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ON C^m-BOUNDING SETS

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Abstract

Let E be a quasi-complete locally convex space and A a subset of E. It is shown that if every real-valued C^∞-function in the weak topology of E is bounded on A, then A is relatively weakly compact. Furthermore, if all real-valued C^∞-functions on E are bounded on A, then A is relatively compact in the associated semi-weak topology of E.

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The purpose of this paper is to examine the C^m-bounding sets ($m = 0, 1, \dots, \infty$) in a real Hausdorff locally convex space (lcs), with emphasis on their connection with relatively compact sets in various topologies. We say that a subset A of a lcs E is C^m-*bounding* if every $f \in C^m(E)$ is bounded on A. Here, $C^m(E)$ denotes the set of all real-valued C^m-functions on E in the sense of Lloyd [13], whose definition of differentiability coincides with that of Fréchet on Banach spaces. C^∞-bounding sets have recently been investigated by Kriegl and Nel in [11].

Since $E' \subset C^m(E) \subset C(E)$ for every lcs E, the class of C^m-bounding sets is somewhere between the classes of bounded and relatively compact sets. The C^0-bounding subsets of a lcs E are relatively compact if E is quasi-complete or metrizable or, by [10], if E is realcompact. Moreover, Valdivia proves in [15] that every set, which is C^0-bounding in the weak topology of a quasi-complete lcs, is relatively weakly compact. We obtain a generalization of this result to C^∞-bounding sets.

For the study of C^∞-bounding sets in Banach spaces it is convenient to consider the finest semi-weak topology compatible with the duality $\langle E, E'\rangle$ [4]. This locally convex topology, which we denote by E_{sw}, is strictly finer than the associated Schwartz topology and the weak topology on E. Our main theorem states that every C^∞-bounding set in a Banach space E is relatively compact in E_{sw}. This should be compared with a deep result due to Bourgain and Diestel [3], by which the limited sets in a Banach space not containing l_1 are relatively weakly compact. Since the C^∞-bounding sets form a proper subclass of the limited sets, it follows from the theory of limited sets that the C^∞-bounding subsets of various Banach spaces are relatively compact. However, using the associated semi-weak structure, we obtain the same results in a general way. Further, we show that the classes of C^∞-bounding and relatively compact sets coincide in Lindelöf as well as in quasi-complete smooth lcs.

The following concept guarantees a rich supply of functions in $C^m(E)$. A lcs E is *C^m-smooth* if there, for every point $a \in E$ and open subset V containing a, exists a function $f \in C^m(E)$ with $f(a) > 0$ and $f(x) = 0$ for $x \notin V$. The class of C^m-smooth spaces is closed under formation of subspaces and arbitrary products [13]. All Schwartz spaces [13] and the Banach spaces $c_0(\Gamma)$, $l_{2n}(\Gamma)$ and $L_{2n}(\mu)$ are well-known examples of C^m-smooth spaces.

Proposition 1. *Let E be a C^m-smooth* lcs. *Then every C^m-bounding subset of E is precompact and hence also relatively compact if, in addition, E is quasi-complete.*

Proof. Assume that $A \subset E$ is not precompact. Then there exist a sequence (x_n) in A and a zero-neighbourhood U in E with $x_n - x_m \notin U$ for $n \neq m$. Let V be an open set in E with $0 \in V$ and $V - V \subset U$. By the C^m-smoothness of E there is for each n a function $f_n \in C^m(E)$ such that $f_n(x_n) = n$ and $f_n(x) = 0$ for $x \notin x_n + V$. Let $f(x) = \sum_{n=1}^\infty f_n(x)$. By construction the sum f is locally finite and thus of class C^m. Since $f(x_n) = n$ for each n, the set A cannot be C^m-bounding.

Let E_m be the set E endowed with the weakest topology making all $f \in C^m(E)$ continuous. Then E_m is a completely regular Hausdorff topological space. Clearly $E = E_0$ and E is C^m-smooth if and only if $E = E_m$.

Proposition 2. *Let E be a* lcs *and suppose that there is a sequence in*

$C^m(E)$ which separates the points of E. Then every C^m-bounding subset A of E is relatively compact in E_m.

PROOF. Let $A \subset E$ be C^m-bounding. The set $f(A) \subset R$ is relatively compact for every $f \in C^m(E)$, thus A must be relatively compact in $R^{C^m(E)}$, by Tychonoff's theorem. Since $C^m(E)$ admits a sequence which separates the points of E, it follows from corollary 4 in [2] that E equals $\operatorname{Hom} C^m(E)$, the set of all real-valued non-zero homomorphisms on $C^m(E)$. Now $\operatorname{Hom} C^m(E)$ is a closed subset of $R^{C^m(E)}$ and therefore A is relatively compact in E_m.

The dual of a lcs E is $\sigma(E', E)$-separable if and only if E' admits a sequence which separates the points of E: Let (l_n) be such a sequence and let $l \in E'$ be arbitrary. For each finite-dimensional subspace F of E, there is an l_F in the linear span of (l_n) with $l_F(x) = l(x)$ for each $x \in F$. Since the net (l_F) converges to l in $\sigma(E', E)$, we conclude that E' is $\sigma(E', E)$-separable. The converse statement is trivial. Now, $\sigma(E, E')$ being C^∞-smooth, we obtain from Proposition 2 the following

COROLLARY 3. *If E is a* lcs *with $\sigma(E', E)$-separable dual, then every C^∞-bounding subset of $(E, \sigma(E, E'))$ is relatively $\sigma(E, E')$-compact.*

A subset A of a lcs E is called *limited* [12], if each equicontinuous $\sigma(E', E)$-null sequence (l_n) in E' converges to zero uniformly on A. Every equicontinuous $\sigma(E', E)$-null sequence (l_n) defines a linear and continuous mapping $T : E \to c_0$, $x \mapsto (l_n(x))_{n \in N}$. Hence, by [1, p. 161], A is limited, if $T(A)$ is relatively compact in c_0 for all $T \in L(E, c_0)$. Thus, by Proposition 1, every C^∞-bounding set in E is limited. However, the converse does not hold in general. According to [12] all precompact sets are limited. Therefore the limited sets in $(E, \sigma(E, E'))$ are exactly the bounded ones for every lcs E. If E is a non-semi-reflexive lcs with $\sigma(E', E)$-separable dual (e.g. l^∞ or c_0), the corollary above ensures the existence of limited sets in lcs which are not C^∞-bounding.

There exist even limited non-C^∞-bounding sets in Banach spaces. Indeed, by Phillips' lemma [1, p. 233], a sequence (l_n) in $(l^\infty)'$ converges uniformly to zero on the closed unit ball B_{c_0} of c_0, if (l_n) converges to zero in $\sigma((l^\infty)', l^\infty)$. This means that the non-compact set $B_{c_0} \subset l^\infty$ is limited in l^∞. Assume that $B_{c_0} \subset l^\infty$ is also C^∞-bounding in l^∞. Since B_{c_0} is closed in $\sigma(l^\infty, (l^\infty)')$, Corollary 3 implies that B_{c_0} is $\sigma(l^\infty, (l^\infty)')$-compact. But, since $\sigma(l^\infty, (l^\infty)')$ and $\sigma(c_0, l_1)$ coincide on B_{c_0}, we conclude that B_{c_0} is

$\sigma(c_0, l_1)$-compact, which is a contradiction. Hence $B_{c_0} \subset l^\infty$ is a limited non-C^∞-bounding set in l^∞.

If we leave the use of Corollary 3 out of account, the proof of the next theorem has much in common with the corresponding proof in the case $m = 0$ as carried out in [8, p. 24]. But since the result is important and will be used in the sequel, we include a full detailed proof for the sake of completeness.

Let $\langle E, E' \rangle$ be a dual pair. We say that two sets $A \subset E$ and $M \subset E'$ have the *interchangeable double limit property*, if for every sequence (x_m) in A and every sequence (l_n) in M the double limits $\lim_m \lim_n \langle x_m, l_n \rangle$ and $\lim_n \lim_m \langle x_m, l_n \rangle$ are equal if they exist. The Eberlein-Grothendieck theorem [8, p. 15] states that a subset A of a lcs E is bounded and has the interchangeable double limit property with all equicontinuous subsets of E', if A is relatively $\sigma(E, E')$-compact. The converse implication is true if E is quasi-complete.

THEOREM 4. *Let E be a quasi-complete* lcs. *Then every C^∞-bounding subset A of $(E, \sigma(E, E'))$ is relatively $\sigma(E, E')$-compact.*

PROOF. Take an equicontinuous sequence (l_n) in E' and a sequence (x_m) in A such that the corresponding double limits exist. Let F be the $\sigma(E', E)$-closed linear span of (l_n). Then $F = (E/F^\circ)'$ is $\sigma(F, E/F^\circ)$-separable and the canonical linear mapping $\pi : (E, \sigma(E, E')) \to (E/F^\circ, \sigma(E/F^\circ, F))$ is continuous. Hence $\pi(A)$ is relatively $\sigma(E/F^\circ, F))$-compact by Corollary 3. The sequence (l_n) is equicontinuous in $F = (E/F^\circ)'$, when E/F° has the quotient topology of E. By the interchangeable double limit property for the duality $\langle E/F^\circ, F \rangle$,

$$\lim_m \lim_n \langle x_m, l_n \rangle_{E, E'} = \lim_m \lim_n \langle \pi(x_m), l_n \rangle_{E/F^\circ, F}$$

$$= \lim_n \lim_m \langle \pi(x_m), l_n \rangle_{E/F^\circ, F} = \lim_n \lim_m \langle x_m, l_n \rangle_{E, E'}.$$

Since E is quasi-complete, A is relatively $\sigma(E, E')$-compact.

The theorem above extends a corresponding result of Valdivia for C^0-bounding sets to C^∞-bounding ones [15]. Similar results for limited sets have been obtained under much more restrictive conditions: In [3] Bourgain and Diestel show that in Banach spaces containing no copy of l_1 all limited sets are relatively weakly compact.

Since $l_1(\Gamma)$ and $\sigma(l_1(\Gamma), l^\infty(\Gamma))$ have the same compact sets, we notice that the C^∞-bounding subsets of $l_1(\Gamma)$ are relatively compact also for uncountable sets Γ.

COROLLARY 5. *Let E be a Banach space. If E' admits a sequence that separates the points of E, then every C^m-bounding subset A of E is metrizable and relatively sequentially compact with respect to E_m.*

PROOF. According to Theorem 4, the set $\overline{A}^\sigma$ is compact in $\sigma(E, E')$. Since E' contains a countable set that separates the points of E, the set $\overline{A}^\sigma$ is metrizable in $\sigma(E, E')$, by theorem 10.11 in [1]. The identity mapping $id : (\overline{A}^{E_m}, E_m) \to (\overline{A}^{E_m}, \sigma(E, E'))$ is continuous. By Proposition 2, $\overline{A}^{E_m}$ is compact in E_m, and therefore id is a homeomorphism. Thus $\sigma(E, E')$ and E_m coincide on $\overline{A}^{E_m}$ and the corollary follows.

PROPOSITION 6. *Let E be a* lcs *such that every C^m-bounding subset of E is relatively compact in E. Then E and E_m have the same convergent sequences and compact sets.*

PROOF. Since the functions in $C^m(E)$ are continuous with respect to E_m, every compact set A in E_m is C^m-bounding in E. Therefore A is compact in E. Further, let $x_n \to x$ in E_m. Then $B = \{x_n : n \in N\} \cup \{x\}$ is compact in E_m and hence also in E. The topologies of E and E_m coincide on B and consequently $x_n \to x$ in E.

REMARK. Let E be a Banach space and assume that l^∞ and $(l^\infty)_m$ define the same convergent sequences. Then it can easily be proved using Corollary 5 and proposition 4 in [6] that every C^m-bounding subset of E is relatively compact in E.

The semi-weak lcs are exactly those which are isomorphic to subspaces of some product $c_0{}^I$ [4]. Every lcs E defines a semi-weak space E_{sw} which is obtained by endowing the set E with the topology of uniform convergence on the $\sigma(E', E)$-closed absolutely convex hulls of all sets of the form $\{l_n : n \in N\}$, where (l_n) is an equicontinuous sequence in E' which converges to zero in $\sigma(E', E)$. The topology of E_{sw} is the finest semi-weak topology, which is coarser than the topology of E [4]. Note that $E' = (E_{sw})'$ and that the topology of E_∞ is finer than E_{sw}, since E_{sw} is C^∞-smooth.

THEOREM 7. *If E is a quasi-complete* lcs, *then every C^∞-bounding subset A of E is relatively compact in E_{sw}.*

PROOF. Since A is C^∞-bounding in E, Theorem 4 yields that $\overline{A}^\sigma$ is weakly compact and therefore complete in $\sigma(E, E')$. Now $E' = (E_{sw})'$ so, by theorem 3.24 in [10], the set $\overline{A}^\sigma$ is complete in E_{sw}. Since E_{sw}

is C^∞-smooth, A is precompact in E_{sw}, by Proposition 1. The zero-neighbourhoods in E_{sw} are $\sigma(E, E')$-closed, so we conclude that $\overline{A}^\sigma$ is precompact and hence also compact in E_{sw}.

Theorem 7, our main result, lifts the relative compactness of the C^∞-bounding sets on E from the weak topology, as in Theorem 4, to the finer semi-weak topology. In fact, the lifting works for any C^∞-smooth locally convex topology compatible with the duality $\langle E, E'\rangle$. The semi-weak space E_{sw} is in general strictly finer than the associated Schwartz space E_{sz} (by proposition 5.10 in [4] a Banach space E is finite-dimensional if $E_{sw} = E_{sz}$). Furthermore, using the associated semi-weak topology we obtain in a unified manner various results, proved for limited sets by different methods.

Corollary 8. *Every C^∞-bounding subset of a Banach space E is relatively sequentially compact in E_{sw}.*

Proof. According to Theorem 7 we have to show that every relatively compact set A in E_{sw} is relatively sequentially compact in E_{sw}. Now the E_{sw}-closure of A is compact in E_{sw} and, by the Eberlein-Šmulian theorem, A is relatively sequentially $\sigma(E, E')$-compact. By compactness, $\sigma(E, E')$ and E_{sw} coincide on $\overline{A}^{E_{sw}}$, and therefore A is relatively sequentially compact in E_{sw}.

Let E be a Banach space and $\mu(E', E)$ the Mackey topology on E'. The topology $\mu(E', E)_{sw}$ is the topology of uniform convergence on the closed absolutely convex hulls of all sequences converging to zero in $\sigma(E, E')$.

Lemma 9. *Let E be a Banach space that contains no subspace isomorphic to l_1. Then $\beta(E', E)$, $\mu(E', E)$ and $\mu(E', E)_{sw}$ have the same convergent sequences.*

Proof. Suppose that $l_n \to 0$ in $\mu(E', E)_{sw}$ but not in $\beta(E', E)$. Then there is an $\varepsilon > 0$ and a sequence (x_n) in B_E such that $|l_n(x_n)| > \varepsilon$ for each n. Let x be in the $\sigma(E, E')$-closure of $\{x_n : n \in N\}$. In [9] Howard proved that a Banach space E contains no subspace isomorphic to l_1 if and only if every bounded subset of E is weakly sequentially dense in its weak closure. Therefore we can find a subsequence (y_n) of (x_n) such that $y_n \to x$ in $\sigma(E, E')$. Since $l_n \to 0$ in $\mu(E', E)_{sw}$ and thus also in $\sigma(E', E)$ there is an $n_0 \in N$ such that $\sup_{k\in N} |l_n(y_k - x)| \le \varepsilon/2$ and $|l_n(x)| \le \varepsilon/2$ for $n \ge n_0$. Now, for $n \ge n_0$, $\varepsilon < |l_n(y_n)| \le |l_n(y_n - x)| + |l_n(x)| \le \varepsilon$, which is a contradiction.

Combining Corollary 8 and Lemma 9 we arrive at the following corollary, which is valid for limited sets by Emmanuele [7].

COROLLARY 10. *Let E be a Banach space that contains no subspace isomorphic to l_1. Then every C^∞-bounding subset of $\beta(E', E)$ is relatively compact in $\beta(E', E)$.*

LEMMA 11. *Let $C(T)$ be the Banach space of continuous functions on a compact, sequentially compact Hausdorff space T with the supremum norm topology. Then $C(T)_{sw}$ and $C(T)$ have the same convergent sequences.*

PROOF. Assume that $f_n \to 0$ in $C(T)_{sw}$ but not in $C(T)$. Then there is an $\varepsilon > 0$ and a sequence (t_n) in T such that $|f_n(t_n)| > \varepsilon$ for each n. Since T is sequentially compact, we can find a convergent subsequence (t_{n_k}) of (t_n) which converges to $t \in T$. Now T can be considered as a subspace of $C(T)'_{co}$ and therefore $t_{n_k} \to t$ in $\sigma(C(T)', C(T))$ and $(t_{n_k} - t)$ is equicontinuous in $C(T)'$. Since $f_n \to 0$ in $C(T)_{sw}$, there exists an $n_0 \in N$ such that $|f_{n_k}(t_{n_k}) - f_{n_k}(t)| \leq \varepsilon/2$ and $|f_{n_k}(t)| \leq \varepsilon/2$ for $n \geq n_0$. As in the previous lemma we thus arrive at a contradiction.

By Diestel [5, p. 238], the next corollary also works for limited sets.

COROLLARY 12. *Let E be a Banach space which is isomorphic to a subspace of $C(T)$ with the supremum norm topology, where T is a compact, sequentially compact Hausdorff space. Then every C^∞-bounding subset of E is relatively compact in E.*

Recall that a Banach space E is called *weakly compactly generated* (WCG), if E is the closed linear span of some weakly compact set. All separable and reflexive Banach spaces as well as the space $c_0(\Gamma)$ are WCG spaces. For a WCG space E the closed unit ball of the dual E' is $\sigma(E', E)$-sequentially compact. Therefore the classes of C^∞-bounding and relatively compact sets coincide in WCG spaces.

In every infinite-dimensional Banach space E there exist bounded non-C^∞-bounding sets. Indeed, let B be the unit ball of E and assume that B is C^∞-bounding. By Theorem 4, B is relatively $\sigma(E, E')$-compact and therefore E is reflexive. But then B is even relatively compact in E, by Corollary 12, forcing E to be finite-dimensional. It should be pointed out that this observation also could be obtained from the Josefson-Nissenzweig theorem [5].

COROLLARY 13. *Let E be a quasi-complete* lcs *that is isomorphic to a subspace of a product $\prod_{\iota\in I} E_\iota$, where each E_ι is a WCG Banach space. Then every C^∞-bounding subset of E is relatively compact in E.*

PROOF. Let $A \subset E$ be C^∞-bounding. Then $pr_\iota(A) \subset E_\iota$ is C^∞-bounding and hence relatively compact for each $\iota \in I$, since E_ι is WCG. Therefore A is relatively compact in $\prod_{\iota\in I} E_\iota$ and thus also in E.

A lcs is said to be *transseparable*, if it is a subspace of a topological product of separable normed spaces [14]. Since the completion of a separable normed space is separable, *every C^∞-bounding subset of a quasi-complete transseparable lcs is relatively compact.* According to [14], a lcs E is transseparable if and only if for every zero-neighbourhood U in E there is a countable subset M of E with $E = M + U$. Thus the class of transseparable spaces contains not only the semi-weak ones but all separable and Lindelöf locally convex spaces as well. For Lindelöf spaces, however, we can characterize the C^∞-bounding sets without quasi-completeness by using the interpolation result below.

PROPOSITION 14. *Let E be a Lindelöf* lcs. *Assume that (x_n) is a sequence of distinct elements in E without cluster points. Then there exists a function $f \in C^\infty(E)$ such that $f(x_n) = n$ for each n.*

PROOF. As the sequence (x_n) has no cluster points in E, there is a convex open neighbourhood A_n of each x_n such that $x_k \notin \overline{A_n}$ for $k \neq n$. By the same argument there exists a convex open neighbourhood B_x of every x outside the sequence (x_n) such that $\overline{B_x} \cap \{x_n : n \in N\} = \varnothing$. Since E is Lindelöf, there is a sequence (B_k) of convex open sets with $E = \bigcup_{n=1}^\infty A_n \cup B_n$ and $\overline{B_k} \cap \{x_n : n \in N\} = \varnothing$ for each k. Let $W_n = \bigcup_{i=1}^n A_i \cup B_i$. Hence (W_n) is an increasing open covering of E, where the closures $\overline{W_n}$ are $\sigma(E, E')$-closed sets with $x_k \notin \overline{W_n}$ for $k > n$. Let f_1 be the constant function with the value 1. Since the weak topology is C^∞-smooth, we can find a sequence (f_n) of functions in $C^\infty(E)$ such that

$$f_n(\overline{W_{n-1}}) = \{0\} \quad \text{and} \quad f_n(x_n) = n - 1 - \sum_{i=2}^{n-1} f_i(x_n), \ n = 2, 3, \dots .$$

Let $f(x) = \sum_{n=1}^\infty f_n(x)$. By construction, f is locally finite and thus a function in $C^\infty(E)$. Furthermore $f(x_n) = n$ for each n.

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Since the compact and countably compact sets are the same in Lindelöf spaces we arrive at

COROLLARY 15. *Every C^{∞}-bounding set in a Lindelöf lcs is relatively compact.*

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