consisting of strongly countable-dimensional (or finite-dimensional) subsets* [9, *Theorem* 6 *and Corollary* 10].

THEOREM 3 (K. Morita). *Let* $f\colon X \to Y$ *be a closed map, and let* n *be a non-negative integer such that* $\dim f^{-1}(y) \leqslant n$ *for each* $y \in Y$. *Then* $\dim X \leqslant \dim Y + n$ [4, *Theorem* 2].

Theorem 3, due to K. Morita, has natural analogues for lfd and scd spaces, which we establish in Theorem 4 below after establishing the following notational conveniences and elementary observations:

Let $f\colon X \to Y$ be a continuous surjection. If $\mathscr{F}$ is a cover of X, then $f[\mathscr{F}] = \{f[F] | F \in \mathscr{F}\}$ is a cover of X. If $\mathscr{G}$ is a cover of Y then $f^{-1}[\mathscr{G}] = \{f^{-1}[G] | G \in \mathscr{G}\}$ is a cover of X and $ff^{-1}[\mathscr{G}] = \mathscr{G}$. If f is closed and $\mathscr{F}$ is closed then so is $f[\mathscr{F}]$. If $\mathscr{G}$ is closed then $f^{-1}[\mathscr{G}]$ is closed. Corresponding remarks hold with closed replaced by open. If $\mathscr{G}$ is countable, locally finite, or σ-locally finite, then so is $f^{-1}[\mathscr{G}]$ respectively countable, locally finite, or σ-locally finite. If $\mathscr{F}$ is countable then so is $f[\mathscr{F}]$. If f is closed then $f[\mathscr{F}]$ is closure-preserving whenever $\mathscr{F}$ is closure-preserving.

THEOREM 4. *Let* $f\colon X \to Y$ *be a closed map and let* n *be a non-negative integer such that* $\dim f^{-1}(y) \leqslant n$ *for each* $y \in Y$. *Then X is lfd if Y is lfd, and X is scd if Y is scd.*

PROOF. If Y is lfd, then by Theorem 1, Y has a locally finite closed cover $\mathscr{F}$ consisting of finite-dimensional sets. For each $F \in \mathscr{F}$, $f|f^{-1}[F]$ (the restriction of f to the set $f^{-1}[F]$) is a closed map and $\dim(f|f^{-1}[F])^{-1}(y) \leqslant n$ for $y \in F$. Thus by Theorem 3 $\dim f^{-1}[f] \leqslant \dim F + n < \infty$. But $f^{-1}[\mathscr{F}]$ is a locally finite closed cover of X, so X is lfd by Theorem 1.

If Y is scd, then we choose the cover $\mathscr{F}$ to be a σ-locally finite closed cover and proceed as above, using Theorem 2 instead of Theorem 1.

If X in Theorem 4 is locally compact, f need not be closed.

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Theorem 5. *Let X be locally compact and $f\colon X \to Y$ a map such that there is a non-negative integer n such that* $\dim f^{-1}(y) \leqslant n$ *for each $y \in Y$. Then X is lfd if Y is lfd, and X is scd if Y is scd.*

Proof. Let $\mathscr{F}$ be a locally finite cover of X by compact sets. Each member of $f[\mathscr{F}]$ is lfd (scd) when Y is lfd (scd). For each $F \in \mathscr{F}$, $f|F$ is closed, so it follows from Theorem 4 that each member of $\mathscr{F}$ is lfd (scd) when Y is lfd (scd), and the theorem follows from Theorems 1 and 2.

Theorem 6. *Let $F\colon X \to Y$ be an open and closed function such that for each $y \in Y, f^{-1}(y)$ is not dense-in-itself. Then Y is lfd if X is lfd, and Y is scd if X is scd.*

Proof. We assume X is lfd, and let $y \in Y$. It suffices to show that y has a finite-dimensional neighborhood, so we need only examine the case where $\{y\}$ is not open in Y.

If $\operatorname{Int} f^{-1}(y) \neq \varnothing$, then $\{y\} = f[\operatorname{Int} f^{-1}(y)]$ is an open set. Hence, $\operatorname{Int} f^{-1}(y) = \varnothing$ and consequently $f^{-1}(y) = \operatorname{Bdry} f^{-1}(y)$. Since X and Y are metric spaces and f is closed, $\operatorname{Bdry} f^{-1}(y)$ is compact (see [8, Theorem VI.12, p. 214]).

Each $x \in f^{-1}(y)$ has a finite-dimensional neighborhood, and, since $f^{-1}(y)$ is compact, finitely many of these neighborhoods cover $f^{-1}(y)$. Thus, $f^{-1}(y)$ is contained in a finite-dimensional open set U.

Since f is closed, there is a neighborhood V of y such that $f^{-1}(y) \subset f^{-1}[V] \subset U$. Hence $f|f^{-1}[V]$ is an open and closed map onto V with each of its point inverses not dense-in-itself. Since $\dim f^{-1}[V] \leqslant \dim U < \infty$, by a theorem of R. E. Hodel [2, Theorem 3.4], $\dim V \leqslant \dim f^{-1}[V] < \infty$. Thus y has a finite-dimensional neighborhood.

We now assume X is scd, so $X = \bigcup_{n=1}^{\infty} Z_n$ where each Z_n is a finite-dimensional closed subset of X, and we shall prove Y is scd. For each $i = 1, 2, \ldots$ we let $X_i = \{x \mid d(x, f^{-1}(f(x)) - \{x\}) \geqslant \frac{1}{i}\}$, $Y_i = f[X_i]$, and $g_i = f|X_i$. Hodel has shown [2, proof of Theorem 3.2, where Y need not be locally compact if f is assumed to be continuous] that $g_i\colon X_i \to Y_i$ is a closed, locally one-to-one (that is, every point of X_i has a neighborhood N for which $g_i|N$ is one-to-one) mapping, $Y = \bigcup_{i=1}^{\infty} Y_i$, and each Y_i is a closed subset of Y. If $X_{i,n} = X_i \cap Z_n$ it follows [2, Lemma 3.1 where again Y need not be locally compact if f is continuous] that $Y_{i,n} = g_i[X_{i,n}]$ is closed in Y and has dimension $\leqslant \dim X_{i,n} \leqslant \dim Z_n < \infty$, so $Y = \bigcup_{i=1}^{\infty} Y_i = \bigcup_{i,n=1}^{\infty} Y_{i,n}$ is scd.

Corollary 7. *Let $f\colon X \to Y$ be an open and closed map such that $f^{-1}(y)$ is not dense-in-itself and* $\dim f^{-1}(y) = 0$ *for each $y \in Y$. Then X is lfd if and only if is lfd, and X is scd if and only if Y is scd.*

Proof. The necessity follows from Theorem 6, and the sufficiency from Theorem 4.

Lemma 8. *Let A be a closed subset of X such that* $\dim X - A < \infty$. *Then X is lfd if and only if A is lfd, and X is scd if and only if A is scd.*

Proof. If A is lfd, it has a locally finite cover $\mathscr{F}$ of finite-dimensional subsets [9, Lemma 1] and $\mathscr{F}$ is locally finite in X since A is closed. Hence, $\mathscr{F} \cup \{A\}$ is a locally finite cover of X consisting of finite-dimensional subsets of X, so X is lfd [9, Lemma 1 again].

Now let A be scd. Since $A - A$ is open, it is the countable union of closed finite-dimensional subsets of X, so $X = A \cup (X - A)$ is scd (see [9, Theorem 8]).

Corollary 9. *Let f: $X \to Y$ be an open, closed, and countable-to-one mapping. Then X is lfd if and only if Y is lfd, and X is scd if and only if Y is scd.*

Proof. The sufficiency follows as in Corollary 7. For the necessity, let $A = \{y \in Y | \{y\}$ is not open in $Y\}$. The set A is closed, and it follows as in the proof of Theorem 6 that $f | f^{-1}[A]$ is open and closed with compact point inverses. But a countable compact subset of a metric space cannot be dense-in-itself, so A is lfd (scd) by Theorem 6. By Lemma 8, then, Y is lfd (scd).

The following adjustments can be made to Corollary 7 when the mapping is not required to be closed.

Theorem 10. *Let f: $X \to Y$ be an open map such that $f^{-1}(y)$ is discrete for each $y \in Y$. Then X is lfd if Y is lfd, and X is scd if Y is scd.*

Proof. Let $\mathscr{F}$ be a closed cover of Y. Then, for each $F \in \mathscr{F}$, $f | f^{-1}[F]$ is an open map onto F with discrete point inverses. Hence, by [2, Theorem 2.9], $\dim f^{-1}[F] \leqslant \dim F$ for each $F \in \mathscr{F}$. The theorem now follows from Theorems 1 and 2.

Theorem 11. *Let f: $X \to Y$ be an open, finite-to-one map. Then X is lfd if and only if Y is lfd, and X is scd if and only if Y is scd.*

Proof. The sufficiency is clear from Theorem 10. Now let X be lfd and let $y \in Y$. Choose a point $x \in f^{-1}(y)$ and a finite-dimensional open neighborhood U

of x. Then $f|U$ is an open and finite-to-one map onto $f[U]$, so $\dim f[U] = \dim U$ [5, Theorem 4.1]. Thus $f[U]$ is a finite-dimensional neighborhood of y.

We now assume X is scd, and for all $n = 1, 2, \ldots$ we define $W_n = \{y \in Y | f^{-1}(y)$ consists of exactly n points$\}$, $Z_n = f^{-1}[W_n]$, and $f_n = f|Z_n$; we note f_n: $Z_n \to W_n$ is an open, exactly n-to-one mapping. Now if $y \in Y$ then $y \in W_n$ for some n, so $f^{-1}(y) = \{x_1, \ldots, x_n\}$. For each $i = 1, \ldots, n$ we choose a neighborhood U_i of x_i such that $U_i \cap U_j = \varnothing$ whenever $i \neq j$, and define the neighborhood V of y by $V = \bigcap_{i=1}^{n} f_n[U_i]$; then f_n is a homeomorphism from each $U_i \cap f^{-1}[V]$ onto V, so V and therefore W_n is scd (see [9, Theorem 7]).

Since f is open it is not difficult to see that each union $\bigcup_{j=n}^{\infty} W_j$ is open in Y; this implies each union $\bigcup_{j=1}^{n} W_n$ is closed, hence each W_n is an F_σ subset of Y. Let $W_n = \bigcup_{j=1}^{\infty} W_{n,j}$ where each $W_{n,j}$ is closed. Clearly each $W_{n,j}$ is scd, so $Y = \bigcup_{n=1}^{\infty} W_n = \bigcup_{n,j=1}^{\infty} W_{n,j}$ is scd.

A space is called *countable-dimensional* if it is the countable union of finite-dimensional subspaces (here the subspaces need not be closed). A. V. Arhangel'skii [1] and J. Nagata [6] have developed analogues to Theorems 10 and 11 for the class of countable-dimensional spaces.

Other invariances can be given in terms of the order of a closed map (if f: $X \to Y$, then $\operatorname{ord} f \leqslant n$ if and only if $f^{-1}(y)$ consists of at most n points for each $y \in Y$). We first state the following theorem of K. Morita [3] and prove a slight generalization.

THEOREM 12 (K. Morita). *Let f: $X \to Y$ be a closed map such that* $\operatorname{ord} f \leqslant k + 1$ *for some non-negative integer k. Then* $\dim Y \leqslant \dim X + k$.

THEOREM 13. *Let f: $X \to Y$ be a closed map and let k be a non-negative integer such that for each $x \in X$ there is an open neighborhood U_x for which* $\operatorname{ord} f|U_x \leqslant k + 1$. *Then* $\dim X \leqslant \dim Y \leqslant \dim X + k$.

PROOF. For each $x \in X$, choose an open neighborhood U_x such that $\operatorname{ord} f|U_x \leqslant k + 1$. Then $\{U_x | x \in X\}$ has a locally finite closed refinement $\mathscr{F}$. Since $f|F$ is closed and $\operatorname{ord} f|F \leqslant k + 1$ for each $F \in \mathscr{F}$, by Theorem 12 $\dim f[F] \leqslant \dim F + k$ for each $F \in \mathscr{F}$. Also, since $\dim f^{-1}(y) = 0$ for each $y \in Y$, $\dim F \leqslant \dim f[F]$ for each $F \in \mathscr{F}$ by Theorem 3. Because $\mathscr{F}$ and $f[\mathscr{F}]$ are both closure-preserving, the Sum Theorem [7, page 18] gives $\dim X = \sup\{\dim F | F \in \mathscr{F}\} \leqslant \sup\{\dim f[F] | F \in \mathscr{F}\} = \dim Y \leqslant \sup\{\dim F | F \in \mathscr{F}\} + k = \dim X + k$.

COROLLARY 14. *Let f: $X \to Y$ be a closed map and let k be a non-negative integer such that for each $y \in Y$, there is an $x_y \in f^{-1}(y)$ and a neighborhood U_y of x_y with* $\operatorname{ord} f|U_y \leqslant k + 1$. *Then* $\dim Y \leqslant \dim X + k$.

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PROOF. Let $A = \bigcup\{U \subset X | U$ is X-open and ord $f|U \leqslant k + 1\}$. Then A is open and by hypothesis $f[A] = Y$. Since an open set is an F_σ, there are X-closed sets $C_1, C_2, \ldots$ such that $U = \bigcup_{i=1}^{\infty} C_i$. For each i, $f|C_i$ satisfies the hypothesis of Theorem 13, so $\dim f[C_i] \leqslant \dim C_i + k \leqslant \dim X + k$, thus $\dim Y = \dim \bigcup_{i=1}^{\infty} f[C_i] \leqslant \dim X + k$.

It need not be the case in Corollary 14 that $\dim X \leqslant \dim Y$. Let (Z, τ) be any topological space with $\dim Z = 1$, and let p be any point not in Z. Let $X = Z \cup \{p\}$, and as a basis for the topology of X use $\tau \cup \{\{p\}\}$. Then the constant map from X onto $\{p\}$ satisfies the hypothesis of Corollary 14, but $\dim X > \dim\{p\}$.

In a previous work [10] the author and J. W. Walker made the following definitions. We say a map $f\colon X \to Y$ has *strong local order* if and only if for each $x \in X$ there is a neighborhood U_x and a positive integer n_x such that ord $f|U_x \leqslant n_x$. The map has *weak local order* if and only if for each $y \in Y$ there is a point $x_y \in f^{-1}(y)$, a neighborhood U_y of x_y and a positive integer n_y such that ord $f|U_y \leqslant n_y$. These properties generalize those given in the hypotheses of Theorems 13 and 14 above, and provide new invariance theorems given below as Theorems 15 and 17.

THEOREM 15. *Let $f\colon X \to Y$ be a closed map with strong local order. Then X is lfd if and only if Y is lfd.*

We need the following results before proceeding.

LEMMA 15.1. *Let $f\colon X \to Y$ be closed, and define $B = \bigcup\{\text{Bdry}\, f^{-1}(y) | y \in Y\}$. Then $Y - f[B]$ is open*, $\dim(Y - f[B]) \leqslant 0$ *and $f|B$ is a perfect map.*

PROOF. Since all spaces are metric, point inverses for $f|B$ must be compact (see [8, Theorem VI.12, p. 214]). Now let $y \in Y - f[B]$, and note that $f^{-1}(y)$ is open in X; since f is closed, this implies y is an isolated point of Y. Hence $Y - f[B]$ is open and discrete, so $\dim(Y - f[B]) \leqslant 0$.

LEMMA 15.2. *If $f\colon X \to Y$ is a perfect map and $\mathscr{F}$ is a locally finite (σ-locally finite) collection of subsets of X, then $f[\mathscr{F}]$ is a locally finite (σ-locally finite) collection of subsets of Y.*

PROOF. Let $y \in Y$. Each $x \in f^{-1}(y)$ has an open neighborhood U_x which intersects at most finitely many elements of $\mathscr{F}$. Since $f^{-1}(y)$ is compact, finitely many of these neighborhoods cover $f^{-1}(y)$. Thus there is an open set U which contains $f^{-1}(y)$ and intersects at most finitely many elements of $\mathscr{F}$.

Since f is closed, there is an open set V in Y such that $f^{-1}(y) \subset f^{-1}[V] \subset U$. Then V is a neighborhood of y which intersects at most finitely many elements of $f[\mathscr{F}]$, since $V \cap f[F] \neq \varnothing$ implies $f^{-1}[V] \cap F \neq \varnothing$ for any $F \in \mathscr{F}$.

Proof of Theorem 15. Suppose Y is lfd. Since a discrete set is 0-dimensional, it is sufficient by Theorem 4 to show that point inverses under f are discrete. Let $y \in Y$, and let $x_1 \in f^{-1}(y)$. There is an open neighborhood U_1 of x_1 and a positive integer n_1 such that ord $f|U_1 \leqslant n_1$. Thus $f^{-1}(y) \cap U_1$ contains at most n_1 elements, say $f^{-1}(y) \cap U_1 = \{x_1, x_2, \ldots, x_k\}$, $1 \leqslant k \leqslant n_1$. Then $V = U_1 - \{x_2, \ldots, x_k\}$ is X-open and $V \cap f^{-1}(y) = \{x_1\}$. Thus $\{x_1\}$ is open in $f^{-1}(y)$, and so $f^{-1}(y)$ is discrete.

Conversely, suppose X is lfd. Let $B = \bigcup\{\text{Bdry} f^{-1}(y) | y \in Y\}$. By Lemmas 15.1 and 8 it is sufficient to show that $f[B]$ is lfd. Thus without loss of generality we may assume that $X = B$ and f is a perfect map.

By Theorem 1, X has a locally finite closed cover $\mathscr{F}$ consisting of finite-dimensional sets. From the paracompactness of X and the definition of strong local order, there is a locally finite closed cover of $\mathscr{G}$ of X such that for each $G \in \mathscr{G}$ there is an integer n_G such that ord $f|G \leqslant n_G$.

Let $\mathscr{H} = \{F \cap G | F \in \mathscr{F}$ and $G \in \mathscr{G}\}$. By Lemma 15.2 $f[\mathscr{H}]$ is a locally finite closed cover of Y. For each $H \in \mathscr{H}$, $f[H]$ is finite-dimensional, since $f|H$ satisfies the hypothesis of Theorem 12 and H is finite-dimensional. Thus Y is lfd by Theorem 1.

Corollary 16. *Let f: $X \to Y$ be a closed map with strong local order. Then X is scd if and only if Y is scd.*

Proof. If $\mathscr{F}$ is a countable closed cover of X consisting of lfd subsets, then $f[\mathscr{F}]$ is a countable closed cover of Y by lfd subsets by Theorem 15. Hence Y is scd by Theorem 2.

Conversely, let $\mathscr{G}$ be a countable closed cover of Y consisting of lfd subsets. Since $\mathscr{G} = f f^{-1}[\mathscr{G}]$, $f^{-1}(\mathscr{G})$ is a countable closed cover of X by lfd subsets by Theorem 15, so X is scd by Theorem 2.

Theorem 17. *Let f: $X \to Y$ be a closed map with weak local order. If X is scd, then Y is scd.*

Proof. Let $A = \bigcup\{U \subset X | U$ is open and ord $f|U < \infty\}$. Then A is open, $f|A$ has strong local order, and by the definition of weak local order $f[A] = Y$. Since A is an F_σ subset of X, there are X-closed sets $C_1, C_2, C_2, \ldots$ such that $A = \bigcup_{i=1}^{\infty} C_i$. But for each i, $f|C_i$ satisfies the hypotheses of Corollary 16, so $f[C_i]$ is scd since C_i is scd. But each $f[C_i]$ is closed in Y, so $Y = \bigcup_{i=1}^{\infty} f[C_i]$ must be scd.

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Comparison with Theorem 15 might lead one to expect a converse to Theorem 17, but none is available. For example, let (Z, τ) be a space which is not scd, $p \notin Z$, and $X = Z \cup \{p\}$ with the topology generated by $\tau \cup \{\{p\}\}$ (as in the example following Corollary 14). Then the constant map from X onto p is closed and has weak local order, but X is not scd.

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