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Hodge Integrals and Hurwitz Numbers via Virtual Localization

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Abstract. We give another proof of Ekedahl, Lando, Shapiro, and Vainshtein's remarkable formula expressing Hurwitz numbers (counting covers of $\mathbb{P}^1$ with specified simple branch points, and specified branching over one other point) in terms of Hodge integrals. Our proof uses virtual localization on the moduli space of stable maps. We describe how the proof could be simplified by the proper algebro-geometric definition of a 'relative space'. Such a space has recently been defined by J. Li.

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1. Introduction

Hurwitz numbers count certain covers of the projective line (or, equivalently, factorizations of permutations into transpositions). They have been studied extensively since the time of Hurwitz, and have recently been the subject of renewed interest in physics ([CT]), combinatorics ([D], [A], and the series starting with [GJ]), algebraic geometry (recursions from Gromov–Witten theory, often conjectural), and symplectic geometry (e.g. [LZZ]).

Ekedahl, Lando, Shapiro and Vainshtein have proved a remarkable formula ([ELSV1, Theorem 1.1], [ELSV2, Theorem 1.1]; Theorem 2.2 below) linking Hurwitz numbers to Hodge integrals in a particularly elegant way.

We prove Theorem 2.2 using virtual localization on the moduli space of stable maps, developed in [GP]. In the simplest case, no complications arise, and Theorem 2.2 comes out immediately; Fantechi and Pandharipande proved this case independently [FP, Theorem 2], and their approach inspired ours.

In Section 5, we show that the proof would be much simpler if there were a moduli space for 'relative maps' in the algebraic category (with a good two-term obstruction theory, virtual fundamental class, and hence virtual localization formula). A space with some of these qualities already exists in the symplectic category (see [LR, Section 7] and [IP] for discussion). In the algebraic case, not much is known, although Gathmann has obtained striking results in genus 0 [G]. *Note added in revision:* J. Li has recently defined such a space; see Section 5 for further discussion.

2. Definitions and Statement

2.1. Throughout, we work over $\mathbb{C}$, and we use the following notation. Fix a genus g, a degree d, and a partition $(\alpha_1, \ldots, \alpha_m)$ of d with m parts. Let $b = 2d + 2g - 2$, the 'expected number of branch points of a degree d genus g cover of $\mathbb{P}^1$' by the Riemann-Hurwitz formula. We will identify $\mathrm{Sym}^b\,\mathbb{P}^1$ with $\mathbb{P}^b$ throughout. Let $r = d + m + 2(g-1)$, so a branched cover of $\mathbb{P}^1$, with monodromy type above ∞ given by α, and r other specified simple branch points (and no other branching) has genus g. Let $k = \sum_i(\alpha_i - 1)$, so $r = b - k$. Let H^g_α be the number of such branched covers that are connected. (We do not take the points over ∞ to be labelled.)

2.2. THEOREM ([ELSV1, Theorem 1.1], [ELSV2, Theorem 1.1]). *Suppose g, m are integers ($g \geqslant 0$, $m \geqslant 1$) such that $2g - 2 + m > 0$ (i.e. the functor $\overline{\mathcal{M}}_{g,m}$ forms a Deligne-Mumford stack). Then*

$$H^g_\alpha = \frac{r!}{\#\mathrm{Aut}(\alpha)} \prod_{i=1}^{m} \frac{\alpha_i^{\alpha_i}}{\alpha_i!} \int_{\overline{\mathcal{M}}_{g,m}} \frac{1 - \lambda_1 + \cdots \pm \lambda_g}{\prod(1 - \alpha_i \psi_i)},$$

where $\lambda_i = c_i(\mathbb{E})$ ($\mathbb{E}$ is the Hodge bundle).

Fantechi and Pandharipande's argument applies in the case where there is no ramification above ∞, i.e. $\alpha = (1^d)$.

The reader may check that a variation of our method also shows that

$$H^0_{\alpha_1} = r!\frac{d^{d-2}}{d!}, \qquad H^0_{\alpha_1,\alpha_2} = \frac{r!}{\#\mathrm{Aut}(\alpha_1, \alpha_2)} \cdot \frac{\alpha_1^{\alpha_1}}{\alpha_1!} \cdot \frac{\alpha_2^{\alpha_2}}{\alpha_2!} \cdot d^{d-1}.$$

As these formulas are known by other means ([D] for the first, [A] for the second, [GJ] for both), we omit the proof.

2.3. APPLICATION HURWITZ NUMBERS TO HODGE INTEGRALS

(i) Theorem 2.2 provides a way of computing all Hodge integrals as follows. Define

$$\langle \alpha_1, \ldots, \alpha_m \rangle := \int_{\overline{\mathcal{M}}_{g,m}} \frac{1 - \lambda_1 + \cdots \pm \lambda_g}{\prod(1 - \alpha_i \psi_i)},$$

a symmetric polynomial in the α_i of degree $3g - 3 + m$ whose coefficients are of the form $\int_{\overline{\mathcal{M}}_{g,m}} \psi_1^{d_1} \cdots \psi_m^{d_m} \lambda_k$. It is straightforward to recover the coefficients of a symmetric polynomial in m variables of known degree from a finite number of values, and $\langle \alpha_1, \ldots, \alpha_m \rangle$ can be easily computed (as Hurwitz numbers are

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combinatorial objects that are easily computable, see Section 3.10). Once these integrals are known, all remaining Hodge integrals (i.e. with more λ-classes) can be computed in the usual way, essentially due to Faber. The only other method to get the ψ-intersection numbers is using Kontsevich's theorem, formerly Witten's conjecture [K1]. Both methods of computation are in keeping with Mumford and Faber's philosophy, which is that Hodge integrals should be essentially combinatorial objects. Recently, Okounkov and Pandharipande have shown that Witten's conjecture can be derived from Theorem 2.2 [OP], giving the first algebraic proof.

(ii) Combinatorially straightforward relations among Hurwitz numbers (e.g. 'cut-and-join', see [GJ, Section 2]) yield nontrivial new identities among Hodge integrals.

2.4. APPLICATION: HODGE INTEGRALS TO HURWITZ NUMBERS

There has been much work on the structure of the Hurwitz numbers, including various predictions from physics. Theorem 2.2 is the key step in a machine to verify these structures and predictions, see [GJV].

3. Background: Maps of Curves to Curves

3.1. Following [V1] Section 4.2, define a *special locus* of a map f: $X \to \mathbb{P}^1$ (where X is a nodal curve) as a connected component of the locus in X where f is not étale. (Remark: No result in this section requires the target to be $\mathbb{P}^1$.) Then a special locus is of one of the following forms: (i) a nonsingular point of X that is an m-fold branch point (i.e. analytically locally the map looks like $x \to x^m$, $m > 1$), (ii) a node of X, where the two branches of the node are branch points of order m_1, m_2, or (iii) one-dimensional, of arithmetic genus g, attached to s branches of the remainder of the curve that are c_j-fold branch points ($1 \leqslant j \leqslant s$). The form of the locus, along with the numerical data, will be called the *type*. (For convenience, we will consider a point *not* in a special locus to be of type (i) with $m = 1$.) We will use the fact that special loci of type (ii) are smoothable [V3, Section 2.2].

3.2. RAMIFICATION NUMBER

To each special locus, associate a *ramification number* as follows: (i) $m - 1$, (ii) $m_1 + m_2$, (iii) $2g - 2 + 2s + \sum_{j=1}^{s}(c_j - 1)$. (Warning: in case (i), this is one less than what is normally called the ramification index; we apologize for any possible confusion.) The *total ramification* above a point of $\mathbb{P}^1$ is the sum of the ramification numbers of the special loci mapping to that point. We will use the following two immediate facts: if the map is stable, then the ramification number of each 'special locus' is a positive integer, and each special locus of type (iii) has ramification number at least 2.

3.3. EXTENDED RIEMANN–HURWITZ FORMULA

There is an easy generalization of the Riemann-Hurwitz formula:

$$2p_a(X) - 2 = -2d + \sum r_i$$

where $\sum r_i$ is the sum of the ramification numbers. (The proof is straightforward. For example, consider the complex $f^*\omega^1_{\mathbb{P}^1} \to \omega^1_X$ as in [FP] Section 2.3, and observe that its degree can be decomposed into contributions from each special locus. Alternatively, it follows from the usual Riemann–Hurwitz formula and induction on the number of nodes.)

3.4. BEHAVIOR OF RAMIFICATION NUMBER AND TYPE IN FAMILIES

Ramification number is preserved under deformations. Specifically, consider a pointed one-parameter family of maps (of nodal curves). Suppose one map in the family has a special locus S with ramification number r. Then the sum of the ramification numbers of the special loci in a general map that specialize to S is also r. (This can be shown by either considering the complex $f^*\omega^1_{\mathbb{P}^1} \to \omega^1_X$ in the family or by deformation theory.)

Next, suppose

$$\begin{array}{l} C \to \mathbb{P}^1 \\ \downarrow \\ B \end{array}$$

is a family of *stable* maps parametrized by a nonsingular curve B.

3.5. LEMMA. *Suppose there is a point ∞ of $\mathbb{P}^1$ where the total ramification number of special loci mapping to ∞ is a constant k for all geometric points of B. Then the type of ramification above ∞ is constant, i.e. the number of preimages of ∞ and their types are constant.*

For example, if the general fiber is nonsingular, i.e. only has special loci of type (i), then that is true for all fibers. Although this lemma is intuitively obvious, we could not find a short rigorous proof.

Proof. Let 0 be any point of B, and let $f: X \to \mathbb{P}^1$ be the map corresponding to 0. We will show that the type of ramification above ∞ for f is the same as for the general point of B.

First reduce to the case where the general map has no contracted components. (If the general map has a contracted component E, then consider the complement of the closure of E in the total general family. Prove the result there, and then show that the statement of Lemma 3.5 behaves well with respect to gluing a contracted component.)

Similarly, next reduce to the case where the general map is nonsingular. (First show the result where the nodes that are in the closure of the nodes in the generic

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curve are normalized, and then show that the statement behaves well with respect to gluing a 2-section of the family to form a node.)

Make a base change to separate the special loci of the general fiber (i.e. so they are preserved under monodromy) and also the preimages of ∞ for the general map.

For convenience of notation, restrict attention to one special locus E of f. Assume first that E is of type (iii), so $\dim E = 1$. Let g_E be the arithmetic genus of E. Suppose that r preimages of ∞ of the general fiber (of type (i) by reductions) meet E in the limit, and that these have ramification indices $b_1, \ldots, b_r$. Let s be the number of other branches of X meeting E, and $c_1, \ldots, c_s$ the ramification indices of the branches (as in Section 3.2).

The ramification number of E is $(2g_E - 2) + 2s + \sum_{j=1}^{s}(c_j - 1)$. The total ramification number of the special loci specializing to E is $\sum_{i=1}^{r}(b_i - 1)$. Also, $\sum_{i=1}^{r} b_i = \sum_{j=1}^{s} c_j$ as these represent the local contributions to the degree of the map to $\mathbb{P}^1$. Hence by conservation of ramification number, $(2g_E - 2 + s) + r = 0$. But $r > 0$, and by the stability condition for f, $2g_E - 2 + s > 0$, so we have a contradiction.

If $\dim E$ is 0 (i.e. E is of type (i) or (ii)), then essentially the same algebra works (with the substitution '$g_E = 0$', resulting in $r + s - 2 = 0$, from which $r = s = 1$, from which the type is constant). □

A similar argument shows:

3.6. LEMMA. *Suppose E is a special locus in a specific fiber, and only one special locus E' in the general fiber meets E. Then the types of E and E' are the same.*

3.7. THE FANTECHI–PANDHARIPANDE BRANCH MORPHISM

For any map f from a nodal curve to a nonsingular curve, the ramification number defines a divisor on the target: $\sum_L r_L f(L)$, where L runs through the special loci, and r_L is the ramification number. This induces a set-theoretic map Br: $\overline{\mathcal{M}}_g(\mathbb{P}^1, d) \to \mathrm{Sym}^b \mathbb{P}^1 \cong \mathbb{P}^b$. In [FP], this was shown to be a morphism.

Let p be the point of $\mathrm{Sym}^b \mathbb{P}^1 \cong \mathbb{P}^b$ corresponding to $k(\infty) + (b-k)(0)$, let $L_\infty \subset \mathbb{P}^b$ be the linear space corresponding to points of the form $k(\infty) + D$ (where D is a divisor of degree $r = b - k$), and let ι: $L_\infty \to \mathbb{P}^b$ be the inclusion.

Define M as the stack-theoretic pullback $\mathrm{Br}^{-1} L_\infty$. It carries a virtual fundamental class $[M]^{\mathrm{vir}} = \iota^! [\overline{\mathcal{M}}_g(\mathbb{P}^1, d)]^{\mathrm{vir}}$ of dimension $r = b - k$ (i.e. simply intersect the class $[\overline{\mathcal{M}}_g(\mathbb{P}^1, d)]^{\mathrm{vir}}$ with the codimension k operational Chow class $\mathrm{Br}^*[L_\infty]$; the result is supported on $\mathrm{Br}^{-1} L_\infty$). Denote the *restricted branch map* by br: $M \to L_\infty$. By abuse of notation, we denote the top horizontal arrow in the following diagram by ι as well.

$$\begin{array}{ccc} M & \rightarrow & \overline{\mathcal{M}}_g(\mathbb{P}^1, d) \\ \mathrm{br} \downarrow & & \downarrow \mathrm{Br} \\ L_\infty & \xrightarrow{\iota} & \mathbb{P}^b \end{array}$$

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By the projection formula,

$$\iota_*(\mathrm{br}^*[p] \cap [M]^{\mathrm{vir}}) = \mathrm{Br}^*[p] \cap [\overline{\mathcal{M}}_g(\mathbb{P}^1, d)]^{\mathrm{vir}}. \tag{1}$$

Define M^α as the union of irreducible components of M whose general members correspond to maps from irreducible curves, with ramification above ∞ corresponding to α with the reduced substack structure. (It is not hard to show that M^α is irreducible, by the same group-theoretic methods as the classical proof that the Hurwitz scheme is irreducible. None of our arguments use this fact, so we will not give the details of the proof. Still, for convenience, we will assume irreducibility in our language.)

3.8. Note that $M = \mathrm{Br}^{-1} L_\infty$ contains M^α with some multiplicity m_α, as M^α is of the expected dimension r. The Hurwitz number H^g_α is given by

$$\int_{M^\alpha} \mathrm{br}^*[p].$$

(The proof of [FP, Proposition 2] carries over without change in this case, as does the argument of [V2, Section 3].) This is $1/m_\alpha$ times the cap product of $\mathrm{br}^*[p]$ with the part of the class of $[M]^{\mathrm{vir}}$ supported on M^α.

3.9. LEMMA $m_\alpha = k! \prod \left(\alpha_i^{\alpha_i - 1}/\alpha_i!\right)$.

3.10. In the proof, we will use the combinatorial interpretation of Hurwitz numbers: H^g_α is $1/d!$ times the number of ordered r-tuples $(\tau_1, \dots, \tau_r)$ of transpositions generating S_d, whose product has cycle structure α.

Proof. Fix r general points $p_1, \dots, p_r$ of $\mathbb{P}^1$. Let $L \subset \mathbb{P}^b$ be the linear space corresponding to divisors of the form $p_1 + \cdots + p_r + D$ (where $\deg D = k$). By the Kleiman–Bertini theorem, $(\mathrm{Br}|_{M_\alpha})^{-1} L$ consists of H^g_α reduced points.

Now $L_\infty \subset \mathrm{Sym}\,\mathbb{P}^b$ can be interpreted as a (*real* one-parameter) degeneration of the linear space corresponding to divisors of the form $D' + \sum_{i=1}^k q_i$, where $q_1, \dots, q_k$ are fixed generally chosen points of $\mathbb{P}^1$ and D' is any degree r divisor on $\mathbb{P}^1$.

Choose branch cuts to the points $p_1, \dots, p_r$, $q_1, \dots, q_k, \infty$ from some other point of $\mathbb{P}^1$. Choose a real one-parameter path connecting $q_1, \dots, q_k, \infty$ (in that order), not meeting the branch cuts (see the dashed line in Figure 1). Degenerate the points q_i to ∞ along this path one at a time (so the family parametrizing this degeneration is

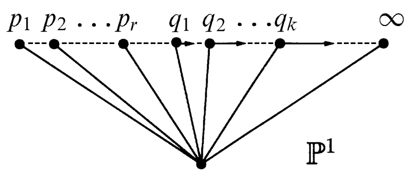

Figure 1. Degenerating the points $q_1, \dots, q_k$ to ∞ one by one, along a real path.

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reducible). If $\sigma_1, \dots, \sigma_k, \sigma_\infty$ are the monodromies around the points $q_1, \dots, q_k, \infty$ for a certain cover, then the monodromy around ∞ after the branch points $q_i, \dots, q_k$ have been degenerated to ∞ (along the path) is $\sigma_i \dots \sigma_k \sigma_\infty$.

At a general point of the family parametrizing this real degeneration (before any of the points q_i have specialized, i.e. the q_i are fixed general points), $\mathrm{Br}^{-1}(L \cap L_\infty)$ is a finite number of reduced points. This number is the Hurwitz number $H^g_{(1^d)}$ [FP, Proposition 2], i.e. $1/d!$ times the number of choices of $b = r + k$ transpositions $\tau_1, \dots, \tau_r, \sigma_1, \dots, \sigma_k$ in S_d such that $\tau_1 \dots \tau_r \sigma_1 \dots \sigma_k$ is the identity and $\tau_1, \dots, \tau_r, \sigma_1, \dots, \sigma_k$ generate S_d.

As we specialize the k branch points $q_1, \dots, q_k$ to ∞ one at a time, some of these points tend to points of M^α; these are the points for which $\tau_1, \dots, \tau_r$ generate S_d, and their product has cycle structure α. The multiplicity m_α is the number of these points that go to each point of M^α. This is the number of choices of k transpositions $\sigma_1, \dots, \sigma_k$ whose product is a *given* permutation ξ with cycle structure α. (Note that this number is independent of the choice of ξ; hence the multiplicity is independent of choice of component of M^α.)

If $k = \sum(\alpha_i - 1)$ transpositions $\sigma_1, \dots, \sigma_k$ multiply to a permutation $\xi = (a_{1,1} \dots a_{1,\alpha_1}) \cdots (a_{m,1} \dots a_{m,\alpha_m})$ (where $\{a_{1,1}, \dots, a_{m,\alpha_m}\} = \{1, \dots, d\}$), then for $1 \leqslant i \leqslant m$, $\alpha_i - 1$ of the transpositions must be of the form $(a_{i,u} a_{i,v})$. (Reason: A choice of $k+1$ points $q_1, \dots, q_k, \infty$, of $\mathbb{P}^1$ and the data $\sigma_1, \dots, \sigma_k, \xi$ defines a degree d branched cover of $\mathbb{P}^1$, simply branched above q_j and with ramification type α above ∞. By the Riemann–Hurwitz formula, the arithmetic genus of this cover is $1 - m$; as the pre-image of ∞ contains m smooth points, the cover has at most m components. Hence, the cover has precisely m components, each of genus 0. The ith component is simply branched at $\alpha_i - 1$ of the points $\{q_1, \dots, q_k\}$ away from ∞.)

The number of ways of factoring an α_i-cycle into $\alpha_i - 1$ transpositions is $\alpha_i^{\alpha_i - 2}$ (straightforward; or see [D] or [GJ, Theorem 1.1]). Hence, m_α is the number of ways of partitioning the k points $q_1, \dots, q_k$ into subsets of size $\alpha_1 - 1, \dots, \alpha_m - 1$, times the number of ways of factoring the α_i-cycles:

$$m_\alpha = \binom{k}{\alpha_1 - 1, \dots, \alpha_m - 1} \prod \alpha_i^{\alpha_i - 2} = k! \prod \left(\frac{\alpha_i^{\alpha_i - 1}}{\alpha_i!} \right). \qquad \square$$

4. Virtual Localization

4.1. VIRTUAL LOCALIZATION PRELIMINARIES

We evaluate the integral of Section 3.8 using virtual localization [GP]. The standard action of $\mathbb{C}^*$ on $\mathbb{P}^1$ (so that the action on the tangent space at 0 has weight 1) induces a natural $\mathbb{C}^*$-action on $\overline{\mathcal{M}}_g(\mathbb{P}^1, d)$, and the branch morphism Br is equivariant with respect to the induced torus action on $\mathrm{Sym}^b \mathbb{P}^1 \cong \mathbb{P}^b$. As a result, we can regard $\mathrm{br}^*[p]$ as an equivariant Chow cohomology class in $A^r_{\mathbb{C}^*} M$. Let $\{F_l\}_{l \in L}$ be the set

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of components of the fixed locus of the torus action on $\overline{\mathcal{M}}_g(\mathbb{P}^1, d)$, where L is some index set. (Note that the connected components of the fixed locus are also irreducible.)

Define F_0 to be the component of the fixed locus whose general point parametrizes a stable map with a single genus g component contracted over 0, and m rational tails mapping with degree α_i $(1 \leqslant i \leqslant m)$ to $\mathbb{P}^1$, totally ramified above 0 and ∞. F_0 is naturally isomorphic to a quotient of $\overline{\mathcal{M}}_{g,m}$ by a finite group. See [K2] or [GP] for a discussion of the structure of the fixed locus of the $\mathbb{C}^*$ action on $\overline{\mathcal{M}}_g(\mathbb{P}^1, d)$.

By the virtual localization formula, we can explicitly write down classes $\mu_l \in A_*^{\mathbb{C}^*}(F_l)_{(1/t)}$ such that $\sum_l i_*(\mu_l) = [M]^{\text{vir}}$ in $A_*^{\mathbb{C}^*}(M)$. Here, and elsewhere, i is the natural inclusion. It is important to note that the μ_l are uniquely determined by this equation. This follows from the Localization Theorem 1 of [EG] (extended to Deligne–Mumford stacks by [Kr]), which says that pushforward gives an isomorphism between the localized Chow group of the fixed locus and that of the whole space.

In order to pick out the contribution to this integral from a single component F_0, we introduce more refined classes. We denote the irreducible components of M by M_n, and arbitrarily choose a representation $[M]^{\text{vir}} = \sum_n i_*\Gamma_n$, where $\Gamma_n \in A_*^{\mathbb{C}^*}(M_n)$. For a general component, we can say little about these classes, but for our distinguished irreducible component M^α the corresponding Γ_α is necessarily $m_\alpha[M^\alpha]$. (Note that M^α has the expected dimension, so the Chow group in that dimension is generated by the fundamental class.)

Next, we localize each of the Γ_n. Define $\eta_{l,n}$ in $A_*^{\mathbb{C}^*}(F_l)_{(1/t)}$ by

$$\sum_l i_*\eta_{l,n} = \Gamma_n \tag{2}$$

Once again (by [EG, Kr]), the $\eta_{l,n}$ are uniquely defined; this will be used in Lemma 4.4. Also, $\sum_n \eta_{l,n} = \mu_l$ (as the μ_l are uniquely determined).

4.2. LEMMA. *The equivariant class* $\text{br}^*[p]$ *restricts to* 0 *on any component of the fixed locus whose general map has total ramification number greater than k above* ∞.

Proof. Restricting the branch morphism to such a component, we see that it gives a constant morphism to a point in $\mathbb{P}^b$ other than p. Consequently, the pull-back of the class p must vanish. □

4.3. LEMMA. $\int_{\Gamma_n} \text{br}^*[p] = 0$ *for any irreducible component* M_n *whose general point corresponds to a map which has a contracted component away from* ∞.

Proof. A general cycle $\gamma \in L_\infty$ representing p is the sum of r distinct points plus the point ∞ exactly k times. However, a contracted component always gives a multiple component of the branch divisor (Section 3.2), so the image of M_n cannot meet a general point. □

4.4. LEMMA. $\eta_{l,n} = 0$ *if* $F_l \cap M_n = \emptyset$.

Proof. Since Γ_n is an element of $A_*^{\mathbb{C}^*}(M_n)$, there exist classes $\tilde{\eta}_{l,n}$ in the localized equivariant Chow groups of the fixed loci of M_n satisfying Equation (2). Pushing

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these forward to the fixed loci of M gives classes in the Chow groups of the F_l satisfying the same equation. By uniqueness, these must be the $\eta_{l,n}$. By this construction, it follows that they can only be nonzero if F_l meets M_n. □

4.5. LEMMA. *No irreducible component of M can meet two distinct components of the fixed locus with total ramification number exactly k above ∞.*

Proof. To each map $f: X \to \mathbb{P}^1$ with total ramification number exactly k above ∞, associate a graph as follows. The connected components of the preimage of ∞ correspond to red vertices; they are labelled with their type. The connected components of $Y = \overline{X \setminus f^{-1}(\infty)}$ (where the closure is taken in X) correspond to green vertices; they are labelled with their arithmetic genus. Points of $Y \cap f^{-1}(\infty)$ correspond to edges connecting the corresponding red and green points; they are labelled with the ramification number of $Y \to \mathbb{P}^1$ at that point. Observe that this associated graph is constant in connected families where the total ramification over ∞ is constant, essentially by Lemma 3.5.

If an irreducible component M' of M meets a component of the fixed locus with total ramification number exactly k above ∞, then the general map in M' has total ramification k above ∞. (Reason: the total ramification is at most k as it specializes to a map with total ramification exactly k; and the total ramification is at least k as it is a component of M.) There is only one component of the fixed locus that has the same associated graph as the general point in M', proving the result. □

4.6. LEMMA. *The map parametrized by a general point of any irreducible component of M other than M^α which meets F_0 must have a contracted component not mapping to ∞.*

Proof. Let M' be an irreducible component of M other than M^α. As in the proof of Lemma 4.5, a general map $f: X \to \mathbb{P}^1$ of M' has total ramification exactly k above ∞. By Lemma 3.5, we know the type of the special loci above ∞: they are nonsingular points of the source curve, and the ramification numbers are given by $\alpha_1, \ldots, \alpha_m$.

As $M' \neq M^\alpha$, X is singular. If f has a special locus of type (iii), then we are done. Otherwise, f has only special loci of type (ii), and none of these map to ∞. But then these type (ii) special loci can be smoothed while staying in M (Section 3.1), contradicting the assumption that f is a general map in a component of M. □

4.7. PROPOSITION

$$m_\alpha \int_{M^\alpha} \mathrm{br}^*[p] = \int_{F_0} \mathrm{br}^*[p] \cap \mu_0.$$

It is the class μ_0 that the Virtual Localization Theorem of [GP] allows us to calculate explicitly. Thus this proposition is the main ingredient in giving us an explicit formula for the integral we want to compute.

Proof. Now $\Gamma_\alpha = m_\alpha[M^\alpha]$, so by definition of $\eta_{l,\alpha}$, $m_\alpha[M^\alpha] = \sum_l i_* \eta_{l,\alpha}$.

By Lemma 4.3, M^α meets only one component of the fixed locus which has total ramification number k, F_0. Along with Lemmas 4.2 and 4.4, this implies that

$$m_\alpha \int_{M^\alpha} \mathrm{br}^*[p] = \int_{F_0} \mathrm{br}^*[p] \cap \eta_{0,\alpha}.$$

In other words, the only component of the fixed locus which contributes to this integral is F_0. Since $\mu_0 = \sum_n \eta_{0,n}$, the proposition will follow if we can show that

$$\int_{F_0} \mathrm{br}^*[p] \cap \eta_{0,n} = 0$$

for $n \neq \alpha$, i.e. that no other irreducible component of M contributes to the localization term coming from F_0.

If $F_0 \cap M_n = \emptyset$, this is true by Lemma 4.4. Otherwise, by Lemma 4.6, the general map in M_n has a contracted component, so by Lemma 4.3 $\int_{\Gamma_n} \mathrm{br}^*[p] = 0$. By Equation (2),

$$\sum_l \int_{F_l} \mathrm{br}^*[p] \cap \eta_{l,n} = 0.$$

If F_l generically corresponds to maps that have total ramification number greater than k above ∞, then $\mathrm{br}^*[p] \cap \eta_{l,n} = 0$ by Lemma 4.3. If $l \neq 0$ and F_l generically corresponds to maps that have total ramification number k above ∞, then $\mathrm{br}^*[p] \cap \eta_{l,n} = 0$ by Lemma 4.5, as M_n meets F_0. Hence $\int_{F_0} \mathrm{br}^*[p] \cap \eta_{0,n} = 0$ as desired. $\square$

4.8. *Proof of Theorem 2.2.* All that is left is to explicitly write down the right hand side of Proposition 4.7. By Equation (1), this integral can be interpreted as the contribution of F_0 to the integral of $\mathrm{Br}^*[p]$ against the virtual fundamental class of $\overline{\mathcal{M}}_g(\mathbb{P}^1, d)$, divided by m_α. Since this means we are trying to compute an equivariant integral over the entire space of maps to $\mathbb{P}^1$, we are in exactly the situation discussed in [GP]. Let γ be the natural morphism from $\overline{\mathcal{M}}_{g,m}$ to F_0. The degree of γ is $\#\mathrm{Aut}(\alpha) \prod \alpha_i$. The pullback under γ of the inverse euler class of the virtual normal bundle is computed to be

$$(-1)^{d+m} c(\mathbb{E}^\vee) \left(\prod \frac{1}{1 - \alpha_i \psi_i} \cdot \frac{\alpha_i^{2\alpha_i}}{(\alpha_i!)^2} \right).$$

The class $\mathrm{br}^*[p]$ is easy to evaluate. Since br is constant when restricted to F_0, this class is pure weight, and is given by the product of the weights of the $\mathbb{C}^*$ action on $T_p\mathbb{P}^b$. These weights are given by the nonzero integers from $-k$ to $b-k$ inclusive. The integral over F_0 is just the integral over $\overline{\mathcal{M}}_{g,m}$ divided by the degree of γ. We conclude that

$$m_\alpha \int_{[M^\alpha]} br^*[p] = \frac{k!(b-k)!}{\#\mathrm{Aut}(\alpha) \prod \alpha_i} \cdot \prod \frac{\alpha_i^{2\alpha_i}}{(\alpha_i!)^2} \cdot \int_{\overline{\mathcal{M}}_{g,m}} \frac{c(\mathbb{E}^\vee)}{\prod (1 - \alpha_i \psi_i)}.$$

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(We use $(-1)^{k+d+m} = 1$.) Dividing by m_α (calculated in Lemma 3.9) yields the desired formula. □

5. A Case for an Algebraic Definition of a Space of 'Relative Stable Maps'

A space of 'relative stable maps' has been defined in the symplectic category (see [LR, IP]), but hasn't yet been properly defined in the algebraic category (with the exception of Gathmann's work in genus 0, [G]).

The proof of Theorem 2.2 would become quite short were such a space $\mathcal{M}$ to exist with expected properties, namely the following. Fix d, g, α, m, k, r as before (see Section 2.1).

(1) $\mathcal{M}$ is a proper Deligne–Mumford stack, which contains as an open substack U the locally closed substack of $\overline{\mathcal{M}}_g(\mathbb{P}^1, d)$ corresponding to maps to $\mathbb{P}^1$ where the pre-image of ∞ consists of m smooth points appearing with multiplicity $\alpha_1, \ldots, \alpha_m$.
(2) There is a Fantechi–Pandharipande branch map $\mathrm{Br}\colon M \to \mathrm{Sym}^b\,\mathbb{P}^1$. The image will be contained in L_∞, so we may consider the induced map br to $L_\infty \cong \mathrm{Sym}^r\,\mathbb{P}^1$. Under this map, the fiber of $k(\infty) + r(0)$ is precisely F_0.
(3) There is a $\mathbb{C}^*$ equivariant perfect obstruction theory on $\mathcal{M}$ which when restricted to U is given (relatively over $\mathcal{M}_g$) by $R\pi_*(f^*(T\mathbb{P}^1) \otimes \mathcal{O}(-\sum \alpha_i x_i))$.

With these axioms, the proof would require only Section 4.8.

All of these requirements are reasonable. However, as a warning, note that the proof of Proposition 4.7 used special properties of the class $\mathrm{br}^*[p]$ (Lemmas 4.2–4.6).

Note added in revision: J. Li has recently defined a space, with the desired properties [L1, L2]. As predicted in the submitted version of this paper, Li's space in this case can be interpreted in this case as a combination of Kontsevich's space $\overline{\mathcal{M}}_g(\mathbb{P}^1, d)$ and the space of twisted maps introduced by Abramovich and Vistoli [AV, Section 3].

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