BOOLEAN NEAR-RINGS

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In this paper we introduce the concept of Boolean near-rings. Using any Boolean ring with identity, we construct a class of Boolean near-rings, called special, and determine left ideals, ideals, factor near-rings which are Boolean rings, isomorphism classes, and ideals which are near-ring direct summands for these special Boolean near-rings.

Blackett [6] discusses the near-ring of affine transformations on a vector space where the near-ring has a unique maximal ideal. Gonshor [10] defines abstract affine near-rings and completely determines the lattice of ideals for these near-rings. The near-ring of differentiable transformations is seen to be simple in [7]. For near-rings with geometric interpretations, see [1] or [2].

1. Preliminaries. A near-ring is a triple $(N,+,\cdot)$ where $(N,+)$ is a group, $(N,\cdot)$ a semi-group, and $a\cdot(b+c) = (a\cdot b)+(a\cdot c)$, for all $a,b,c \in N$. A normal subgroup $(I,+)$ of $(N,+)$ is a left ideal if $NI \subseteq I$ and is an ideal if $(I,+,\cdot)$ is the kernel of a near-ring homomorphism [5]. For notation and terminology with respect to Boolean rings one might see [11].

It is not clear exactly what a Boolean near-ring should be. The following definition might serve as a first approximation.

Definition 1.1. A near-ring $(B,+,\cdot)$ is Boolean if there exists a Boolean ring $(B,+,\wedge,1)$ with identity such that $\cdot$ is defined in terms of $+$, $\wedge$, and 1 and, for any $b \in B$, $b\cdot b = b$.

A near-ring $(N,+,\cdot)$ is said to be idempotent if $x^2 = x$ for all $x \in N$. If $(R,+,\cdot)$ is an idempotent ring, then for all $a,b \in R$, $a+a = 0$ and $a\cdot b = b\cdot a$. The following example shows that this is not the case for all idempotent near-rings and emphasizes the role of the distributive laws for ring theory:

Example 1.2. Given a non-trivial group $(N,+)$, define multiplication by

$$a \cdot b = b, \quad \text{for all } a, b \in N .$$

Then, $(N,+,\cdot)$ is an idempotent near-ring for which $\cdot$ is not commutative and for which $(N,+)$ need not be of characteristic two. For other examples of idempotent near-rings that need not be Boolean, see [8].

The following theorem gives examples of Boolean near-rings that are not Boolean rings. First recall that in a Boolean ring $(B,+,\wedge,1)$ one can define complementation, $'$, by $a' = a + 1$, and sup, $\vee$, by $a \vee b = (a' \wedge b')'$.

THEOREM 1.3. Let $(B,+,\wedge,1)$ be a Boolean ring with identity. Fix $x \in B$ and define a multiplication on B by $a \cdot b = (a \vee x) \wedge b$. Then $(B,+,\cdot)$ is a Boolean near-ring which is a Boolean ring if and only if $x = 0$.

Proof. For $a, b, c \in B$ we have $a \cdot (b \cdot c) = (a \vee x) \wedge [(b \vee x) \wedge c]$ and $(a \cdot b) \cdot c = \{[(a \vee x) \wedge b] \vee x\} \wedge c = [(a \vee x) \wedge (b \vee x)] \wedge c$, so that $a \cdot (b \cdot c) = (a \cdot b) \cdot c$. Also, $a \cdot (b + c) = (a \vee x) \wedge (b + c) = (a \vee x) \wedge b + (a \vee x) \wedge c$ $= (a \cdot b) + (a \cdot c)$. If $x = 0$, $(B,+,\cdot) = (B,+,\wedge,1)$. Now, $(x + x) \cdot x = 0 \cdot x$ $= (0 \vee x) \wedge x = x$ and $(x \cdot x) + (x \cdot x) = 0$, so that $(B,+,\cdot)$ is not a ring if $x \neq 0$. Also, $b \cdot b = (b \vee x) \wedge b = b$, for all $b \in B$, so that $(B,+,\cdot)$ is a Boolean near-ring.

Note that any e such that $e \wedge x' = x'$ is a left identity for $(B,+,\cdot)$ but $(B,+,\cdot)$ has no right identity unless $x = 0$.

Boolean near-rings of the type defined in the above theorem will be called special and the remainder of this paper will be devoted to discussing topics for special Boolean near-rings as listed in the introduction.

2. Left ideals of special Boolean near-rings.

Definition 2.1. For a special Boolean near-ring $(B,+,\cdot)$ and $t \in B$ define $P(t) = \{a \in B \mid a \wedge t = a\}$. If $S \subseteq B$ and $t \in B$, define $S(t) = \{s \wedge t \mid s \in S\}$. (Note that $S(t) \subseteq P(t)$.)

Recall from [3] that the additive group $(N,+)$ of a near-ring can be written as a group direct sum of the additive group of its maximal sub-Z-ring, $N_Z = \{b \in N \mid a \cdot b = b, \text{ for all } a \in N\}$, and the additive group of its maximal sub-C-ring, $N_C = \{b \in N \mid 0 \cdot b = 0\}$.

PROPOSITION 2.2. Let $(B,+,\cdot)$ be a special Boolean near-ring. The maximal sub-Z-ring of B is $B_Z = \{b \mid b \wedge x = b\} = P(x)$ and the maximal sub-C-ring of B is $B_C = \{b \mid b \wedge x = 0\} = P(x')$.

Proof. The proof is direct so will be omitted.

PROPOSITION 2.3. Let $(B,+,\cdot)$ be a special Boolean near-ring and let L be a left ideal of B. Then $L = L(x) \oplus L(x')$, a direct sum of left ideals, where, if $a_1, b_1 \in L(x)$ and $a_2, b_2 \in L(x')$, then

$(a_1 + b_1)\cdot(a_2 + b_2) = (a_1 \cdot a_2) + (b_1 \cdot b_2)$. Conversely, if $M \subseteq P(x)$ and $N \subseteq P(x')$ are left ideals of B, then their direct sum, $M \oplus N$, is a left ideal of B.

Proof. First we show that $L(x) \subseteq L$ and $L(x') \subseteq L$. Now $x \wedge a = x \cdot a \in L$ for $a \in L$, so that $L(x) \subseteq L$. Also $x' \wedge a = (1 + x) \wedge a = a + x \wedge a \in L$ if $a \in L$, so that $L(x') \subseteq L$.

Now we wish to show that $L(x)$ and $L(x')$ are left ideals of B. $L(x) \subseteq P(x) = B_Z = \{ b \in B \mid a \cdot b = b, \text{ for all } a \in B \}$, so that $BL(x) \subseteq L(x)$. Let $s \wedge x,\ t \wedge x \in L(x)$. Then, $s \wedge x + t \wedge x = (s+t) \wedge x \in L(x)$. Thus $L(x)$ is a left ideal of B. Let $s \wedge x' \in L(x')$ and $b \in B$. Then, $b \cdot (s \wedge x') = [b \wedge (s \wedge x')] \vee [x \wedge (s \wedge x')] = (b \wedge s) \wedge x' \in L(x')$. Thus, $L(x')$ is a left ideal of B.

Certainly $L = L(x) + L(x')$ and $L(x) \cap L(x') = \{0\}$.

Let $a_1, b_1 \in L(x)$ and $a_2, b_2 \in L(x')$. Then, noting that $a_1 \wedge a_2' = a_1, a_1' \wedge a_2 = a_2, b_1 \wedge b_2' = b_1$, and $b_1' \wedge b_2 = b_2$, we have:

$$\begin{aligned}
(a_1 + a_2)\cdot(b_1 + b_2) &= [(a_1 \wedge a_2') \vee (a_1' \wedge a_2)] \cdot [(b_1 \wedge b_2') \vee (b_1' \wedge b_2)] \\
&= [a_1 \vee a_2 \vee x] \wedge [b_1 \vee b_2] = (a_1 \wedge b_1) \vee (x \wedge b_1) \vee (a_2 \wedge b_2) \\
&= (a_1 \wedge b_1) \vee b_1 \vee (a_2 \wedge b_2) = b_1 \vee (a_2 \wedge b_2) \\
&= b_1 \vee (a_2 \wedge b_2) \vee (x \wedge b_2) = b_1 \vee [(a_2 \vee x) \wedge b_2] \\
&= \{ b_1 \wedge [(a_2' \wedge x') \vee b_2'] \} \vee \{ b_1' \wedge [(a_2 \vee x) \wedge b_2] \\
&= b_1 + [(a_2 \vee x) \wedge b_2] = [(a_1 \vee x) \wedge b_1] + [(a_2 \vee x) \wedge b_2] \\
&= (a_1 \cdot b_1) + (a_2 \cdot b_2).
\end{aligned}$$

The proof of the converse follows by the left distributive law.

Proposition 2.3 reduces the problem of finding the left ideals of $(B,+,\cdot)$ to finding the left ideals $M \subseteq P(x)$ and the left ideals $N \subseteq P(x')$.

PROPOSITION 2.4. The left ideals of $(B,+,\cdot)$ in $P(x)$ are just the subgroups of the group $(P(x),+)$.

Proof. If L is a left ideal of B and $L \subseteq P(x)$, then certainly $(L,+)$ is a subgroup of $(P(x),+)$. Conversely, suppose $(L,+)$ is a subgroup of $(P(x),+)$ and suppose $a \in L$. Then, for $b \in B$, $b \cdot a = a$, by

Proposition 2.2. Hence $BL \subseteq L$ so that L is a left ideal of B.

LEMMA 2.5. *If* L *is a left ideal of* $(B,+,\cdot)$ *with* $L \subseteq P(x')$ *then, for any* $a \in L$, $P(a) \subseteq L$.

Proof. We have $b \cdot a \in L$, for all $b \in B$. Hence $b \cdot a = (b \vee x) \wedge a = b \wedge a \in L$, for all $b \in B$, so that $P(a) \subseteq L$.

COROLLARY 2.6. *If* $a \in P(x')$, *then* $P(a)$ *is a left ideal of* B.

Proof. If $t \in P(a)$, then $b \cdot t = b \wedge t \in P(a)$, for any $b \in B$.

PROPOSITION 2.7. *If* $\{L_i\}_{i \in I}$ *is a family of left ideals of* $(B,+,\cdot)$, *then* $L = \sum_{i \in I} L_i$ *is also a left ideal of* B.

Proof. Let $b \in B$ and $a_i \in L_i$. Then $b \cdot \sum_{i=1}^{n} a_i = \sum_{i=1}^{n} b \cdot a_i \in L$, since each $b \cdot a_i \in L_i$.

Definition 2.8. By a *filter* F in $(B,+,\cdot)$ we mean a filter in $(B,+,\wedge,1)$. The *dual* of a filter F is $F' = \{a' \mid a \in F\}$.

PROPOSITION 2.9. *If* F *is a filter in* $(B,+,\cdot)$ *and if* $F' \subseteq P(x')$, *then* F' *is a left ideal of* $(B,+,\cdot)$. *Conversely, if* $L \subseteq P(x')$ *is a left ideal of* $(B,+,\cdot)$, *then* $L' = \{a' \mid a \in L\}$ *is a filter in* $(B,+,\cdot)$.

Proof. This follows immediately from the observation that $a \in P(x')$ implies $b \cdot a = b \wedge a$ for all $b \in B$ and the fact that duals of filters in $(B,+,\wedge,1)$ are equivalent to ideals in $(B,+,\wedge,1)$.

To summarize Propositions 2.3, 2.4, and 2.9, we have the following theorem.

THEOREM 2.10. *Let* $(B,+,\cdot)$ *be a special Boolean near-ring with multiplication determined by* x. *Then,* L *is a left ideal of* $(B,+,\cdot)$ *if and only if there is a subgroup* $M \subseteq P(x)$ *and a set* $N \subseteq P(x')$ *such that* N' *is a filter in* $(B,+,\cdot)$ *and* $L = M \oplus N$.

3. *Ideals of special Boolean near-rings.* Again, in this section, $(B,+,\cdot)$ will denote a special Boolean near-ring with multiplication determined by x. The ideals of a near-ring $(N,+,\cdot)$ are just kernels of near-ring homomorphisms. We state here, without proof, a theorem due to Blackett [5] that characterizes all the ideals of a near-ring.

THEOREM 3.1. *The ideals of a near-ring* $(N,+,\cdot)$ *are just the normal subgroups* $(T,+)$ *of* $(N,+)$ *such that*

(a) $NT \subseteq T$ (i.e. T *is a left ideal*), *and*

(b) $(n+t)m - nm \in T$ *if* $n, m \in N$ *and* $t \in T$.

LEMMA 3.2. *If* $k \in P(x)$ *and* $a, b \in B$, *then* $(a+k)\cdot b + a\cdot b = 0$.

Proof. Suppose $k \in P(x)$. Then $x' \in P(k')$ so that $k' \vee x = 1$. Now $(a+k) = (a \wedge k') \vee (a' \wedge k)$ so that $(a+k) \vee x = (a \wedge k') \vee (a' \wedge k) \vee x$ $= [(a \vee x) \wedge (k' \vee x)] \vee [(a' \vee x) \wedge (k \vee x)] = (a \vee x) \vee [(a' \wedge x) \vee x] = (a \vee x)$. Thus, $(a+k)\cdot b + a\cdot b = a\cdot b + a\cdot b = 0$.

COROLLARY 3.3. *The ideals* $I \subseteq P(x)$ of $(B, +, \cdot)$ *are exactly the left ideals* $I \subseteq P(x)$ of $(B, +, \cdot)$.

LEMMA 3.4. *If* $k \in P(x')$ *and* $a, b \in B$, *then* $(a+k)\cdot b + a\cdot b = k \wedge b$.

Proof. Using $k \wedge x' = k$, $k' \wedge x = x$, and $k \wedge x = 0$, we have:

$$\begin{aligned}(a+k)\cdot b + a\cdot b &= \{[(a \wedge k') \vee (a' \wedge k)] \vee x + (a \vee x)\} \wedge b\\ &= [\{[x \vee (a \wedge k') \vee (k \wedge a')] \wedge (a' \wedge x')\} \vee \{x' \wedge (a \wedge k')' \wedge\\ &\quad (k \wedge a')' \wedge (a \vee x)\}] \wedge b\\ &= [(k \wedge a' \wedge x') \vee \{[(x' \wedge a') \vee (x' \wedge k)] \wedge [(k' \wedge x) \vee a]\}] \wedge b\\ &= [(k \wedge a') \vee \{[(x \vee a)' \vee k] \wedge (x \vee a)\}] \wedge b = [(k \wedge a') \vee (k \wedge a)] \wedge b\\ &= k \wedge b.\end{aligned}$$

COROLLARY 3.5. *The ideals* $I \subseteq P(x')$ *of* $(B, +, \cdot)$ *are exactly the left ideals* $I \subseteq P(x')$ *of* $(B, +, \cdot)$.

THEOREM 3.6. *The ideals of* $(B, +, \cdot)$ *are exactly the left ideals of* $(B, +, \cdot)$.

Proof. That the sum of two ideals is an ideal follows from $(n+a+b)\cdot m - n\cdot m = (n+a+b)\cdot m - (n+a)\cdot m + (n+a)\cdot m - n\cdot m$. Now the proof is an immediate consequence of Corollaries 3.3 and 3.5 and Proposition 2.3.

4. *Ideals* I *such that* B/I *is a Boolean ring*. If I is a proper ideal of the near-ring N of affine transformations of a vector space, then N/I is isomorphic to a ring of linear transformations of that vector space [13]. In this section we determine which ideals I of a special Boolean near-ring $(B, +, \cdot)$ have the property that B/I is a Boolean ring. The following lemma is crucial.

LEMMA 4.1. *If* $a, b, c \in B$, *then* $(a+b)\cdot c + a\cdot c + b\cdot c = x \wedge c$.

Proof. $(a+b)\cdot c + a\cdot c + b\cdot c = \{[(a+b) \vee x] + (a \vee x) + (b \vee x)\} \wedge c$. Now $(a+b) \vee x + (a \vee x) + (b \vee x)$

$= (a+b) \vee x + \{[(a \vee x) \wedge b' \wedge x'] \vee [a' \wedge x' \wedge (b \vee x)]\}$
$= (a+b) \vee x + \{[(a \wedge b') \vee (a' \wedge b)] \wedge x'\} = (a+b) \vee x + (a+b) \wedge x'$
$= \{[(a+b) \vee x] \wedge [(a+b) \wedge x']'\} \vee \{[(a+b) \vee x]' \wedge [(a+b) \wedge x']\}$
$= \{[(a+b) \wedge (a+b)'] \vee x\} \vee \{(a+b)' \wedge x' \wedge (a+b)\} = (0 \vee x) \vee (0 \wedge x')$
$= x.$

Hence, $(a+b) \cdot c + a \cdot c + b \cdot c = x \wedge c.$

THEOREM 4.2. Let I be an ideal of $(B, +, \cdot)$. Then B/I is a Boolean ring if and only if $P(x) \subseteq I$.

Proof. Suppose B/I is a Boolean ring. Then the right distributive law holds so that

(*) $[(a+I) + (b+I)](c+I) = (a+I)(c+I) + (b+I)(c+I).$

Thus, $(a+b) \cdot c + I = (a \cdot c + b \cdot c) + I$. Hence, $(a+b) \cdot c + a \cdot c + b \cdot c = x \wedge c \in I$, by Lemma 4.1. Since c is arbitrary, we have $P(x) \subseteq I$. Conversely, if $P(x) \subseteq I$, then equation (*) is valid if and only if $(a+b) \cdot c + a \cdot c + b \cdot c \in I$. But $(a+b) \cdot c + a \cdot c + b \cdot c \in I$, by Lemma 4.1.

5. Isomorphism of special Boolean near-rings. Using Theorem 1.3 we can construct, from an arbitrary Boolean ring, β Boolean near-rings where β is the cardinality of the underlying set. Which of these Boolean near-rings are isomorphic? We answer this question in the following theorem.

THEOREM 5.1 Let $(B, +, \wedge, 1)$ be a Boolean ring with identity. Let $x, y \in B$ define special Boolean near-rings $(B, +, \cdot_x)$ and $(B, +, \cdot_y)$ as in Theorem 1.3, respectively. Then the following are equivalent:

(a) $(B, +, \cdot_x)$ is isomorphic to $(B, +, \cdot_y)$;
(b) $P(x')$ is isomorphic to $P(y')$ as subrings of $(B, +, \wedge, 1)$;
(c) $P(x)$ is isomorphic to $P(y)$ as subrings of $(B, +, \wedge, 1)$;
(d) there exists an automorphism α of $(B, +, \wedge, 1)$ such that $\alpha(x) = y$.

Proof. (a) $\to$ (b). First note, using Proposition 2.2, that $P(x)$ $P(x')$ are, respectively, the maximal sub-Z-ring and maximal sub-C-ring of $(B, +, \cdot_x)$. A similar statement holds for $P(y)$, $P(y')$ of $(B, +, \cdot_y)$. Since $(B, +, \cdot_x)$ has an ideal decomposition $P(x) \oplus P(x')$ and $(B, +, \cdot_y)$ has an ideal decomposition $P(y) \oplus P(y')$ (Proposition 2.3 and Theorem 3.6), any isomorphism of $(B, +, \cdot_x)$ onto $(B, +, \cdot_y)$ can be restricted to an isomorphism of $(P(x'), +, \cdot_x)$ onto $(P(y'), +, \cdot_y)$, since isomorphisms of near-rings take maximal sub-C-rings onto maximal sub-C-rings. (This last statement follows from [12, Proposition 1].) If $a, b \in P(x')$, then $a \cdot_x b = (a \vee x) \wedge b = (a \wedge b) \vee (x \wedge b) = a \wedge b$, so multiplication $\cdot_x$ in $P(x')$ is identical to that in $(B, +, \wedge, 1)$. Similarly, multiplication $\cdot_y$ in $P(y')$ is identical to that in $(B, +, \wedge, 1)$. Hence $P(x')$ is isomorphic to $P(y')$ as subrings of $(B, +, \wedge, 1)$.

(b) $\rightarrow$ (c). Now $P(x)$, $P(x')$, $P(y)$, $P(y')$ are all ideals in the ring $(B, +, \wedge, 1)$ and we have $B = P(x') \oplus P(x) = P(y') \oplus P(y)$, hence $P(y') \oplus P(x) \cong P(x') \oplus P(x) = P(y') \oplus P(y)$. Consequently $P(x) \cong P(y)$.

(c) $\rightarrow$ (d). Now x and y act as identities in the subrings $P(x)$ and $P(y)$, respectively. A proof analogous to that of (b) $\rightarrow$ (c) shows that (c) $\rightarrow$ (b). So we have the existence of isomorphisms $\alpha_1 : P(x) \rightarrow P(y)$ and $\alpha_2 : P(x') \rightarrow P(y')$. Now $\alpha_1(x) = y$ since x and y are identities. Define $\alpha : B \rightarrow B$ by $\alpha(b) = \alpha_1(b_1) + \alpha_2(b_2)$ where $b = b_1 + b_2$ with $b_1 \in P(x)$ and $b_2 \in P(x')$. It is direct to see that α is our required automorphism.

(d) $\rightarrow$ (a). $\alpha(b \cdot_x c) = \alpha[(b \vee x) \wedge c] = [\alpha(b) \vee y] \wedge \alpha(c) = \alpha(b) \cdot_y \alpha(c)$. This completes the proof of the theorem.

6. <u>Ideals that are direct summands of $(B, +, \cdot)$</u>. Let $(B, +, \cdot)$ be the special Boolean near-ring determined by $x \in B$. We have seen in Proposition 2.2 that $B = P(x) \oplus P(x')$ and we have seen from Propositions 2.4 and 2.9 and Theorem 3.6 that $P(x)$ and $P(x')$ are near-ring ideals. In this section we classify those ideals that are direct summands. We will make use of the following:

LEMMA 6.1. *Let $(B, +, \wedge, 1)$ be a Boolean ring with identity 1, and let A be an ideal of B. Then A is a direct summand if and only if $A = P(x)$ for some $x \in B$.*

Proof. For $x \in B$, $B = P(x) \oplus P(x')$. Conversely, suppose $B = A \oplus C$ where A and C are ideals. Now $1 = x + x'$, $x \in A$ and $x' \in C$. Let $a \in A$. Then $a = a \wedge 1 = (a \wedge x) + (a \wedge x')$ and $a \wedge x' = 0$, since $x' \in C$. Hence $a = a \wedge x$ which implies that $a \in P(x)$. Consequently $A \subseteq P(x)$. But $P(x) \subseteq A$ since $x \in A$. Hence $A = P(x)$.

Let A be an ideal of $(B, +, \cdot)$. As seen from Theorem 2.3 and Theorem 3.6, $A = A(x) \oplus A(x')$ where $A(x) = \{a \wedge x \mid a \in A\}$ and $A(x') = \{a \wedge x' \mid a \in A\}$. Suppose $B = A \oplus C$ where C is also an ideal. Then $B = A(x) \oplus C(x) \oplus A(x') \oplus C(x')$ and $P(x) = A(x) \oplus C(x)$ and $P(x') = A(x') \oplus C(x')$. So the problem of determining which ideals are direct summands has been reduced to finding the ideals that are direct summands of $P(x)$ and $P(x')$, respectively. Since $(P(x'), +, \cdot, x')$ is a Boolean ring with identity x', we know from Lemma 6.1 the ideals

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that are direct summands of $P(x')$. We now have only to find the ideals $I \subseteq P(x)$ that are direct summands of $(P(x), +, \cdot)$. Recall from Proposition 2.4 and Theorem 3.6 that the ideals $I \subseteq P(x)$ of $(B, +, \cdot)$ are exactly the subgroups. We shall see that they are all direct summands.

Suppose $P(x) = M \oplus N$ where M and N are ideals of $(B, +, \cdot)$. Then M and N are subgroups of $(B, +)$ and are direct summands of $(P(x), +)$. We now will see that the converse is also true. Since each subgroup $M \subseteq P(x)$ is bounded and is a pure subgroup, then a theorem of Prüfer [9, Theorem 24.5] shows that M is a direct summand. We have already seen that each subgroup is also an ideal. In summary we have the following:

THEOREM 6.2. *An ideal I of $(B, +, \cdot)$ is a direct summand if and only if $I = P(t) \oplus M$ where $P(t) \subset P(x')$ is an ideal and $M \subseteq P(x)$ is a subgroup, hence an ideal.*

Remark. In this paper we have considered only one type of Boolean near-ring. It would be of interest to classify all Boolean near-rings according to Definition 1.1.

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