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Finite Groups Generated by Involutions on Lagrangian Planes of $\mathbf{C}^2$

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Abstract. We classify finite subgroups of $\mathrm{SO}(4)$ generated by anti-unitary involutions. They correspond to involutions fixing pointwise a Lagrangian plane. Explicit descriptions of the finite groups and the configurations of Lagrangian planes are obtained.

1 Introduction

Finite subgroups of $U(n)$ generated by complex reflections were classified by Shephard and Todd [ST] (see also [C2]). In this paper we consider the problem of classifying finite subgroups of $\widehat{U(2)}$, the group generated by the unitary transformations and their complex conjugate transformations, generated by involutions on Lagrangian planes.

A Lagrangian plane Π is a totally real plane such that $J\Pi$ is orthogonal to Π (here, J is the linear operator defined on $\mathbf{R}^4$ such that $J^2 = -I$ which defines the complex structure). There is a special involution in $\widehat{U(2)}$ associated to a Lagrangian plane. It is an antiholomorphic map which fixes pointwise that plane and acts as a rotation by 180 degrees on its orthogonal plane. We call those involutions *inversions* (see Section 4). One can define them simply as anti-unitary transformations which are involutions.

The interest in those groups arouse from complex hyperbolic geometry [G] and the construction of discrete subgroups of $\mathrm{PU}(2,1)$ (the group of biholomorphisms of the two dimensional complex ball). In fact (see [FZ], [FK]), one obtains natural constructions of those groups as index two subgroups of groups generated by involutions on totally geodesic totally real planes of the complex ball considered as the complex hyperbolic space.

We hope in a future paper to consider the analogous problem in higher dimensions. Dimension two is rather special as one can use the many properties of quaternions to give very explicit descriptions of the groups. Moreover, the proof given here makes appeal to the classification of finite groups of $\mathrm{SO}(4)$. We also hope that those finite subgroups will constitute building blocks of new discrete subgroups of $\mathrm{SU}(2,1)$.

In Sections 2 and 3 we recall the classification of finite subgroups of $\mathrm{SU}(2)$ and $U(2)$ for the reader's benefit. The main classification theorem is Theorem 4.1. In Section 5 we give explicit descriptions of the configurations of Lagrangian planes. Those can be visualized if we consider their intersections with $S^3 \subset \mathbf{C}^2$.

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2 Finite Subgroups of SU(2)

Let H be the space of quaternions $\mathbf{x} = x_0 + x_1\mathbf{i} + x_2\mathbf{j} + x_3\mathbf{k}$. We denote $\tilde{\mathbf{x}} = x_0 - x_1\mathbf{i} - x_2\mathbf{j} - x_3\mathbf{k}$, the conjugate, $S\mathbf{x} = x_0$ and $V\mathbf{x} = x_1\mathbf{i} + x_2\mathbf{j} + x_3\mathbf{k}$. If $S\mathbf{x} = 0$ the quaternion is called pure. Consider the group of unit quaternions isomorphic to $\mathrm{SU}(2)$ which can be identified to S^3.

Using the notation of [C2, p. 68], we enumerate the finite subgroups of $\mathrm{SU}(2)$.

1. the cyclic group C_{n} of order n,
2. the dicyclic group $\langle p, 2, 2\rangle$ (also denoted D_p) of order $4p$,
3. the binary tetrahedral group $\langle 3, 3, 2\rangle$ of order 24,
4. the binary octahedral group $\langle 4, 3, 2\rangle$ of order 48,
5. the binary icosahedral group $\langle 5, 3, 2\rangle$ of order 120.

It will be important in the following to determine those groups generated by pure quaternions.

Proposition 2.1 **(see [C2, Section 7.5])** *The following finite groups are generated by pure quaternions*

1. *C_2 generated by $U = \mathbf{i}$,*
2. *$\langle \mathrm{p}, 2, 2\rangle$ generated by $U_1 = \mathbf{j}$, $U_2 = \exp(\mathbf{i}\pi/\mathrm{p})\mathbf{j}$, in particular $\langle 2, 2, 2\rangle$ is generated by $\mathbf{j}$ and $\mathbf{k}$,*
3. *$\langle 4, 3, 2\rangle$ generated by $\mathbf{p}_1 = \frac{1}{\sqrt{2}}(\mathbf{k} - \mathbf{j})$, $\mathbf{p}_2 = \frac{1}{\sqrt{2}}(\mathbf{i} - \mathbf{k})$, $\mathbf{p}_3 = \frac{1}{\sqrt{2}}(\mathbf{k} + \mathbf{j})$,*
4. *$\langle 5, 3, 2\rangle$ generated by $U_1 = \mathbf{j}$, $U_2 = -\frac{1}{2}(\mathbf{i} + \tau\mathbf{j} - \tau^{-1}\mathbf{k})$, $U_3 = \mathbf{i}$, where $\tau = \frac{1}{2\cos(2\pi/5)}$.*

3 Finite Subgroups of SO(4)

The finite subgroups of $\mathrm{SO}(4)$ are classified in Du Val's book (see [D] and his historical remarks). We use the slightly different coordinates of Coxeter (see [C2]).

We define two right actions of $\mathrm{SU}(2)$, identified to the unit quaternions $\mathbf{Q}$, on the space of quaternions identified to $\mathbf{R}^4$.

1. right screw: $\mathrm{SU}(2) \times \mathbf{R}^4 \to \mathbf{R}^4$, given by $(q, x) \to xq$.
2. left screw: $\mathrm{SU}(2) \times \mathbf{R}^4 \to \mathbf{R}^4$, given by $(q, x) \to \tilde{q}x$.

Proposition 3.1 **(see [D], [C2])** *The map $\mathrm{SU}(2) \times \mathrm{SU}(2) \to \mathrm{SO}(4)$ given by $(q, q') \to (\mathbf{x} \to \tilde{q}\mathbf{x}q')$ is a two to one covering homomorphism.*

We will characterize the transformations of $U(2)$ and their conjugates, a set we denote by $\overline{U}(2)$. Observe that $\overline{U}(2)$ is not a group. We think $x = x_0 + x_1\mathbf{i} + x_2\mathbf{j} + x_3\mathbf{k} = u + v\mathbf{j}$, where $u = x_0 + x_1\mathbf{i}$ and $v = x_2 + x_3\mathbf{i}$ are complex numbers. Restricting the map above to the complex left screws times the unit quaternions, that is, to $U(1) \times \mathrm{SU}(2)$ where $U(1) = \{z = x_0 + x_1\mathbf{i} \in \mathbf{Q}\}$, we get $U(2) \subset \mathrm{SO}(4)$.

Proposition 3.2 **(see [Cr], [D], [C2])** *The map $U(1) \times \mathrm{SU}(2) \to U(2)$ given by $(\epsilon, q) \to \epsilon x q$ is a two to one covering homomorphism.*

Now, observe that if u is a complex number $\mathbf{j}u\mathbf{j} = -\overline{u}$. So one obtains the following:

Proposition 3.3 *The map $U(1) \times \mathrm{SU}(2) \to \overline{U}(2)$ given by $(\epsilon, q) \to \epsilon\mathbf{j}x\mathbf{j}q$ is a two to one covering.*

The description of a finite group $\mathbf{G}$ of $\mathrm{SU}(2) \times \mathrm{SU}(2)$ is based on the groups $\mathbf{L} = \{l \in \mathrm{SU}(2) \mid (l, r) \in \mathbf{G}$ for a certain $r \in \mathrm{SU}(2)\}$ and $\mathbf{R} = \{r \in \mathrm{SU}(2) \mid (l, r) \in \mathbf{G}$ for a certain $l \in \mathrm{SU}(2)\}$. We consider the normal subgroups $\mathbf{L}_K = \{l \in \mathbf{L} \mid (l, 1) \in \mathbf{G}\}$ and $\mathbf{R}_K = \{r \in \mathbf{R} \mid (1, r) \in \mathbf{G}\}$. We must have $\mathbf{L}/\mathbf{L}_K \equiv \mathbf{R}/\mathbf{R}_K$. A choice of this isomorphism defines a subgroup of $\mathrm{SU}(2) \times \mathrm{SU}(2)$ denoted $(\mathbf{L}/\mathbf{L}_K; \mathbf{R}/\mathbf{R}_K)$. The same discussion with the same notation is valid for subgroups of $U(2)$, but now $\mathbf{L} \in U(1)$ and $\mathbf{R} \in \mathrm{SU}(2)$.

***Example 3.1* (See [D, p. 55])** Consider $\mathbf{L} = \mathbf{C}_{mr} \subset U(1)$ and $\mathbf{R} = \mathbf{C}_{nr} \subset \mathrm{SU}(2)$, cyclic groups. We have that $p^j\mathbf{C}_m \to q^{sj}\mathbf{C}_n$ (where p, q are generators of $\mathbf{C}_{mr}$ and $\mathbf{C}_{nr}$) defines an isomorphism $\mathbf{C}_{mr}/\mathbf{C}_m \to \mathbf{C}_{nr}/\mathbf{C}_n$, if s is prime to r. For different values of $s < r/2$ we obtain distinct groups denoted by $(\mathbf{C}_{mr}/\mathbf{C}_m; \mathbf{C}_{nr}/\mathbf{C}_n)_s$. Observe that if $\mathbf{G} \subset U(1) \times \mathrm{SU}(2)$ covers a group in $U(2)$, $-1 \in \mathbf{C}_{mr}$, that is, mr is even and, also, m and n are both odd or both even.

Example 3.2 In the case $\mathbf{L} = \mathbf{L}_K = \mathbf{C}_{2m} \subset U(1)$ and $\mathbf{R} = \mathbf{R}_K = \langle p, q, r\rangle \subset \mathrm{SU}(2)$, the quotient $\mathbf{L}/\mathbf{L}_K$ being trivial, the group is $\mathbf{G} = \mathbf{C}_{2m} \times \langle p, q, r\rangle$ and is denoted $\langle p, q, r\rangle_m$.

***Theorem 3.1* (see [D], [C2])** *The finite groups of $U(2)$ are*

1. $(C_{2m}/C_f; C_{2n}/C_g)_d$ *of order* $gm = fn$ $(f \equiv g (\mathrm{mod}\, 2)$, $(d, 2m/f) = 1$ *and* $d < m/f)$
2. $\langle p, 2, 2\rangle_m$ $(4mp)$
3. $(C_{4m}/C_{2m}; \langle p, 2, 2\rangle/C_{2p})$ $(4mp)$ *and* $(C_{4m}/C_m; \langle p, 2, 2\rangle/C_p)$, *$m$ and p odd,* $(2mp)$
4. $(C_{4m}/C_{2m}; \langle 2p, 2, 2\rangle/\langle p, 2, 2\rangle)$ $(8mp)$
5. $\langle 3, 3, 2\rangle_m$ $(24m)$
6. $(C_{6m}/C_{2m}; \langle 3, 3, 2\rangle/\langle 2, 2, 2\rangle)$ $(24m)$
7. $\langle 4, 3, 2\rangle_m$ $(48m)$
8. $(C_{4m}/C_{2m}; \langle 4, 3, 2\rangle/\langle 3, 3, 2\rangle)$ $(48m)$
9. $\langle 5, 3, 2\rangle_m$ $(120m)$.

4 Finite Groups Generated by Involutions

4.1 Reflections versus Inversions

Definition 4.1 $\widehat{U(2)}$ is the subgroup of $\mathrm{SO}(4)$ generated by $U(2)$ and $\overline{U}(2)$.

In fact, $\overline{U}(2)$ is generated by $U(2)$ and one element of $\overline{U}(2)$ which is not the identity.

Definition 4.2 A complex reflection is a finite order element (different from the identity) of $U(2)$ fixing pointwise a complex line.

In the following we will call a complex line a complex plane as opposed to a real plane.

Definition 4.3 An inversion is an element (different from the identity) of $\overline{U}(2)$ fixing pointwise a totally real plane.

Proposition 4.1 *An inversion is an involution. It acts on the orthogonal totally real plane as a half-turn. Moreover Lagrangian planes are in one to one correspondence to involutions.*

Proof It is enough to show that the set of anti-unitary transformations fixing a specific Lagrangian plane contains only one element which is an involution. Take the Lagrangian plane fixed by the conjugation $(z_1, z_2) \to (\overline{z}_1, \overline{z}_2)$. The plane is $\{(z_1, z_2) \mid \operatorname{Im}(z_1) = \operatorname{Im}(z_2) = 0\}$ and the conjugation acts on the orthogonal plane as a half turn. Other anti-unitary transformation are compositions of the conjugation and unitary maps. As the Lagrangian plane is fixed by the conjugation, the unitary map should also fix that plane. The only possibility is that the unitary map be the identity. ■

Using 3.3 we may characterize inversions as in the following

Proposition 4.2 *The map $x \to \epsilon \mathbf{j} x \mathbf{j} q \in \overline{U}(2)$ is an inversion if and only if $q = q_0 + q_1\mathbf{i} + q_3\mathbf{k} \in \mathbf{Q}$, that is, the $\mathbf{j}$-component of q is 0. This implies that the space of inversions is 2-1 covered by $S^1 \times S^2$.*

Proof We impose that the square of an element of $\overline{U}(2)$ be the identity. Using 3.3 we get $\epsilon\mathbf{j}\epsilon\mathbf{j}x\mathbf{j}q\mathbf{j}q = -\overline{\epsilon}\epsilon x(-\overline{u} - \overline{v}\mathbf{j})(u + v\mathbf{j})$, where we wrote $q = u + v\mathbf{j} \in \mathbf{Q}$. Writing $q = q_0 + q_1\mathbf{i} + q_2\mathbf{j} + q_3\mathbf{k}$, and substituting above we obtain $q_2 = 0$. ■

In other words the inversions are described by $x \to \iota x \mathbf{p} \in \overline{U}(2)$, where $\iota = \epsilon\mathbf{j}$ (ϵ complex) and $\mathbf{p} = p_1\mathbf{i} + p_2\mathbf{j} + p_3\mathbf{k}$ is a pure quaternion.

4.2 Finite Groups Generated by Inversions

The finite groups of $U(2)$ generated by reflections were enumerated by Shephard and Todd ([ST], see also [C1]). In this section we classify the finite groups of $\widehat{U(2)}$ generated by inversions. Observe that by Proposition 4.2 inversions are associated to pairs $(\epsilon\mathbf{j}, \mathbf{p})$, with $\mathbf{p}$ a pure quaternion. The product of two such inversions $I_{(\epsilon_1\mathbf{j},\mathbf{p}_1)} \circ I_{(\epsilon_2\mathbf{j},\mathbf{p}_2)}$ is an element of $U(2)$ given by the pair $\big(-\epsilon_1\overline{\epsilon}_2, (-\mathbf{p}_1.\mathbf{p}_2 + \mathbf{p}_1 \times \mathbf{p}_2)\big)$.

Example 4.1 **(Groups Generated by Two Inversions)** If we take the pair $\mathbf{r}_1 = (\mathbf{j}, \mathbf{j})$ and $\mathbf{r}_2 = \big(\exp(-\mathbf{i}\pi/\mathrm{q})\mathbf{j}, \exp(-\mathbf{i}s\pi/\mathrm{p})\mathbf{j}\big)$, we obtain $\big(\exp(\mathbf{i}\pi/\mathrm{q}), \exp(\mathbf{i}s\pi/\mathrm{p})\big)$ as generator of the index two unitary subgroup. If we impose $(s, 2p) = 1$, we obtain the

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group $(C_{2q}/C_{q_1}; C_{2p}/C_{p_1})_s$ where $d = (p, q)$ and we put $p_1 = p/d$, $q_1 = q/d$. Observe that

$$(C_{2q}/C_{q_1}; C_{2p}/C_{p_1})_s \subset (D_q/C_{q_1}; D_p/C_{p_1})_s.$$

It is an index two subgroup of the subgroup of $\widehat{U(2)}$ of order $2sp_1q_1$ (see [D, types 1 and 11, p. 57]). On the other hand, if we start with $\mathbf{r}_1 = (-\mathbf{j}, \mathbf{j})$ (with p and q odd), one obtains the groups $(C_{2q}/C_{2q_1}; C_{2p}/C_{2p_1})_s \subset (D_q/C_{2q_1}; D_p/C_{2p_1})_s$ (see [D, types $1'$ and $11'$]).

Example 4.2 **(Groups Generated by Three Inversions I)** Take $\mathbf{r}_1 = (\mathbf{j}, \mathbf{j})$, $\mathbf{r}_2 = \big(\exp(-\mathbf{i}\pi/\mathrm{q})\mathbf{j}, \mathbf{j}\big)$ and $\mathbf{r}_3 = \big(\mathbf{j}, \exp(-\mathbf{i}\pi/\mathrm{p})\mathbf{j}\big)$. In that case the group generated by these inversions contains, as an index two subgroup, the group $(C_{2q}/C_{2q}; C_{2p}/C_{2p})$. Here we don't suppose that $(p, q) = 1$ otherwise the unitary subgroup would be generated by $\mathbf{r}_2$ and $\mathbf{r}_3$. On the other hand, if we start with $\mathbf{r}_1 = (-\mathbf{j}, \mathbf{j})$ (with p and q odd), one obtains the group $(C_{2q}/C_q; C_{2p}/C_p)$.

Example 4.3 **(Groups Generated by Three Inversions II)** Take $\mathbf{r}_1 = (\mathbf{j}, \mathbf{j})$, $\mathbf{r}_2 = \big(\exp(-\mathbf{i}\pi/\mathrm{mr})\mathbf{j}, \exp(-\mathbf{i}s\pi/\mathrm{nr})\mathbf{j}\big)$ and $\mathbf{r}_3 = \big(\mathbf{j}, \exp(-\mathbf{i}\pi/\mathrm{n})\mathbf{j}\big)$, with $(s, r) = 1$. In that case the group generated by these inversions contains, as an index two subgroup, the group $(C_{2mr}/C_{2m}; C_{2nr}/C_{2n})_s$. Several of these groups, as in Example 4.1, can be generated with two inversions. The groups $(C_{2mr}/C_m; C_{2nr}/C_n)_s$ with n and m odd, are obtained using $\mathbf{r}_3 = \big(-\mathbf{j}, \exp(-\mathbf{i}\pi/\mathrm{n})\mathbf{j}\big)$.

Example 4.4 **(Groups Generated by Three Inversions III)** Three inversions are given by three pairs $(\epsilon_1\mathbf{j}, \mathbf{p}_1)$, $(\epsilon_2\mathbf{j}, \mathbf{p}_2)$, $(\epsilon_3\mathbf{j}, \mathbf{p}_3)$. The subgroup $\mathbf{L}$ is generated by $\mathbf{p}_1$, $\mathbf{p}_2$, $\mathbf{p}_3$. The subgroups of $\mathrm{SU}(2)$ generated by pure quaternions are enumerated above. As an example, we take $\langle 4, 3, 2\rangle$ a group generated by $\mathbf{p}_1 = \frac{1}{\sqrt{2}}(\mathbf{k} - \mathbf{j})$, $\mathbf{p}_2 = \frac{1}{\sqrt{2}}(\mathbf{i} - \mathbf{k})$, $\mathbf{p}_3 = \frac{1}{\sqrt{2}}(\mathbf{k} + \mathbf{j})$. In this case (see [C2, p. 79]) $a = \mathbf{p}_1\mathbf{p}_2$, $b = \mathbf{p}_2\mathbf{p}_3$ and $c = \mathbf{p}_3\mathbf{p}_1$ are generators of $\langle 3, 3, 2\rangle$. If we take $\epsilon_1 = \exp(\pi\mathbf{i}/m)$, $\epsilon_2 = 1$ and $\epsilon_3 = \exp(-\pi\mathbf{i}/m)$ we obtain an index two unitary subgroup

$$(C_{2m}/C_{2m}; \langle 3, 3, 2\rangle/\langle 3, 3, 2\rangle) \subset (D_m/C_{2m}; \langle 4, 3, 2\rangle/\langle 3, 2, 2\rangle)$$

of a subgroup of $\widehat{U(2)}$ of order $48m$ (see types 5 and 16 of [D]).

Theorem 4.1 *The finite subgroups of $\widehat{U(2)}$ generated by inversions are the following, where we indicate their index two subgroup in $U(2)$.*

1. - $(C_{2mr}/C_{2m}; C_{2nr}/C_{2n})_s \subset (D_{mr}/C_{2m}; D_{nr}/C_{2n})_s$, *of order* $4mns$ *(1 and 11 of [D]).*
   - $(C_{2mr}/C_m; C_{2nr}/C_n)_s \subset (D_{mr}/C_m; D_{nr}/C_n)_s$, *of order* $2mns$ *($1'$ and $11'$ of [D]).*
2. $(C_{2m}/C_{2m}; \langle 3, 3, 2\rangle/\langle 3, 3, 2\rangle) \subset (D_m/C_{2m}; \langle 4, 3, 2\rangle/\langle 3, 2, 2\rangle)$ *of order* $48m$ *(types 5 and 16 of [D]).*
3. $(C_{2m}/C_{2m}; \langle 4, 3, 2\rangle/\langle 4, 3, 2\rangle) \subset (D_m/D_m; \langle 4, 3, 2\rangle/\langle 4, 3, 2\rangle)$ *of order* $96m$ *(types 7 and 15 of [D]).*

4. $(C_{4m}/C_{2m}; \langle 4,3,2\rangle/\langle 3,3,2\rangle) \subset (D_{2m}/D_m; \langle 4,3,2\rangle/\langle 3,3,2\rangle)$ *of order* $96m$ *(types 8 and 17 of [D]).*
5. $(C_{2m}/C_{2m}; \langle 5,3,2\rangle/\langle 5,3,2\rangle) \subset (D_m/D_m; \langle 5,3,2\rangle/\langle 5,3,2\rangle)$ *of order* $240m$ *(types 9 and 19 of [D]).*

Proof The proof follows from the classification of [D] and by a case by case verification. Observe that, for a group generated by inversions, $C_{2m} \subset D_m$ is the subgroup of index two that should appear in the first component of Du Val's notation. This restricts the many possibilities in his list. Analogously, the second component should be generated by pure quaternions, excluding $\langle 3,3,2\rangle$.

We recall [C2] that $\langle 5,3,2\rangle$ and $\langle 4,3,2\rangle$ are generated by three pure quaternions U_1, U_2 and U_3 satisfying $A^p = B^3 = C^2 = ABC = -1$ ($p = 4,5$) with $A = U_1U_2$, $B = U_2U_3$, $C = U_3U_1$ also generating the groups. Define the following generators $\mu_1 = (\mathbf{j}, U_1)$, $\mu_2 = (-\mathbf{j}, U_2)$ and $\mu_3 = \big((\exp \mathbf{i}\pi/m)\mathbf{j}, U_3\big)$. We claim that $\langle \mu_1,\mu_2,\mu_3\rangle = D_m \times \langle p,3,2\rangle$; observe that if we call $a = \mu_1\mu_2$, $b = \mu_2\mu_3$ and $c = \mu_3\mu_1$, one obtains $c^2 = (\exp \mathbf{i}\pi/m, -1)$, $c^2b^2 = (1, B^{-1})$ and $a(c^2b^2)^{-1} = (1,A)(1,B) = (1,C)$. We conclude that $1 \times \langle p,3,2\rangle$ is in the group and therefore $\langle \mu_1,\mu_2,\mu_3\rangle = D_m \times \langle p,3,2\rangle$. This proves the cases 19 and 15. The case 17 is obtained considering the following generators: $\mu_1 = (-\mathbf{j},\mathbf{j})$, $\mu_2 = \big((\exp \mathbf{i}\pi/2m)\mathbf{j}, (\mathbf{k}-\mathbf{j})/\sqrt{2}\big)$ and $\mu_3 = \big((\exp \mathbf{i}\pi/2m)\mathbf{j}, (\mathbf{i}-\mathbf{k})/\sqrt{2}\big)$ in $(D_{2m}/D_m; \langle 4,3,2\rangle/\langle 3,3,2\rangle)$. By a computation, one shows that $(\mu_1\mu_2\mu_3)^3 = (\mathbf{j},1)$ and $(\mu_1\mu_3)^2 = (\exp \mathbf{i}\pi/m, -1)$. We conclude that $D_m \times 1 \subset \langle \mu_1,\mu_2,\mu_3\rangle$ and therefore the elements generate the whole group. The other cases were constructed in the examples above. Finally, verifying the orders of the remaining groups, we obtain only the cases above. The construction of each group belonging to those cases shows, in particular, that each group is generated by, at most, three inversions. ∎

Observe that the first class of groups (types 1 and 1′ in Du Vals' notation) correspond to type 1 of Coxeter's notation (see Theorem 3.1 above).

5 Configurations of Lagrangian Planes

5.1 S^3 and the Cayley Transform

We will consider the action of $\widehat{U(2)}$ on the sphere $S^3 \subset \mathbf{C}^2$. Recall that, in matrix notation, an element $x \to \epsilon x q \in U(2)$ corresponds to

$$\epsilon \begin{bmatrix} a & -\overline{c} \\ c & \overline{a} \end{bmatrix}$$

with $a = q_0 + q_1\mathbf{i}$ and $c = q_2 + q_3\mathbf{i}$ acting on (w_1, w_2) where $x = w_1 + w_2\mathbf{j}$.

Each inversion fixes a totally real plane in $\mathbf{C}^2$ called an $\mathbf{R}$-plane or Lagrangian plane, which intersects S^3 along a circle called $\mathbf{R}$-circle. The intersection of S^3 with a complex plane is called a $\mathbf{C}$-circle. One may verify the following proposition which gives a redundant representation, but which is sufficient for our calculational purposes.

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Proposition 5.1 *In quaternionic coordinates,* $\mathbf{R}$*-circles are given by* $\epsilon \exp(\mathbf{j}\theta)\mathbf{q}$ *and* $\mathbf{C}$*-circles by* $\exp(\mathbf{i}\theta)\mathbf{q}$*, where* $\epsilon \in U(1) \subset \mathbf{C}$*,* $\mathbf{q} \in \mathrm{SU}(2)$.

Proof The proposition follows from the observation that the action of $U(2)$ is transitive on $\mathbf{R}$-circles and $\mathbf{C}$-circles. It suffices, then, to describe one in each class. Clearly $\exp(\mathbf{i}\theta)$ is a $\mathbf{C}$-circle in quaternionic coordinates. Also it is easy to see that $\exp(\mathbf{j}\theta)$ describes an $\mathbf{R}$-circle. ∎

The mapping

$$C\colon (w_1, w_2) \mapsto \left(z_1 = \frac{iw_1}{1+w_2}, z_2 = i\frac{1-w_2}{1+w_2} \right)$$

is usually referred to as the *Cayley transform*. It maps the unit ball

$$B = \{w \in \mathbf{C}^2 : |w_1|^2 + |w_2|^2 < 1\}$$

biholomorphically onto

$$V = \{z \in \mathbf{C}^2 : \mathrm{Im}(z_2) > |z_1|^2\}.$$

The Cayley transform leads to a generalized form of the *stereographic projection*. This mapping $\pi\colon S^3 \setminus \{-e_2\} \to \mathbf{R}^3$, where $S^3 = \partial B$ and $e_2 = (0,1) \in \mathbf{C}^2$, is defined as the composition of the Cayley transform restricted to $S^3 \setminus \{-e_2\}$ followed by the projection

$$(z_1, z_2) \mapsto \big(z_1, \mathrm{Re}(z_2)\big).$$

The stereographic projection π can be extended to a one to one mapping from S^3 onto the one-point compactification $\overline{\mathbf{R}}^3$ of $\mathbf{R}^3 = \{(z,t) \mid z \in \mathbf{C}, t \in \mathbf{R}\} = H$. Its inverse function is given by

$$\pi^{-1}(z,t) = \left(\frac{-2iz}{1+|z|^2 - it}, \frac{1-|z|^2+it}{1+|z|^2-it} \right).$$

Observe that the x-axis in $\mathbf{R}^3$ corresponds to the intersection of S^3 with the real plane $\mathrm{Re}(w_1) = 0$, $\mathrm{Im}(w_2) = 0$. Also, the y-axis corresponds to the intersection of S^3 with the real plane $\mathrm{Im}(w_1) = 0$, $\mathrm{Im}(w_2) = 0$.

5.2 Inversions

The *antipodal map* of H is defined on $H \setminus \{(0,0)\}$ by

$$\hat{s}\colon (z,t) \mapsto \left(\frac{-z}{|z|^2 - it}, -\frac{t}{|z|^4 + t^2} \right).$$

Note that $\hat{s} = \pi \circ s \circ \pi^{-1}$, where s is the involution

$$s\colon (w_1, w_2) \mapsto (-w_1, -w_2), \quad (w_1, w_2) \in \mathbf{C}^2.$$

It corresponds to the map $x \to \exp(i\pi)x1$.

The map $\hat{\jmath}$ defined by

$$\hat{\jmath}\colon (z,t) \mapsto (-\overline{z}, -t),$$

corresponds to

$$\pi^{-1} \circ \hat{\jmath} \circ \pi(w_1, w_2) = (\overline{w}_1, \overline{w}_2).$$

It corresponds to the map $x \to \exp(i\pi)\mathbf{j}x\mathbf{j} = -\mathbf{j}x\mathbf{j}$.

Also, define

$$I_0\colon (z,t) \mapsto \left(\frac{-\overline{z}}{|z|^2 + it}, \frac{t}{|z|^4 + t^2}\right)$$

which corresponds to

$$\pi^{-1} \circ I_0 \circ \pi(w_1, w_2) = (\overline{w}_1, -\overline{w}_2).$$

I_0 leaves pointwise fixed the standard $\mathbf{R}$-circle $\mathbf{R}_0$ (see [G] for details)

$$r^2 + it = -e^{-2i\theta}$$

where $z = re^{i\theta}$. In cylindrical coordinates $\mathbf{R}_0$ is given by

$$r = \sqrt{-\cos(2\theta)}, \quad t = \sin(2\theta).$$

In quaternionic coordinates, this $\mathbf{R}$-circle is parametrized by $\alpha \to \exp(\alpha\mathbf{k})$. Using this parametrization, after the Cayley transform, one obtains the following expression for $\mathbf{R}_0$

$$\left(\frac{\sin(2\alpha)}{2\bigl(1+\sin^2(\alpha)\bigr)}, \frac{\cos(\alpha)}{1+\sin^2(\alpha)}, \frac{2\sin(\alpha)}{1+\sin^2(\alpha)}\right).$$

Observe that the relation between the two parametrizations is $\sin(\alpha) = 1/\tan(\theta)$.

Proposition 5.2 *The transformation $\epsilon\mathbf{j}x\mathbf{j}q$ is an inversion on the image of the $\mathbf{R}$-circle fixed by $\mathbf{j}x\mathbf{j}$, by the transformation $\epsilon_1 x q_1$, where $\epsilon_1^2 = \epsilon$ and $q_1^2 = q$.*

Proof Recall that by Proposition 4.2 $x \to \epsilon\mathbf{j}x\mathbf{j}q \in \overline{U}(2)$ is an inversion if and only if $q = q_0 + q_1\mathbf{i} + q_3\mathbf{k} \in \mathbf{Q}$, that is, the $\mathbf{j}$-component of q is 0. Using the matrix notation, an element $x \to \epsilon\mathbf{j}x\mathbf{j}q \in \overline{U}(2)$ corresponds to

$$\epsilon \begin{bmatrix} a & -\overline{c} \\ c & \overline{a} \end{bmatrix}$$

with $a = q_0 + q_1\mathbf{i}$ and $c = q_3\mathbf{i}$ acting on $(-\overline{w}_1, -\overline{w}_2)$ where $x = w_1 + w_2\mathbf{j}$. Observe now that, if $\epsilon_1^2 = \epsilon$ and $q_1^2 = q$, $\epsilon\mathbf{j}x\mathbf{j}q = \epsilon_1\mathbf{j}\overline{\epsilon}_1\mathbf{j}\mathbf{j}x\mathbf{j}\mathbf{j}\overline{q}_1\mathbf{j}q_1$, where we used the fact that $\overline{q} = \tilde{q}$ if the $\mathbf{j}$-component of q is missing. So we conclude that the transformation is conjugated to $\mathbf{j}x\mathbf{j}$ by $\epsilon_1 x q_1$. ∎

Observe that the fixed $\mathbf{R}$-circle by $\mathbf{j}x\mathbf{j}$ is, in quaternionic coordinates, $\mathbf{i}\exp(\mathbf{j}\alpha)$.

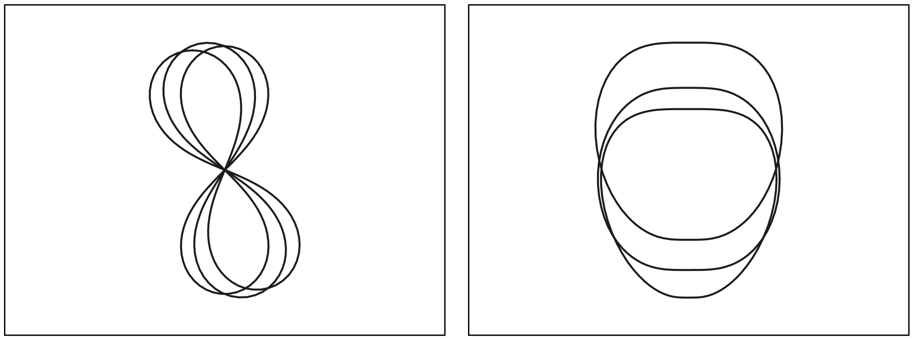

Figure 1: Views of the configuration with three generators fixing the vertical axis with $q = 4$, $p = 2$.

Remark 5.1 The map $x \to \epsilon x a$, with $a \in \mathbf{C}$, represented by

$$R_\epsilon = \begin{bmatrix} \epsilon a & 0 \\ 0 & \epsilon \overline{a} \end{bmatrix}$$

acting on (w_1, w_2) where $x = w_1 + w_2\mathbf{j}$ transforms an $\mathbf{R}$-circle with center on the vertical axis to another one with center on the vertical axis. In fact those transformations form the group fixing setwise the complex disc $w_1 = 0$ whose intersection with S^3 is precisely the vertical axis after applying the Cayley transform. This is the case in the following three examples.

Example 5.1 **(Two Generator Groups)** If we take the pair $\mathbf{r}_1 = (\mathbf{j}, \mathbf{j})$ and $\mathbf{r}_2 = \big(\exp(-\mathbf{i}\pi/\mathrm{q})\mathbf{j}, \exp(-\mathbf{i}s\pi/\mathrm{p})\mathbf{j}\big)$ as in 4.1, we obtain $(\epsilon_1, q_1) = (1, 1)$ and $(\epsilon_2, q_2) = \big(\exp(\mathbf{i}\pi/2\mathrm{q}), \exp(\mathbf{i}s\pi/2\mathrm{p})\big)$.

Example 5.2 **(Three Generator Groups I)** As in 4.2 we take $\mathbf{r}_1 = (\mathbf{j}, \mathbf{j})$, $\mathbf{r}_2 = \big(\exp(-\mathbf{i}\pi/\mathrm{q})\mathbf{j}, \mathbf{j}\big)$ and $\mathbf{r}_3 = \big(\mathbf{j}, \exp(-\mathbf{i}\pi/\mathrm{p})\mathbf{j}\big)$. In that case we obtain $(\epsilon_1, q_1) = (1, 1)$, $(\epsilon_2, q_2) = \big(\exp(\mathbf{i}\pi/2\mathrm{q}), 1\big)$ and $(\epsilon_3, q_3) = \big(1, \exp(\mathbf{i}\pi/2\mathrm{p})\big)$. See Figure 1 where a configuration of three $\mathbf{R}$-circles is drawn.

Example 5.3 **(Three Generator Groups III)** From Example 4.4 we have $\mathbf{p}_1 = \frac{1}{\sqrt{2}}(\mathbf{k} - \mathbf{j}) = \mathbf{j}\exp(-3\mathbf{i}\pi/4)$, $\mathbf{p}_2 = \frac{1}{\sqrt{2}}(\mathbf{i} - \mathbf{k}) = \mathbf{j}\Big(\exp\big(\frac{1}{\sqrt{2}}(\mathbf{k} + \mathbf{i})\pi/2\big)\Big) = \mathbf{j}\big(\frac{1}{\sqrt{2}}(\mathbf{k} + \mathbf{i})\big)$, $\mathbf{p}_3 = \frac{1}{\sqrt{2}}(\mathbf{k} + \mathbf{j}) = \mathbf{j}\exp(-\mathbf{i}\pi/4)$ and $\epsilon_1 = \exp(\pi\mathbf{i}/m)$, $\epsilon_2 = 1$, $\epsilon_3 = \exp(-\pi\mathbf{i}/m)$. So we obtain $\mathbf{q}_i$ and η_i such that $\mathbf{q}_i^2 = -\mathbf{j}\mathbf{p}_i$ and $\eta_i^2 = \epsilon_i$ to be $\mathbf{q}_1 = \exp(-3\mathbf{i}\pi/8)$, $\mathbf{q}_2 = \exp\big(\frac{1}{\sqrt{2}}(\mathbf{k} + \mathbf{i})\pi/4\big) = \frac{1}{\sqrt{2}} + \frac{\mathbf{i}}{2} + \frac{\mathbf{k}}{2}$, $\mathbf{q}_3 = \exp(-\mathbf{i}\pi/8)$ and $\eta_1 = \exp(\pi\mathbf{i}/2m)$, $\eta_2 = 1$, $\eta_3 = \exp(-\pi\mathbf{i}/2m)$. The $\mathbf{R}$-circles obtained using those generators are represented in Figures 2, 3, and for $m = 1$ and $m = 2$, where the standard $\mathbf{R}$-circle rotated by $\pi/2$ is also represented in dotted lines.

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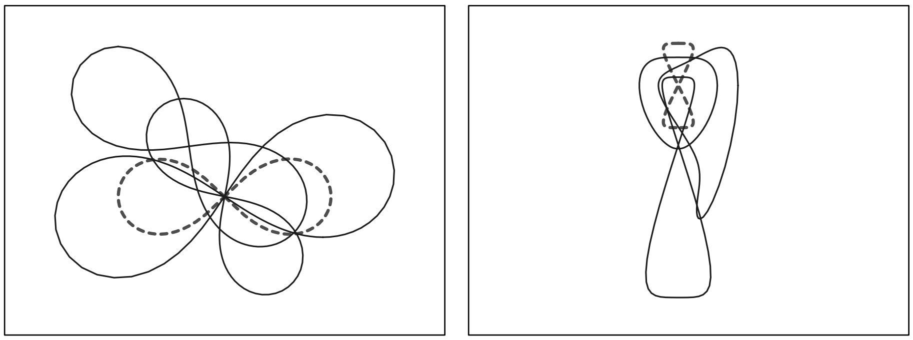

Figure 2: Two views of the configuration with three generators $m = 1$.

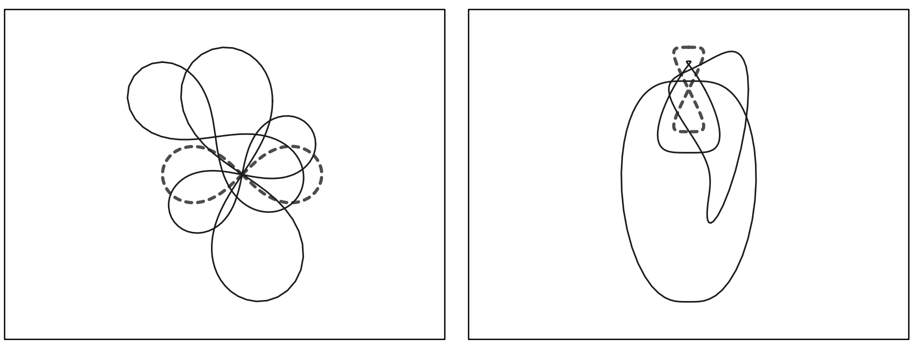

Figure 3: Two views of the configuration with three generators $m = 2$.

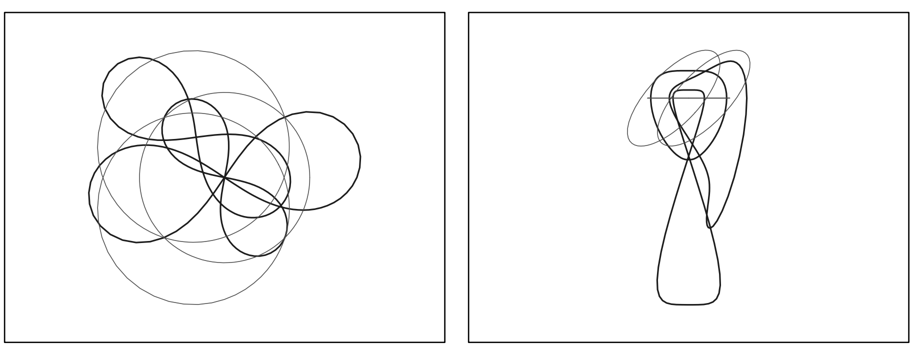

Figure 4: Configuration of three Lagrangian planes showing invariant **C**-circles: two views.

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Example 5.4 **(Three Generator Groups III. Invariant C-Circles)** The invariant **C**-circles associated to an element of $U(2)$ are the intersections of the eigenspaces of that element with S^3. For a transformation $x \to \epsilon x\mathbf{p}$, the eigenspace is completely determined by the transformation $x \to x\mathbf{p}$.

We determine now for each element $\mathbf{g}_1 = \mathbf{p}_2 \circ \mathbf{p}_3$, $\mathbf{g}_2 = \mathbf{p}_3 \circ \mathbf{p}_1$ and $\mathbf{g}_3 = \mathbf{p}_1 \circ \mathbf{p}_2$ the invariant **C**-circles. From a calculation (*cf.* [C2, p. 79]) we get

$$\mathbf{g}_1 = \mathbf{p}_2 \circ \mathbf{p}_3 = \mathbf{j} \exp\left(\frac{1}{\sqrt{2}}(\mathbf{k}+\mathbf{i})\pi/2\right) \mathbf{j} \exp(-\mathbf{i}\pi/4) = 1/2 + \mathbf{i}/2 - \mathbf{j}/2 + \mathbf{k}/2$$

$$\mathbf{g}_2 = \mathbf{p}_3 \circ \mathbf{p}_1 = \mathbf{j} \exp(-\mathbf{i}\pi/4) \mathbf{j} \exp(-3\mathbf{i}\pi/4) = \mathbf{i}$$

$$\mathbf{g}_3 = \mathbf{p}_1 \circ \mathbf{p}_2 = \mathbf{j} \exp(-3\mathbf{i}\pi/4) \mathbf{j} \frac{1}{\sqrt{2}}(\mathbf{k}+\mathbf{i}) = 1/2 + \mathbf{i}/2 + \mathbf{j}/2 + \mathbf{k}/2.$$

To find the invariant **C**-circles we find the eigenvalues of

$$\begin{bmatrix} a & -\bar{c} \\ c & \bar{a} \end{bmatrix}$$

with $a = q_0 + q_1\mathbf{i}$ and $c = q_2 + q_3\mathbf{i}$, where $\mathbf{q} = q_0 + q_1\mathbf{i} + q_2\mathbf{j} + q_3\mathbf{k}$. We find the eigenvalues for $\mathbf{g}_1$, $\mathbf{g}_2$, $\mathbf{g}_3$, respectively, $\exp(\mathbf{i}\pi/3)$, $\mathbf{i}$, $\exp(\mathbf{i}\pi/3)$ and their complex conjugates:

- for $\mathbf{g}_1$ we find eigenvectors $\frac{1}{\sqrt{6\pm 2\sqrt{3}}}(1 \pm \sqrt{3} + \mathbf{j} + \mathbf{k})$, so the **C**-circles are given by $\frac{\exp \mathbf{i}\theta}{\sqrt{6\pm 2\sqrt{3}}}(1 \pm \sqrt{3} + \mathbf{j} + \mathbf{k})$
- for $\mathbf{g}_2$ we find eigenvectors 1 and $\mathbf{j}$ having invariant **C**-circles $\exp \mathbf{i}\theta$ and $\exp \mathbf{i}\theta\mathbf{j}$
- for $\mathbf{g}_3$ the eigenvectors are $\frac{1}{\sqrt{6\pm 2\sqrt{3}}}(1 \pm \sqrt{3} + \mathbf{j} - \mathbf{k})$ having invariant **C**-circles $\frac{\exp \mathbf{i}\theta}{\sqrt{6\pm 2\sqrt{3}}}(1 \pm \sqrt{3} + \mathbf{j} - \mathbf{k})$.

In Figure 4 we see the invariant **C**-circles above corresponding to the eigenvalues $\exp(\mathbf{i}\pi/3)$, $\mathbf{i}$, $\exp(\mathbf{i}\pi/3)$ in the configuration for $m = 1$. Observe that the **C**-circles do not depend on m.

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