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# Holomorphic functions of slow growth on coverings of pseudoconvex domains in Stein manifolds

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Abstract

We apply the methods developed in our previous work to study holomorphic functions of slow growth on coverings of pseudoconvex domains in Stein manifolds. In particular, we extend and strengthen certain results of Gromov, Henkin and Shubin on holomorphic $L^2$ functions on coverings of pseudoconvex manifolds in the case of coverings of Stein manifolds.

## 1. Introduction

**1.1** Let $M$ be a complex manifold satisfying

$$M \subset\subset \widetilde{M} \subset N \text{ and the natural map } \pi_1(M) \to \pi_1(N) \text{ is an isomorphism;} \tag{1.1}$$

here $M$ and $\widetilde{M}$ are open connected subsets of a complex manifold $N$, $\widetilde{M}$ is Stein, and $\pi_1(X)$ stands for the fundamental group of $X$. Condition (1.1) is valid, e.g., for $M$ a strictly pseudoconvex domain or an analytic polyhedra in a Stein manifold. It implies that the group $\pi_1(N)$ is finitely generated. In [Bru06a] we presented a method to construct integral representation formulas for holomorphic functions of slow growth defined on unbranched coverings of $M$. Using such formulas we established that some known results for holomorphic functions on $M$ can be extended to similar results for holomorphic functions of slow growth on coverings of $M$. In this paper we continue to study holomorphic functions of slow growth on coverings of $M$ and apply the methods developed in [Bru06a] to extend and strengthen certain results of Gromov, Henkin and Shubin [GHS98] on holomorphic $L^2$ functions on coverings of pseudoconvex manifolds in the case of coverings of Stein manifolds.

**1.2** The presentation in this paper is focused on several problems and results formulated in [GHS98]. To describe them, we first recall some definitions.

Let $M \subset\subset N$ be a domain with a smooth boundary $bM$ in an $n$-dimensional complex manifold $N$, that is,

$$M = \{z \in N : \rho(z) < 0\} \tag{1.2}$$

where $\rho$ is a real-valued function of class $C^2(\Omega)$ in a neighbourhood $\Omega$ of the compact set $\overline{M} := M \cup bM$ such that

$$d\rho(z) \neq 0 \quad \text{for all } z \in bM. \tag{1.3}$$

Let $z_1, \dots, z_n$ be complex local coordinates in $N$ near $z \in bM$. Then the tangent space $T_zN$ at $z$ is

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identified with $\mathbb{C}^n$. By $T_z^c(bM) \subset T_zN$ we denote the complex tangent space to $bM$ at $z$, i.e.,

$$T_z^c(bM) = \left\{ w = (w_1, \dots, w_n) \in T_z(N) : \sum_{j=1}^{n} \frac{\partial \rho}{\partial z_j}(z) w_j = 0 \right\}. \tag{1.4}$$

The *Levi form* of $\rho$ at $z \in bM$ is a hermitian form on $T_z^c(bM)$ defined in the local coordinates by the formula

$$L_z(w, \overline{w}) = \sum_{j,k=1}^{n} \frac{\partial^2 \rho}{\partial z_j \partial \overline{z}_k}(z) w_j \overline{w}_k. \tag{1.5}$$

The manifold $M$ is called *pseudoconvex* if $L_z(w, \overline{w}) \geqslant 0$ for all $z \in bM$ and $w \in T_z^c(bM)$. It is called *strictly pseudoconvex* if $L_z(w, \overline{w}) > 0$ for all $z \in bM$ and all $w \neq 0$, $w \in T_z^c(bM)$.

Equivalently, strictly pseudoconvex manifolds can be described as those which locally, in a neighbourhood of any boundary point, can be presented as strictly convex domains in $\mathbb{C}^n$. It is also known (see [Car60, Rem56]) that any strictly pseudoconvex manifold admits a proper holomorphic map with connected fibres onto a normal Stein space.

Diminishing, if necessary, $N$ we may assume that $\pi_1(M) = \pi_1(N)$ for $M$ defined by (1.2) and (1.3). Let $r : N_G \to N$ be the regular covering of $N$ with (discrete) transformation group $G$. Then $M_G := r^{-1}(M)$ is a regular covering of $M$ (with the same transformation group). It is a domain in $N_G$ with the smooth boundary $bM_G := r^{-1}(bM)$. By $\overline{M}_G := M_G \cup bM_G$ we denote the closure of $M_G$ in $N_G$.

Let $X$ be a subspace of the space $\mathcal{O}(M_G)$ of all holomorphic functions on $M_G$. A point $z \in bM_G$ is called a *peak point* for $X$ if there exists a function $f \in X$ such that $f$ is unbounded on $M_G$ but bounded outside $U \cap M_G$ for any neighbourhood $U$ of $z$ in $N_G$.

A point $z \in bM_G$ is called a *local peak point* for $X$ if there exists a function $f \in X$ such that $f$ is unbounded in $U \cap M_G$ for any neighbourhood $U$ of $z$ in $N_G$ and there exists a neighbourhood $U$ of $z$ in $N_G$ such that for any neighbourhood $V$ of $z$ in $N_G$ the function $f$ is bounded on $U \setminus V$.

The Oka–Grauert theorem [Gra58a] states that if $M$ is strictly pseudoconvex and $bM$ is not empty, then every $z \in bM$ is a peak point for $\mathcal{O}(M)$. In general, it is not known whether the similar statement is true for boundary points of $M_G$ with an infinite $G$.

Let $dV_{M_G}$ be the Riemannian volume form on $M_G$ obtained by a Riemannian metric pulled back from $N$. By $\mathcal{H}^2(M_G)$ we denote the Hilbert space of holomorphic functions $g$ on $M_G$ with norm

$$\left( \int_{z \in M_G} |g(z)|^2 \, dV_{M_G}(z) \right)^{1/2}.$$

In [GHS98], the von Neumann $G$-dimension $\dim_G$ was used to measure the space $\mathcal{H}^2(M_G)$. In particular, in [GHS98, Theorem 0.2] the following result was proved.

THEOREM A. *If $M$ is strictly pseudoconvex, then:*

(a) $\dim_G \mathcal{H}^2(M_G) = \infty$*; and*

(b) *each point in $bM_G$ is a local peak point for $\mathcal{H}^2(M_G)$.*

In [GHS98, Theorem 0.5] a similar result was established for a covering $M_G$ of a pseudoconvex manifold $M$ with a strictly plurisubharmonic $G$-invariant function existing in a neighbourhood of $bM_G$. Finally, in [GHS98, § 4] the following open problems were formulated.

Suppose that $M$ is strictly pseudoconvex.

(1) Does there exist a finite number of functions in $\mathcal{H}^2(M_G) \cap C(\overline{M}_G)$ which separate all points in $bM_G$?



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(2) Assume that $\dim_{\mathbb{C}} M = 2$. Does there exist $f \in \mathcal{H}^2(M_G) \cap C(\overline{M}_G)$ such that $f(x) \neq 0$ for all $x \in bM_G$?

(3) Is it true that for every Cauchy–Riemann (CR)-function $f \in L^2(bM_G) \cap C(bM_G)$ in the case $\dim_{\mathbb{C}} M_G > 1$ there exists $f' \in \mathcal{H}^2(M_G) \cap C(\overline{M}_G)$ such that $f'|_{bM_G} = f$?

Here $L^2(bM_G)$ is defined similar to $\mathcal{H}^2(M_G)$ with respect to the volume form on $bM_G$ obtained by a Riemannian metric pulled back from $N$. Also, recall that $f \in C(bM_G)$ is called a CR-function if for every smooth $(n, n-2)$-form $\omega$ with a compact support one has

$$\int_{bM_G} f \wedge \overline{\partial}\omega = 0.$$

If $f$ is smooth this is equivalent to the fact that $f$ is a solution of the tangential CR-equations: $\overline{\partial}_b f = 0$ (see, e.g., [KR65]).

The present paper deals with the above results and problems in the case of coverings of $M$ satisfying condition (1.1).

## 2. Formulation of main results

**2.1** We start with some results related to question (2) of § 1.2.

Let $M$ be a manifold satisfying condition (1.1) and $M'$ be an unbranched covering of $M$. Condition (1.1) implies that there is a covering $r : N' \to N$ of $N$ such that $M'$ is a domain in $N'$ (i.e. $\pi_1(M') = \pi_1(N')$). As above, $\overline{M'}$ denotes the closure of $M'$ in $N'$.

Let $\phi : N' \to \mathbb{R}$ be a function uniformly continuous with respect to the path metric induced by a Riemannian metric pulled back from $N$.

Theorem 2.1. *There exist a function $f_\phi \in \mathcal{O}(M') \cap C(\overline{M'})$ and a constant*[1] *$C = C(\phi, M', N')$ such that*

$$|f_\phi(z) - \phi(z)| < C \quad \text{and} \quad |df_\phi(z)| < C \quad \text{for all } z \in M'.$$

(Here the norm $|\omega(z)|$ of a differential form $\omega$ at $z \in M'$ is determined with respect to the Riemannian metric pulled back from $N$.)

As a corollary of this result we answer an extended version of question (2) of § 1.2 for coverings of manifolds $M$ satisfying condition (1.1). Namely, let $d$ be the path metric on $N'$ obtained by the pullback of a Riemannian metric defined on $N$. Fix a point $o \in M'$ and set

$$d_o(x) := d(o, x), \quad x \in N'.$$

From the triangle inequality it follows that the function $\phi(x) := d_o(x)$, $x \in N'$, satisfies the hypothesis of Theorem 2.1.

Corollary 2.2. *Let $f := f_\phi$ be the function from Theorem 2.1 for $\phi = d_o$. Then there exists a constant $\alpha > 0$ such that $F = e^{-\alpha f} \in \mathcal{H}^2(M') \cap C(\overline{M'})$.*

(Note that here $F(x) \neq 0$ for all $x \in M'$ and there are no restrictions on $\dim_{\mathbb{C}} M$.)

**2.2** In this part we formulate our results related to Theorem A.

Let $M'$ be an unbranched covering of $M$ satisfying (1.1). Let $\psi : M' \to \mathbb{R}_+$ be a continuous function and $dV_{M'}$ be the Riemannian volume form on $M'$ obtained by a Riemannian metric pulled back from $N$. For an open set $D \subset M$ we introduce the Banach space $H^p_\psi(D')$, $1 \leqslant p \leqslant \infty$,

[1] Here and below the notation $C = C(\alpha, \beta, \gamma, \dots)$ means that the constant depends only on the parameters $\alpha, \beta, \gamma, \dots$.



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of holomorphic functions $g$ on $D' := r^{-1}(D) \subset M'$ with norm

$$\left(\int_{z\in M'} |g(z)|^p \psi(z)\, dV_{M'}(z)\right)^{1/p}.$$

Let $r : N' \to N$ be the covering of $N$ satisfying (1.1) such that $\pi_1(N') = \pi_1(M')$. Then $M'$ $(= r^{-1}(M))$ is a domain in $N'$. Suppose that $\psi : N' \to \mathbb{R}_+$ is such that $\log\psi$ is uniformly continuous with respect to the path metric induced by a Riemannian metric pulled back from $N$. We set $\phi := \log\psi$ and consider the holomorphic function $f_\phi$ from Theorem 2.1. This theorem implies that for $\widetilde{C} := e^C$

$$\frac{1}{\widetilde{C}}\psi(z) \leqslant |e^{f_\phi(z)}| \leqslant \widetilde{C}\psi(z), \quad z \in \overline{M'}. \tag{2.1}$$

Therefore, the following result holds.

Proposition 2.3. *For any open set $D \subset M$ and every $p \in [1,\infty)$ the map $L_\psi : H^p_\psi(D') \to H^p_1(D')$, $L_\psi(g) = g \cdot e^{f_\phi/p}$, is an isomorphism of Banach spaces.*

Let us now formulate an extension of Theorem A.

Suppose that $M$ is a strictly pseudoconvex domain in a complex manifold $N$ such that $\pi_1(M) = \pi_1(N)$ and $N$ is a domain in a Stein manifold. Let $r : N' \to N$ be an unbranched covering of $N$ and $M' = r^{-1}(M)$ be the corresponding covering of $M$. Let $bM' = r^{-1}(bM)$ be the boundary of $M'$ in $N'$.

Theorem 2.4. *Each point in $bM'$ is a peak point for $\mathcal{O}(M')$ and for every $H^p_1(M')$, $1 \leqslant p < \infty$.*

From Theorem 2.4 and Proposition 2.3 we get (for $\psi$ as in Proposition 2.3) the following corollary.

Corollary 2.5. *If $\log\psi$ is bounded from below, then each $z \in bM'$ is a peak point for $H^p_\psi(M')$, $1 \leqslant p < \infty$.*

*Remark* 2.6. The main ingredient of the proof of Theorem 2.4 is uniform estimates for solutions of certain $\overline{\partial}$-equations on $M'$. In fact, similar estimates are valid on coverings of so-called non-degenerate pseudoconvex polyhedrons on Stein manifolds (see [SH80] and [Heu83] for their definition). This class contains, in particular, piecewise strictly pseudoconvex domains and non-degenerate analytic polyhedrons on Stein manifolds. Also, every $M$ from this class satisfies condition (1.1). Let $M'$ be a covering of such $M$ and $z \in bM'$ be such that $M' \cap U$ is strictly pseudoconvex for a neighbourhood $U \subset N'$ of $z$. Then, arguing as in the proof of Theorem 2.4, one obtains that $z$ *is a peak point for $\mathcal{O}(M')$ and for every $H^p_1(M')$, $1 \leqslant p < \infty$.*

**2.3** In this section we discuss some results related to question (3) of § 1.2.

Let $r : N' \to N$ be a covering of $N$ satisfying (1.1). As before we set $M' = r^{-1}(M) \subset N'$. Consider a function $\psi : N' \to \mathbb{R}_+$ such that $\log\psi$ *is uniformly continuous with respect to the path metric induced by a Riemannian metric pulled back from $N$*. For such $\psi$ and every $x \in M$, we introduce the Banach space $l_{p,\psi,x}(M')$, $1 \leqslant p \leqslant \infty$, of functions $g$ on $r^{-1}(x) \subset M'$ with norm

$$|g|_{p,\psi,x} := \left(\sum_{y\in r^{-1}(x)} |g(y)|^p \psi(y)\right)^{1/p}. \tag{2.2}$$

Next, for an open set $D \subset M$ we introduce the Banach space $\mathcal{H}_{p,\psi}(D')$, $1 \leqslant p \leqslant \infty$, of functions $f$ holomorphic on $D' := r^{-1}(D) \subset M'$ with norm

$$|f|^D_{p,\psi} := \sup_{x\in D} |f|_{p,\psi,x}. \tag{2.3}$$



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Clearly, one has a continuous embedding $\mathcal{H}_{p,\psi}(D') \hookrightarrow H^p_\psi(D')$. Let $U \subset N$ be an open set containing $\overline{D}$ and $U' = r^{-1}(U)$. Then for $\psi$ as above using the mean value property for plurisubharmonic functions one can easily show that *for each* $p \in [1,\infty]$ *the restriction* $f \mapsto f|_{D'}$ *induces a linear continuous map* $H^p_\psi(U') \to \mathcal{H}_{p,\psi}(D')$. (However, it is unknown *whether the image of* $\mathcal{H}_{p,\psi}(D')$ *is dense in* $H^p_\psi(D')$ *for each* $p \in [1,\infty]$.) Also, for such $\psi$ from the results proved in [Bru06a] follow that *holomorphic functions from* $\mathcal{H}_{p,\psi}(M')$ *separate all points in* $M'$ *(for each* $p \in [1,\infty]$*)*.

Let us formulate the main result of this section.

Let $M \subset\subset \widetilde{M} \subset N$ be manifolds satisfying condition (1.1) with $\dim_{\mathbb{C}} M \geqslant 2$. Suppose that $D \subset\subset M$ is an open subset whose boundary $bD$ is a connected $C^k$ submanifold of $M$ ($1 \leqslant k \leqslant \infty$). For a covering $r : N' \to N$ we set $D' = r^{-1}(D)$ and $bD' = r^{-1}(bD)$.

Theorem 2.7. *For every CR-function* $f \in C^s(bD')$, $0 \leqslant s \leqslant k$, *satisfying*

$$f|_{r^{-1}(x)} \in l_{p,\psi,x}(M') \quad \text{for all } x \in D \text{ and } \sup_{x \in bD} |f|_{p,\psi,x} < \infty$$

*there exists a function* $f' \in \mathcal{H}_{p,\psi}(D') \cap C^s(\overline{D'})$ *such that* $f'|_{bD'} = f$.

*Remark* 2.8. (1) The converse to this theorem is always true: the restriction of every $f' \in \mathcal{H}_{p,\psi}(D') \cap C^s(\overline{D'})$ to $bD'$ is a CR-function satisfying the hypotheses of the theorem.

(2) We will also prove (see (6.6)) that for some $c = c(M', M, \psi, p)$

$$|f'|^D_{p,\psi} \leqslant c \sup_{x \in bD} |f|_{p,\psi,x}.$$

(3) An interesting question is whether the space of CR-functions of Theorem 2.7 is $L_p$-dense in the space of $L_p$ CR-functions on $bD'$ where the $L_p$ norm on $bD'$ is defined by the Riemannian volume form obtained by the pullback of the Riemannian metric on $N$.

As a corollary we obtain an analog of the Hartogs extension theorem (see [Boc43]). We formulate it for functions of the maximal possible growth for which our method works.

Suppose that $M$ satisfies (1.1). Let $D \subset M$ be a domain and $K \subset\subset D$ be a compact set such that $U := D \setminus K$ is connected. Consider a covering $r : M' \to M$ and set $D' = r^{-1}(D)$, $U' = r^{-1}(U)$. By $d_o$, $o \in M'$, we denote the distance function on $M'$ as in Corollary 2.2.

Corollary 2.9. *There exists a constant* $c > 0$ *such that for every* $f \in \mathcal{O}(U')$ *satisfying for some* $c_2 > 0$ *and* $0 < c_1 < c$ *the inequality*

$$|f(z)| \leqslant e^{c_2 e^{c_1 d_o(z)}}, \quad z \in U',$$

*there is* $f' \in \mathcal{O}(D')$ *such that*

$$|f'(z)| \leqslant e^{c_3 e^{c_1 d_o(z)}}, \quad z \in D', \quad \text{and} \quad f'|_{U'} = f;$$

*where* $c_3$ *depends on* $c_2, c_1, c, M, M'$ *only.*

We do not know whether a similar extension result holds for functions $f$ growing faster than those of the corollary.

**2.4** Finally, we formulate a result related to question (1) of § 1.2. First, we recall some definitions of the theory of flat vector bundles (see, e.g., [Oni67]).

Let $X$ be a complex manifold and $\rho : \pi_1(X) \to GL_k(\mathbb{C})$ be a homomorphism of its fundamental group. We set $G := \pi_1(X)/\operatorname{Ker}\rho$. It is well known (see, e.g., Example 3.2(b) below) that to any such $\rho$ corresponds a complex flat vector bundle $E_\rho$ on $X$ (i.e. a bundle constructed by a locally constant cocycle). We call $E_\rho$ the *bundle associated with* $\rho$. Assume that $\rho$ is such that $E_\rho$ is topologically



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trivial, i.e. is isomorphic in the category of continuous bundles to the bundle $X \times \mathbb{C}^k$. Every such $\rho$ can be obtained as the monodromy of the equation $dF = \omega F$ on $X$ where $\omega$ is a matrix-valued 1-form on $X$ satisfying $d\omega - \omega \wedge \omega = 0$. By $\mathcal{T}(X)$ we denote the class of quotient groups $G$ obtained by representations $\rho$ as above.

Now, let $r : M_G \to M$ be the regular covering of $M$ satisfying condition (1.1) with transformation group $G$. Let $G_1 \subset G$ be a subgroup of a finite index. Then there is a finite covering $r_1 : M_1 \to M$ whose fibre is the quotient set $G/G_1$ such that $M_G$ is also the regular covering of $M_1$ with transformation group $G_1$.

Theorem 2.10. *Assume that $G_1 \in \mathcal{T}(M_1)$. Then there is a finite number of functions in $\mathcal{H}_{2,1}(M_G) \cap C(\overline{M}_G)$ that separate all points in $M_G$.*

*Remark* 2.11. (1) We will see from the proof that the functions in Theorem 2.10 can be taken even from $\mathcal{H}_{2,\psi}(M_G)$ where $\psi : M_G \to \mathbb{R}_+$ has a double exponential growth.

(2) As the group $G$ in Theorem 2.10 one can take, e.g., a finitely generated free group (see, e.g., [Oni67]) or a polycyclic group (see, e.g., [Rag72]). If $\dim_{\mathbb{C}} M = 1$, then, since $M$ is homotopically equivalent to a one-dimensional CW-complex (see, e.g., [GR77]), every quotient group $G$ obtained by a linear representation $\rho$ belongs to $\mathcal{T}(M)$.

### 2.5 Remark added in February 2006

The present paper was written in February–March of 2005. Since then, the author obtained several new results related to questions posed in [GHS98] on holomorphic $L^2$-functions on coverings $M'$ of strongly pseudoconvex (not necessarily Stein) manifolds $M$. In particular, in connection with [GHS98, Theorem 0.2] (see Theorem A) the following important question was asked (see [GHS98, p. 3]): '*A natural question arises: is the cocompact group action* (on $\overline{M'}$) *really relevant for the existence of many holomorphic $L^2$-functions* (on $M'$) *or is it just an artifact of the chosen methods which require a use of von Neumann algebras?*' and further '*It is not clear how to formulate conditions assuring that $\dim L^2\mathcal{O}(M') = \infty$ without any group action*' (this is $\dim H_1^2(M') = \infty$ in our notation). In [Bru06b] it was shown that the regularity of $M'$ is irrelevant for the existence of many holomorphic $L^2$-functions on $M'$. Moreover, a substantial extension of the above result of [GHS98] was also proved. To formulate our result, let $C_M \subset M$ be the union of all compact complex subvarieties of $M$ of complex dimension $\geqslant 1$. It is known that if $M$ is strongly pseudoconvex, then $C_M$ is a compact complex subvariety of $M$. Let $z_i$, $1 \leqslant i \leqslant m$, be distinct points in $M \setminus C_M$ and $\psi : N' \to \mathbb{R}_+$ be as in § 2.3.

Theorem 2.12 [Bru06b, Theorem 1.1]. *If $M$ is strongly pseudoconvex, then:*

(a) *for any $f_i \in l_{2,\psi,z_i}(M')$, $1 \leqslant i \leqslant m$, there exists $F \in H_\psi^2(M')$ such that $F|_{z_i'} = f_i$, $1 \leqslant i \leqslant m$;*

(b) *if $\psi$ is such that $\log \psi$ is bounded from below on $N'$, then each point in $bM'$ is a peak point for $H_\psi^2(M')$.*

Also, in [Bru06c, Bru06d] new Hartogs type theorems on coverings of strongly pseudoconvex manifolds were obtained that, in a sense, extend Theorem 2.7.

## 3. Preliminary results

**3.1** First, we recall some basic facts from the theory of bundles (see, e.g., [Hir66]).

Let $X$ be a complex analytic space and $S$ be a complex analytic Lie group with unit $e \in S$. Consider an effective holomorphic action of $S$ on a complex analytic space $F$. Here *holomorphic*



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*action* means a holomorphic map $S \times F \to F$ sending $s \times f \in S \times F$ to $sf \in F$ such that $s_1(s_2 f) = (s_1 s_2)f$ and $ef = f$ for any $f \in F$. *Efficiency* means that the condition $sf = f$ for some $s$ and any $f$ implies that $s = e$.

Definition 3.1. A complex analytic space $W$ together with a holomorphic map (projection) $\pi : W \to X$ is called a holomorphic bundle on $X$ with structure group $S$ and fibre $F$, if there exists a system of coordinate transformations, i.e., if:

(1) there is an open cover $\mathcal{U} = \{U_i\}_{i\in I}$ of $X$ and a family of biholomorphisms $h_i : \pi^{-1}(U_i) \to U_i \times F$, that map 'fibres' $\pi^{-1}(u)$ onto $u \times F$;

(2) for any $i, j \in I$ there are elements $s_{ij} \in \mathcal{O}(U_i \cap U_j, S)$ such that
$$(h_i h_j^{-1})(u \times f) = u \times s_{ij}(u)f \quad \text{for any } u \in U_i \cap U_j,\ f \in F.$$

A holomorphic bundle $\pi : W \to X$ whose fibre is a Banach space $F$ and the structure group is $GL(F)$ (the group of linear invertible transformations of $F$) is called a holomorphic Banach vector bundle. A holomorphic section of a holomorphic bundle $\pi : W \to X$ is a holomorphic map $s : X \to W$ satisfying $\pi \circ s = \mathrm{id}$.

We will use the following construction of holomorphic bundles (see, e.g., [Hir66, ch. 1]).

Let $S$ be a complex analytic Lie group and $\mathcal{U} = \{U_i\}_{i\in I}$ be an open cover of $X$. By $Z^1_{\mathcal{O}}(\mathcal{U}, S)$ we denote the set of holomorphic $S$-valued $\mathcal{U}$-cocycles. By definition, $s = \{s_{ij}\} \in Z^1_{\mathcal{O}}(\mathcal{U}, S)$, where $s_{ij} \in \mathcal{O}(U_i \cap U_j, S)$ and $s_{ij}s_{jk} = s_{ik}$ on $U_i \cap U_j \cap U_k$. Consider the disjoint union $\bigsqcup_{i\in I} U_i \times F$ and for any $u \in U_i \cap U_j$ identify the point $u \times f \in U_j \times F$ with $u \times s_{ij}(u)f \in U_i \times F$. We obtain a holomorphic bundle $W_s$ on $X$ whose projection is induced by the projection $U_i \times F \to U_i$. Moreover, any holomorphic bundle on $X$ with structure group $S$ and fibre $F$ is isomorphic (in the category of holomorphic bundles) to a bundle $W_s$.

*Example* 3.2. (a) Let $M$ be a complex manifold. For any subgroup $H \subset \pi_1(M)$ consider the unbranched covering $r : M(H) \to M$ corresponding to $H$. We will describe $M(H)$ as a holomorphic bundle on $M$.

First, assume that $H \subset \pi_1(M)$ is a normal subgroup. Then $M(H)$ is a regular covering of $M$ and the quotient group $G := \pi_1(M)/H$ acts holomorphically on $M(H)$ by deck transformations. It is well known that $M(H)$ in this case can be thought of as a *principal fibre bundle* on $M$ with fibre $G$ (here $G$ is equipped with the discrete topology). Namely, let us consider the map $R_G(g) : G \to G$, $g \in G$, defined by the formula
$$R_G(g)(q) = q \cdot g^{-1}, \quad q \in G.$$
Then for an open cover $\mathcal{U} = \{U_i\}_{i\in I}$ of $M$ by sets biholomorphic to open Euclidean balls in some $\mathbb{C}^n$ there is a locally constant cocycle $c = \{c_{ij}\} \in Z^1_{\mathcal{O}}(\mathcal{U}, G)$ such that $M(H)$ is biholomorphic to the quotient space of the disjoint union $V = \bigsqcup_{i\in I} U_i \times G$ by the equivalence relation: $U_i \times G \ni x \times R_G(c_{ij})(q) \sim x \times q \in U_j \times G$. The identification space is a holomorphic bundle with projection $r : M(H) \to M$ induced by the projections $U_i \times G \to U_i$. In particular, when $H = e$ we obtain the definition of the universal covering $M_u$ of $M$.

Assume now that $H \subset \pi_1(M)$ is not necessarily normal. Let $X_H = \pi_1(M)/H$ be the set of cosets with respect to the (left) action of $H$ on $\pi_1(M)$ defined by left multiplications. By $[Hq] \in X_H$ we denote the coset containing $q \in \pi_1(M)$. Let $A(X_H)$ be the group of all homeomorphisms of $X_H$ (equipped with the discrete topology). We define the homomorphism $\tau : \pi_1(M) \to A(X_H)$ by the formula:
$$\tau(g)([Hq]) := [Hqg^{-1}], \quad q \in \pi_1(M).$$
Set $Q(H) := \pi_1(M)/\mathrm{Ker}(\tau)$ and let $\widetilde{g}$ be the image of $g \in \pi_1(M)$ in $Q(H)$. By $\tau_{Q(H)} : Q(H) \to A(X_H)$ we denote the unique homomorphism whose pullback to $\pi_1(M)$ coincides with $\tau$.



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Consider the action of $H$ on $V = \bigsqcup_{i\in I} U_i \times \pi_1(M)$ induced by the left action of $H$ on $\pi_1(M)$ and let $V_H = \bigsqcup_{i\in I} U_i \times X_H$ be the corresponding quotient set. Define the equivalence relation $U_i \times X_H \ni x \times \tau_{Q(H)}(\widetilde{c}_{ij})(h) \sim x \times h \in U_j \times X_H$ with the same $\{c_{ij}\}$ as in the definition of $M(e)$. The corresponding quotient space is a holomorphic bundle with fibre $X_H$ biholomorphic to $M(H)$.

(b) We retain the notation of example (a). Let $B$ be a complex Banach space and $GL(B)$ be the group of invertible bounded linear operators $B \to B$. Consider a homomorphism $\rho : G \to GL(B)$. Without loss of generality we assume that $\operatorname{Ker}(\rho) = e$, for otherwise we can pass to the corresponding quotient group. The *holomorphic Banach vector bundle* $E_\rho \to M$ *associated with* $\rho$ is defined as the quotient of $\bigsqcup_{i\in I} U_i \times B$ by the equivalence relation $U_i \times B \ni x \times \rho(c_{ij})(w) \sim x \times w \in U_j \times B$ for any $x \in U_i \cap U_j$. Let us illustrate this construction by an example.

Let $\phi : X_H \to \mathbb{R}^+$ ($X_H := \pi_1(M)/H$) be a function satisfying

$$\phi(\tau(h)(x)) \leqslant c_h \phi(x), \quad x \in X_H,\ h \in \pi_1(M), \tag{3.1}$$

where $c_h$ is a constant depending on $h$. By $l_{p,\phi}(X_H)$, $1 \leqslant p \leqslant \infty$, we denote the Banach space of complex functions $f$ on $X_H$ with norm

$$\|f\|_{p,\phi} := \left(\sum_{g\in X_H} |f(g)|^p \phi(g)\right)^{1/p}. \tag{3.2}$$

Then according to (3.1) the map $\rho$ defined by the formula $[\rho(g)(f)](x) := f(\tau(g)(x))$, $g \in \pi_1(M)$, $x \in X_H$, is a homomorphism of $\pi_1(M)$ into $GL(l_{p,\phi}(X_H))$. By $E_{p,\phi}(X_H)$ we denote the holomorphic Banach vector bundle associated with this $\rho$.

**3.2** We retain the notation of Example 3.2. Let $r : M' \to M$ be a covering where $M' = M(H)$ (i.e. $\pi_1(M') = H$). Assume that $M$ satisfies condition (1.1), i.e. $M \subset\subset N$ and $\pi_1(M) = \pi_1(N)$. Then there is an embedding $M(H) \hookrightarrow N(H)$. (Without loss of generality we consider $M(H)$ as an open subset of $N(H)$.) Let $\{V_i\}_{i\in I}$ be a finite acyclic open cover of $\overline{M}$ by relatively compact sets. We set $U_i := V_i \cap M$ and consider the open cover $\mathcal{U} = \{U_i\}_{i\in I}$ of $M$. Then as in Example 3.2(a) we can define $M(H)$ by a cocycle $c = \{c_{ij}\} \in Z^1_{\mathcal{O}}(\mathcal{U}, \pi_1(M))$.

Further, let $\psi : N(H) \to \mathbb{R}_+$ be a function such that $\log\psi$ is uniformly continuous with respect to the path metric induced by a Riemannian metric pulled back from $N$. Fix a point $z_0 \in M$ and identify $r^{-1}(z_0)$ with $z_0 \times X_H$ ($X_H := \pi_1(M)/H$). We define the function $\phi : X_H \to \mathbb{R}_+$ by the formula

$$\phi(x) := \psi(z_0, x), \quad x \in X_H.$$

It was proved in [Bru06a, Lemma 2.3] that $\phi$ *satisfies inequality* (3.1). Then the bundle $E_{p,\phi}(X_H)$ is well defined. By definition, any holomorphic section of this bundle is determined by a family $\{f_i(z, g)\}_{i\in I}$ of holomorphic functions on $U_i$ with values in $l_{p,\phi}(X_H)$ satisfying

$$f_i(z, \tau(c_{ij})(h)) = f_j(z, h) \quad \text{for any } z \in U_i \cap U_j.$$

We introduce the Banach space $B_{p,\phi}(X_H)$ of *bounded holomorphic sections* $f = \{f_i\}_{i\in I}$ of $E_{p,\phi}(X_H)$ with norm

$$|f|_{p,\phi} := \sup_{i\in I, z\in U_i} ||f_i(z,\cdot)||_{p,\phi}. \tag{3.3}$$

(Here $\|\cdot\|_{p,\phi}$ is the norm on $l_{p,\phi}(X_H)$, see (3.2).)

Further, let $f \in \mathcal{H}_{p,\psi}(M(H))$ (see § 2.2 for the definition). We define the family $\{f_i\}_{i\in I}$ of functions on $U_i$ with values in the space of functions on $X_H$ by the formula

$$f_i(z, g) := f(z, g), \quad z \in U_i,\ i \in I,\ g \in X_H. \tag{3.4}$$



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It was established in [Bru06a, Proposition 2.4] that *the correspondence* $f \mapsto \{f_i\}_{i\in I}$ *determines an isomorphism of Banach spaces* $D : \mathcal{H}_{p,\psi}(M(H)) \to B_{p,\phi}(X_H)$. (Here $D$ is an isometry for $\psi \equiv 1$.)

Next, suppose that $\{x_n\}_{n\geqslant 1} \subset M$ converges to $x \in M$. Then for sufficiently large $n$ we can arrange $r^{-1}(x_n)$ and $r^{-1}(x)$ in sequences $\{y_{in}\}_{i\geqslant 1}$ and $\{y_i\}_{i\geqslant 1}$ such that every $\{y_{in}\}$ converges to $y_i$ as $n \to \infty$. For such $n$ we define maps $\tau_n(x) : r^{-1}(x) \to r^{-1}(x_n)$ so that $\tau_n(y_i) = y_{in}$, $i \in \mathbb{N}$. Below, $\tau_n^*$ denotes the transpose map generated by $\tau_n$ on functions defined on $r^{-1}(x_n)$ and $r^{-1}(x)$.

Definition 3.3. Let $X \subset M$ be a subset. We say that a function $f$ on $r^{-1}(X)$ belongs to the class $C_{p,\psi}(r^{-1}(X))$ if

(1) $f|_{r^{-1}(x)} \in l_{p,\psi,x}(M')$ for all $x \in X$; and

(2) for any $x \in X$ and any sequence $\{x_n\} \subset X$ converging to $x$ the sequence of functions $\{\tau_n^*(f|_{r^{-1}(x_n)})\}$ converges to $f|_{r^{-1}(x)}$ in the norm of $l_{p,\psi,x}(M')$.

By $C^b_{p,\psi}(r^{-1}(X))$ we denote the Banach space of functions $f \in C_{p,\psi}(r^{-1}(X))$ with norm

$$|f|^X_{p,\psi} := \sup_{x\in X} |f|_{r^{-1}(x)}|_{p,\psi,x}. \tag{3.5}$$

Note that if $X \subset M$ is compact, then $|f|^X_{p,\psi} < \infty$ for every $f \in C_{p,\psi}(r^{-1}(X))$.

Comparing with the above definition of $D$ one determines a similar map for $C^b_{p,\psi}(r^{-1}(X))$. This gives an isomorphism $D : C^b_{p,\psi}(r^{-1}(X)) \to CB^X_{p,\phi}(X_H)$ where $CB^X_{p,\phi}(X_H)$ is the Banach space of *bounded continuous sections* of $E_{p,\phi}(X_H)|_X$ with norm defined as in (3.3).

**3.3** Most of our proofs are based on [Bru06a, Theorem 1.3]. In its proof we use the above isomorphisms $D$ and Cartan's A and B theorems for coherent Banach vector sheaves (see [Bun68]). Let us formulate this result.

Suppose that $r : M' \to M$ is a covering with $M$ satisfying (1.1). We define $\psi : M' \to \mathbb{R}_+$ as in § 3.2. Also, we define $\mathcal{H}_{p,\psi}(M')$ and $l_{p,\psi,x}(M')$ as in § 2.3. For Banach spaces $E$ and $F$ by $\mathcal{B}(E, F)$ we denote the space of all linear bounded operators $E \to F$ with norm $\|\cdot\|$.

Theorem 3.4. *For any* $p \in [1,\infty]$ *there is a family* $\{L_z \in \mathcal{B}(l_{p,\psi,z}(M'), \mathcal{H}_{p,\psi}(M'))\}_{z\in M}$ *holomorphic in* $z$ *such that*

$$(L_z h)(x) = h(x) \quad \text{for any } h \in l_{p,\psi,z}(M') \text{ and } x \in r^{-1}(z).$$

*Moreover,*

$$\sup_{z\in M} \|L_z\| < \infty.$$

The following facts are simple corollaries of this result.

Suppose that $X \subset M$ and $f \in C^b_{p,\psi}(r^{-1}(X))$. We define the function $F$ on $X \times M'$ by the formula

$$F(x, z) := (L_x(f|_{r^{-1}(x)}))(z), \quad x \times z \in X \times M'. \tag{3.6}$$

Then $F$ is continuous and $F(x,\cdot) \in \mathcal{H}_{p,\psi}(M')$ for every $x$. Moreover, if $X$ is open and $f \in \mathcal{H}_{p,\psi}(r^{-1}(X))$, then $F \in \mathcal{O}(X \times M')$ and the map $X \to \mathcal{H}_{p,\psi}(M')$, $x \mapsto F(x,\cdot)$, is holomorphic.

We can also express $F$ in local coordinates. Namely, take $x \in X$ and let $U \subset M$ be a neighbourhood of $x$ biholomorphic to an open Euclidean ball. Then $r^{-1}(U) = \bigsqcup_{y\in r^{-1}(x)} V_y$ and there are biholomorphisms $s_y : U \to V_y$ such that $r \circ s_y = \mathrm{id}$. Now, the restriction of $f \in C^b_{p,\psi}(r^{-1}(X))$ to $r^{-1}(U \cap X)$ can be written as

$$f(z) = \sum_{y\in r^{-1}(x)} f(z)\chi_y(z), \quad z \in r^{-1}(U \cap X), \tag{3.7}$$

where $\chi_y$ is the characteristic function of $V_y$. Let us introduce the functions $\widetilde{f}_y$, $y \in r^{-1}(x)$, by the formulas

$$\widetilde{f}_y(v) = f(s_y(v)), \quad v \in X \cap U.$$

Then we have

$$f(z) = \sum_{y \in r^{-1}(x)} \widetilde{f}_y(v)\chi_y(z), \quad v = r(z) \in U \cap X. \tag{3.8}$$

Consider the series

$$\sum_{y \in r^{-1}(x)} \widetilde{f}_y(v) L_v(\chi_y|_{r^{-1}(v)}), \quad v \in U \cap X, \tag{3.9}$$

with $L_v$ as in Theorem 3.4.

Proposition 3.5. *For $p \in [1, \infty)$ the series in (3.9) converges in $\mathcal{H}_{p,\psi}(M')$ to $F(v, \cdot) := L_v(f|_{r^{-1}(v)})$ uniformly on every compact subset of $U \cap X$. If $p = \infty$ and $f \in C^b_{1,1}(r^{-1}(X))$ then this series also converges in $\mathcal{H}_{\infty,\psi}(M')$ to $F(v, \cdot)$ uniformly on every compact subset of $U \cap X$.*

*Proof.* Suppose that $p \in [1, \infty)$ and $f \in C^b_{p,\psi}(r^{-1}(X))$. Let $C \subset U \cap X$ be a compact subset. By the definition the function $\Phi : U \cap X \to l_{p,\psi}(X_H)$, $z \mapsto \widetilde{f}_\cdot(z)$, is continuous (here we identify $r^{-1}(x)$ with $X_H$). Thus $\Phi(C) \subset l_{p,\psi}(X_H)$ is compact. Fix a family $\{X_i\}_{i \in \mathbb{N}}$ of finite subsets of $X_H$ such that $X_i \subset X_{i+1}$ for any $i$ and $\bigcup_{i=1}^\infty X_i = X_H$. Let $V_i \subset l_{p,\psi}(X_H)$ be a finite-dimensional subspace generated by functions $\delta_z$ on $X_H$ with $z \in X_i$. Here $\delta_z(v) = 1$ if $v = z$ and $\delta_z(v) = 0$ if $v \neq z$. Then $\bigcup_{i=1}^\infty V_i$ is everywhere dense in $l_{p,\psi}(X_H)$ (since $1 \leqslant p < \infty$). This and compactness of $\Phi(C)$ imply that for any $\epsilon > 0$ there exists an integer $l$ such that $\Phi(C) \subset V_l + B_\epsilon$ where $B_\epsilon$ is the open ball in $l_{p,\psi}(X_H)$ centered at 0 of radius $\epsilon$. By $p_l : l_{p,\psi}(X_H) \to V_l$ we denote the projection sending $v = \sum_{x \in X_H} v_x \delta_x \in l_{p,\psi}(X_H)$ to $\sum_{x \in X_l} v_x \delta_x \in V_l$ (here all $v_x \in \mathbb{C}$). Then for $\epsilon$ as above and every $v \in \Phi(C)$ we have $||v - p_l(v)||_{p,\phi} < \epsilon$. From this by (3.8), identifying $r^{-1}(x)$ with $X_H$, we obtain

$$\sup_{v \in C} \bigg\| f|_{r^{-1}(v)} - \sum_{y \in X_l} \widetilde{f}_y(v)\chi_y|_{r^{-1}(v)} \bigg\|_{p,\phi} < \epsilon. \tag{3.10}$$

Thus, by the definition of operators $L_v$ (see Theorem 3.4)

$$\sup_{v \in C} \bigg| F(v, \cdot) - \sum_{y \in X_l} \widetilde{f}_y(v) L_v(\chi_y|_{r^{-1}(v)}) \bigg|^M_{p,\psi} < C\epsilon \tag{3.11}$$

for some constant $C$. This implies the required uniform convergence for $p \in [1, \infty)$.

For $p = \infty$ and $f \in C^b_{1,1}(r^{-1}(X))$ we obtain a new that $\Phi(C) \subset l_{1,1}(X_H)$ is compact. Then in the above notation we easily get $||v - p_l(v)||_{\infty,\phi} < \epsilon$ for any $v \in \Phi(C)$ (because $\|\cdot\|_{\infty,\phi} = \|\cdot\|_{\infty,1} \leqslant \|\cdot\|_{1,1}$). Thus, (3.10) is also valid for $p = \infty$. This gives (3.11) with $p = \infty$. □

## 4. Proofs of Theorem 2.1 and Corollary 2.2

*Proof of Theorem 2.1.* Let $M \subset\subset \widetilde{M} \subset N$ be complex manifolds such that $\pi_1(M) = \pi_1(N)$ and $\widetilde{M}$ is Stein. Consider an unbranched covering $r : N' \to N$ of $N$ and the corresponding coverings $M' = r^{-1}(M)$ and $\widetilde{M}' = r^{-1}(\widetilde{M})$ of $M$ and $\widetilde{M}$. According to Example 3.2(a), $\widetilde{M}'$ is defined on an open cover $\mathcal{U} = \{U_i\}_{i \in I}$ of $\widetilde{M}$ by sets biholomorphic to open Euclidean balls by a locally constant cocycle $\{\widetilde{c}_{ij}\} \in Z^1_{\mathcal{O}}(\mathcal{U}, Q(H))$. (Here we retain the notation of Example 3.2(a) so that $\widetilde{M}' = \widetilde{M}(H)$.) Using this construction we identify $r^{-1}(U_i)$ with $U_i \times X_H$ ($X_H = \pi_1(M)/H$). Also, we choose some points $z_i \in U_i$ and assume that diameters of all $U_i$ in the path metric on $N$ induced by a Riemannian metric are uniformly bounded by a constant.

Let $\phi : N' \to \mathbb{R}$ be a function uniformly continuous with respect to the path metric induced by the Riemannian metric pulled back from $N$. For every $i \in I$ we define a function $\phi_i : r^{-1}(U_i) \to \mathbb{R}$ by the formula

$$\phi_i(z, g) := \phi(z_i, g), \quad z \times g \in r^{-1}(U_i).$$

Then from the uniform continuity of $\phi$ and boundedness of $\operatorname{diam}(U_i)$ for all $i$ we obtain that there exists a constant $c$ such that

$$|\phi(v) - \phi_i(v)| \leqslant c \quad \text{for every } v \in r^{-1}(U_i),\ i \in I. \tag{4.1}$$

Define a locally constant cocycle $\phi_{ij}$ on the open cover $\{r^{-1}(U_i)\}$ of $\widetilde{M}'$ by the formula

$$\phi_{ij}(v) = \phi_i(v) - \phi_j(v) \quad \text{for } v \in r^{-1}(U_i \cap U_j).$$

Then from (4.1) by the triangle inequality we get

$$\sup_{i,j,v} |\phi_{ij}(v)| \leqslant 2c. \tag{4.2}$$

This inequality implies that rewriting cocycle $\{\phi_{ij}\}$ in the coordinates on $\widetilde{M}$ (i.e. taking its direct image $\{r_*(\phi_{ij})\}$ with respect to $r$) we can regard it as a holomorphic cocycle on the cover $\mathcal{U}$ with values in the Banach vector bundle $E_{\infty,1}(X_H)$ with fibre $l_{\infty,1}(X_H)$ defined on $\widetilde{M}$ (see Example 3.2(b)). This correspondence is described in [Bru04, Proposition 2.4]. Since $\mathcal{U}$ is acyclic and $\widetilde{M}$ is Stein, from the above construction, a version of Cartan's B theorem for coherent Banach sheaves (see [Bun68]), and the classical Leray theorem we obtain as in [Bru04] that there are holomorphic functions $f_i \in \mathcal{O}(r^{-1}(U_i))$ such that:

(1) for every compact set $K \subset U_i$,

$$\sup_{y \in r^{-1}(K)} |f_i(y)| < \infty;$$

(2)

$$f_i(z) - f_j(z) = \phi_{ij}(z) \quad \text{for } z \in r^{-1}(U_i \cap U_j).$$

Let $\mathcal{V} = \{V_j\}_{j \in J}$ be a refinement of $\mathcal{U}$ such that every $V_j$ is open and relatively compact in some $U_{i_j}$. Then condition (1) implies that

$$\sup_{y \in r^{-1}(V_j)} |f_{i_j}(y)| < \infty. \tag{4.3}$$

Finally, define a function $\widetilde{f} \in \mathcal{O}(\widetilde{M}')$ by the formula

$$\widetilde{f}(z) := \phi_i(z) - f_i(z), \quad z \in r^{-1}(U_i). \tag{4.4}$$

Since $\overline{M} \subset \widetilde{M}$ is a compact set, there is a finite subcover of $\mathcal{V}$ that covers $\overline{M}$. From here, (4.3) and (4.1) for the restriction $f_\phi := \widetilde{f}|_{\overline{M'}}$ we obtain (for some $C$)

$$|f_\phi(z) - \phi(z)| < C \quad \text{and} \quad |df_\phi(z)| < C \quad \text{for any } z \in M'. \tag{4.5}$$

The proof of the theorem is complete. □

*Proof of Corollary 2.2.* We retain the notation of the proof of Theorem 2.1.

Let $d_o := d(o, \cdot)$ be the distance on $N'$ from a fixed point $o \in M'$ and let $f := f_\phi$ be the function from Theorem 2.1 for $\phi = d_o$. Consider a finite open cover $\{U_i\}_{i=1}^{l}$ of $\overline{M}$ such that every $U_i \subset\subset \widetilde{M}$ is biholomorphic to an open Euclidean ball. As above we identify $r^{-1}(U_i) \subset N'$ with $U_i \times X_H$. Fix an element $e \in X_H$ and set $o_i(z) = (z, e) \in r^{-1}(U_i)$ for every $z \in U_i$, $1 \leqslant i \leqslant l$, and $d_{o_i(z)}(v) := d(o_i(z), v)$, $v \in N'$. Then from compactness of every $\overline{U_i}$ by the triangle inequality we



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get

$$|d_{o_i(z)}(v) - d_o(v)| \leqslant a, \quad 1 \leqslant i \leqslant l, \tag{4.6}$$

for some constant $a$. By $B_{z,i}(R)$ we denote the open ball on $r^{-1}(z)$ of radius $R$ centered at $o_i(z)$ with respect to the induced metric $d|_{r^{-1}(z)}$. Also, by $\#A$ we denote the number of elements of $A$. Now we prove

Lemma 4.1. *There is $k \in \mathbb{N}$ such that*

$$\#B_{z,i}(R) \leqslant e^{kR}, \quad 1 \leqslant i \leqslant l.$$

*Proof.* Let $\widetilde{r} : N_u \to N$ be the universal covering of $N$ and $r' : N_u \to N'$ be the intermediate covering, i.e. $\widetilde{r} = r \circ r'$. We equip $N_u$ with the path metric $\widetilde{d}$ induced by the Riemannian metric pulled back from $N$, the same as in the definition of the metric $d$ on $N'$. Let $\widetilde{o}_i(z) \in \widetilde{r}^{-1}(z)$ be such that $r'(\widetilde{o}_i(z)) = o_i(z)$. By $\widetilde{B}_{\widetilde{z},i}(R)$ we denote the open ball on $\widetilde{r}^{-1}(z)$ of radius $R$ centered at $\widetilde{o}_i(z)$ with respect to the metric $\widetilde{d}|_{\widetilde{r}^{-1}(z)}$. Let us check that $r'(\widetilde{B}_{\widetilde{z},i}(R)) = B_{z,i}(R)$.

Indeed, let $y \in \widetilde{B}_{\widetilde{z},i}(R)$ and $\gamma_y$ be a path joining $\widetilde{o}_i(z)$ and $y$ in $N_u$ whose length is less than $R$ (such a path exists by the definition of $\widetilde{d}$). Then $r'(\gamma_y)$ is a path joining $o_i(z)$ and $r'(y)$ in $N'$. By the definition of the metrics on $N_u$ and $N'$ the length of $r'(\gamma_y)$ does not exceed the length of $\gamma_y$. In particular, it is less than $R$. Thus $d_{o_i(z)}(r'(y)) < R$, i.e., $r'(y) \in B_{z,i}(R)$. Conversely, let $w \in B_{z,i}(R)$ and let $\gamma_w$ be a path in $N'$ joining $o_i(z)$ and $w$ with length less than $R$. By the covering homotopy theorem (see, e.g., [Hu59, ch. III]) there is a path $\widetilde{\gamma}_w \subset N_u$ that covers $\gamma_w$ and joins $\widetilde{o}_i(z)$ with some point $\widetilde{w}$ such that $r'(\widetilde{w}) = w$. Moreover, by the definition, the length of $\widetilde{\gamma}_w$ is the same as the length of $\gamma_w$. In particular, it is less than $R$. Thus, $\widetilde{w} \in \widetilde{B}_{\widetilde{z},i}(R)$. This shows that $r'(\widetilde{B}_{\widetilde{z},i}(R)) = B_{z,i}(R)$. In turn, the latter implies that

$$\#B_{z,i}(R) \leqslant \#\widetilde{B}_{\widetilde{z},i}(R). \tag{4.7}$$

Next, let $A$ be a finite set of generators of $\pi_1(N)$ (recall that condition (1.1) implies that $\pi_1(N)$ is finitely generated). By $d_w$ we denote the word metric on $\pi_1(N)$ with respect to $A$. Now, from compactness of every $\overline{U_i}$ by the Švarc–Milnor lemma (see, e.g., [BH99, p. 140]) we obtain that there exists a constant $c$ such that for any $z \in U_i$, $1 \leqslant i \leqslant l$, and $g, h \in \pi_1(N)$,

$$c^{-1} d_w(g,h) \leqslant \widetilde{d}((z,g),(z,h)) \leqslant c d_w(g,h) \tag{4.8}$$

(Here we identify $\widetilde{r}^{-1}(U_i)$ with $U_i \times \pi_1(N)$ as in Example 3.2(a).) Let $B_R \subset \pi_1(N)$ be the open ball of radius $R$ centered at 1 with respect to $d_w$. Then there is a natural number $\widetilde{k}$ such that

$$\#B_R \leqslant e^{\widetilde{k}R} \quad \text{for any } R \geqslant 0. \tag{4.9}$$

From here, (4.8) and (4.7) we get for $k := \widetilde{k}c$

$$\#B_{z,i}(R) \leqslant e^{kR}, \quad 1 \leqslant i \leqslant l. \qquad \square$$

We proceed with the proof of the corollary. Let us define $\alpha := (k+1)/2$ and prove that $F = e^{-\alpha f} \in \mathcal{H}_{2,1}(M') \cap C(\overline{M'})$. Since $\mathcal{H}_{2,1}(M') \hookrightarrow H_1^2(M')$ (see § 2.3), this implies the required statement.

Let $z \in U_i$ for some $1 \leqslant i \leqslant l$. We will estimate $|F|_{2,1,z}$ (see (2.2)). By the definition using



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inequalities (4.5), (4.6) and Lemma 4.1 we obtain

$$|F|_{2,1,z}^2 = \sum_{y\in r^{-1}(z)} |e^{-\alpha f(y)}|^2 \leqslant e^{2\alpha(a+C)} \cdot \sum_{y\in r^{-1}(z)} e^{-2\alpha d_{o_i(z)}(y)}$$
$$\leqslant e^{2\alpha(a+C)} \cdot \sum_{R=0}^{\infty} e^{-2\alpha R} \cdot \# B_{z,i}(R)$$
$$\leqslant e^{2\alpha(a+C)} \cdot \sum_{R=0}^{\infty} e^{(-2\alpha+k)R} = \frac{e^{2\alpha(a+C)+1}}{e-1}.$$

Therefore,

$$|F|_{2,1}^M := \sup_{z\in M} |F|_{2,1,z} \leqslant \left(\frac{e^{2\alpha(a+C)+1}}{e-1}\right)^{1/2}.$$

This shows that $F \in \mathcal{H}_{2,1}(M') \cap C(\overline{M'})$. □

*Remark* 4.2. (1) Using some construction from [Bru04] one can prove that the constant $C$ in Theorem 2.1 for $\phi = d_o$ (see (4.5)) can be chosen independent of the covering $r : M' \to M$. It depends only on $M$, $\widetilde{M}$ and the Riemannian metric on $N$.

(2) Consider the holomorphic map $f : M' \to \mathbb{C}$ with $f$ as in Corollary 2.2. Then

$$f(M') \subset S := \{z \in \mathbb{C} : |\operatorname{Im} z| < C, -C < \operatorname{Re} z < \infty\}$$

where $C$ is the constant in Theorem 2.1 for $\phi = d_o$. Let $B_t = \{x \in M' : d_o(x) < t\}$ be the open ball in $M'$ centered at $o$ of radius $t$ and $S_R := \{z \in S : \operatorname{Re} z \geqslant R\}$. Then

$$f^{-1}(S_R) \subset M' \setminus B_{R-C} \quad \text{for } R > C \quad \text{and} \quad f^{-1}(S \setminus S_R) \subset B_{R+C}.$$

Using such $f$ one can construct holomorphic functions on $M'$ decreasing faster than the function $F$ from Corollary 2.2. Actually, let $l : \mathbb{R}_+ \to \mathbb{R}_+$ be a continuous function monotonically increasing for $x \geqslant R_0$. Consider a holomorphic function $g$ on $S$ satisfying

$$\log|g(x+iy)| \geqslant l(x) \quad \text{for } x \geqslant R_0 \quad \text{and} \quad \inf_{z\in S} |g(z)| > 0. \tag{4.10}$$

Then one can easily check that the function $G = g \circ f \in \mathcal{O}(M') \cap C(\overline{M'})$ satisfies

$$|G(z)| \geqslant e^{l(d_o(z)-C)} \quad \text{for } d_o(z) \geqslant R_0 + C.$$

In particular, $H = 1/G \in \mathcal{O}(M') \cap C(\overline{M'})$ satisfies (for some $c_1 > 0$)

$$|H(z)| \leqslant c_1 e^{-l(d_o(z)-C)}, \quad z \in \overline{M'} \setminus B_C. \tag{4.11}$$

Observe that the Harnack inequality for positive harmonic functions implies for $g$ as in (4.10) (for some positive $\widetilde{c}_1$, $\widetilde{c}_2$)

$$l(x) \leqslant \log|g(x)| \leqslant \widetilde{c}_1 e^{\widetilde{c}_2 x}, \quad x \geqslant R_0.$$

This and the properties of $f$ impose the following restriction on the decay of $H$:

$$|H(z)| \geqslant e^{-\widetilde{c}_3 e^{\widetilde{c}_2 d_o(z)}}, \quad z \in M'. \tag{4.12}$$

*Example* 4.3. As the function $g$ in (4.10) one can take, e.g., $g(z) = e^{z^n}$ for $n \in \mathbb{N}$ (in this case $l(x) = (1-\epsilon)x^n$ for any $\epsilon > 0$), or $g(z) = e^{C_1 e^{C_2 z}}$ for $C_1 > 0$ and $0 < C_2 < \pi/2C$ with $C$ as above (in this case $l(x) = C_1 \cos(C_2 C) e^{C_2 x}$). For the latter example estimate (4.11) shows that the lower bound (4.12) of the decay of $H$ is attainable.



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## 5. Proof of Theorem 2.4

Suppose that $M \subset\subset N$ are domains in a Stein manifold, $M$ is strictly pseudoconvex and $\pi_1(M) = \pi_1(N)$. Using the Remmert embedding theorem (see, e.g., [GR77]) we may assume without loss of generality that $N$ is a domain in a closed complex submanifold of some $\mathbb{C}^k$. Let $r : N' \to N$ be an unbranched covering of $N$. As usual, we set $M' = r^{-1}(M)$ and $bM' = r^{-1}(bM)$ where $bM$ is the boundary of $M$. We must show that every point in $bM'$ is a peak point for $\mathcal{H}_{p,1}(M')$, $1 \leqslant p \leqslant \infty$. In our proof we use a result on uniform estimates for solutions of certain $\overline{\partial}$-equations on $M'$. To its formulation we first introduce the corresponding class of $(0,1)$-forms on $M'$.

Let $\{V_i\}_{i\in I}$ be a finite acyclic open cover of $\overline{M}$ by relatively compact complex coordinate systems. We set $U_i \cap M$ and consider the open cover $\mathcal{U} = \{U_i\}_{i\in I}$ of $M$. Let $X_H$ be the fibre of $N'$ with $H = \pi_1(N')$. Using the construction of Example 3.2(a) we identify $r^{-1}(U_i)$ with $U_i \times X_H$. Let $\omega$ be a $(0,1)$-form on $M'$. Then in local coordinates on $r^{-1}(U_i)$ it is presented as

$$\omega(v,x) = \sum_{j=1}^{n} a_j(v,x)\, d\overline{v}_j \quad \text{for } v \times x \in U_i \times X_H$$

where $v = (v_1, \dots, v_n)$ are coordinates on $U_i$. Consider every $a_j$ as a function on $U_i$ with values in the space of functions on $X_H$. We assume that for every $i \in I$

$$a_j \in C^\infty(U_i, l_{p,1}(X_H)), \quad 1 \leqslant j \leqslant n. \tag{5.1}$$

Then for such an $\omega$ its direct image $r_*(\omega)$ is a bounded $C^\infty$ form with values in the Banach vector bundle $E_{p,1}(X_H)$ (see § 3.1). Also, we assume that the norm of $\omega$ defined by the formula

$$\|\omega\| := \sup_{i\in I} \max_{1\leqslant j\leqslant n} |a_j|_{p,1}^{U_i} \tag{5.2}$$

is finite. (Recall that $|\cdot|_{p,1}^{U_i}$ are norms on $C_{p,1}(r^{-1}(U_i))$, see (3.5).)

Proposition 5.1. *There is a constant $C > 0$ and for each $\overline{\partial}$-closed $(0,1)$-form $\omega$ satisfying (5.1) there is a function $f \in C^\infty(M') \cap C^b_{p,1}(M')$ such that*

$$\overline{\partial} f = \omega \quad \text{and} \quad |f|_{p,1}^{M} \leqslant C\|\omega\|.$$

*Proof.* We apply the operators $L_z$ from Theorem 3.4 to $\omega$. Namely, let us define a form $\widetilde{\omega}$ on $M$ by the formula

$$\widetilde{\omega}(v,z) := \sum_{j=1}^{n} (L_v a_j(v,\cdot))(z)\, d\overline{v}_j \quad \text{for } v \times z \in U_i \times M',\ i \in I.$$

It is readily seen that $\widetilde{\omega}$ is a bounded $\overline{\partial}$-closed $C^\infty$ form on $M$ with values in $\mathcal{H}_{p,1}(M')$ (i.e. the form with values in the holomorphically trivial Banach vector bundle on $M$ whose fibre is $\mathcal{H}_{p,1}(M')$). We define the norm of $\widetilde{\omega}$ by

$$\|\widetilde{\omega}\| := \sup_{i\in I, v\in U_i} \max_{1\leqslant j\leqslant n} |L_v a_j(v,\cdot)|_{p,1}^{M} \tag{5.3}$$

where $|\cdot|_{p,1}^{M}$ is norm on $\mathcal{H}_{p,1}(M')$. Then according to Theorem 3.4 there is a constant $c$ (independent of $\omega$) such that

$$\|\widetilde{\omega}\| \leqslant c\|\omega\|. \tag{5.4}$$

Further, we use [Heu83, Lemma 1]. According to this lemma *there exist a strictly pseudoconvex domain $W \subset \mathbb{C}^k$ with $C^2$ boundary such that $W \cap N = M$ and a holomorphic map $\pi$ from a neighbourhood $U(\overline{W})$ of $\overline{W}$ onto $U(\overline{W}) \cap N$ such that $\pi(W) = M$ and $\pi|_{U(\overline{W})\cap N}$ is the identity map.*

Using this result we obtain that the pullback $\pi^*\widetilde{\omega}$ with respect to $\pi$ is a bounded $\overline{\partial}$-closed $C^\infty$ form on $W$ with values in $\mathcal{H}_{p,1}(M')$. Moreover, there is a constant $c'$ (depending on $\pi$ and $W$) such



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that

$$\|\pi^*\widetilde{\omega}\| \leqslant c'\|\widetilde{\omega}\|. \tag{5.5}$$

Here, for $\pi^*\widetilde{\omega}(w,\cdot) = \sum_{j=1}^{k} \widetilde{a}_j(w,\cdot)\, d\overline{w}_j$, $w = (w_1, \dots, w_k) \in \mathbb{C}^k$, we define

$$\|\pi^*\widetilde{\omega}\| := \sup_{w\in W} \max_{1\leqslant j\leqslant k} |\widetilde{a}_j(w,\cdot)|_{p,1}^{M}.$$

In [SH80], uniform estimates for solutions of $\overline{\partial}$-equations on so-called pseudoconvex polyhedra were obtained by means of global integral formulas. This class contains, in particular, strictly pseudoconvex domains with $C^2$ boundaries. Note that the estimates in [SH80] remain valid if one solves Banach-valued $\overline{\partial}$-equations. Therefore, from the results of [SH80] we obtain that *there exists a bounded $C^\infty$ function $h$ on $W$ with values in $\mathcal{H}_{p,1}(M')$ such that $\overline{\partial}h = \pi^*\widetilde{\omega}$. Moreover,*

$$\|h\| \leqslant c''\|\pi^*\widetilde{\omega}\| \tag{5.6}$$

*for some $c''$ (depending on $W$ only). Here*

$$\|h\| := \sup_{w\in W} |h(w,\cdot)|_{p,1}^{M}.$$

Finally, define a function $f$ on $M'$ by the formula

$$f(z) := h(r(z), z), \quad z \in M'.$$

Using that $r$ is holomorphic, $\widetilde{\omega}(r(z), z) = \omega(z)$, $z \in M'$, and $(\pi^*\widetilde{\omega})|_M = \widetilde{\omega}$ we easily conclude that $\overline{\partial}f = \omega$. By the definition $f \in C^\infty(M') \cap C_{p,1}^b(M')$ and from (5.4)–(5.6) we have (for some $C$)

$$|f|_{p,1}^{M} \leqslant C\|\omega\|. \qquad \square$$

*Remark* 5.2. (1) A statement analogous to Proposition 5.1 is valid for a similar class of bounded $\overline{\partial}$-closed $(0,q)$-forms on $M'$.

(2) Using the main result of [Heu83] and the estimates from [SH80] one can show that the result of Proposition 5.1 is also valid for coverings of non-degenerate pseudoconvex polyhedrons on Stein manifolds (see [Heu83] and [SH80] for the definition).

We proceed to the proof of Theorem 2.4. Take a point $z \in bM'$ and set $v = r(z) \in bM$. Let $U \subset\subset N$ be a simply connected coordinate neighbourhood of $v$ and let $W \subset N'$ be the neighbourhood of $z$ such that $r : W \to U$ is biholomorphic. Since $M$ is strictly pseudoconvex, $v$ is a peak point for $\mathcal{O}(U \cap M)$ for a sufficiently small $U$. Moreover, for such $U$ we can find $\widetilde{f} \in \mathcal{O}(U \cap M)$ with a peak point at $v$ such that $\widetilde{f} \in L^q(U \cap M)$ for all $1 \leqslant q < \infty$ (see [GHS98, p. 575]). Then $f := (r^*\widetilde{f})|_{W\cap M'}$ has a peak point at $z$ and $f \in L^q(W \cap M')$ for all $1 \leqslant q < \infty$. Next, let $\widetilde{\rho} \in C^\infty(U)$ be a cut-off function that equals 1 in a neighbourhood $O \subset\subset U$ of $v$ and 0 outside $U$. Consider its pullback $\rho := (r^*\widetilde{\rho})|_W \in C^\infty(W)$. Clearly the $(0,1)$-form $\omega = \overline{\partial}(\rho f)$ on $M'$ satisfies the conditions of Proposition 5.1. Then this proposition implies that there exists a function $h \in C^\infty(M') \cap C_{p,1}^b(M')$ such that $\overline{\partial}h = \omega$. Finally, consider the function $h_z := \rho f - h$. Then $h_z$ is holomorphic, has a peak point at $z_0$ and belongs to $H_1^p(M')$ for $1 \leqslant p < \infty$ by the choice of $f$. Also, for $p = \infty$ the function $h_z$ is bounded outside $W$.

The proof of the theorem is complete. $\square$

## 6. Proofs of Theorem 2.7 and Corollary 2.9

*Proof of Theorem 2.7.* Let $M \subset\subset \widetilde{M} \subset N$ be manifolds satisfying condition (1.1) with $\dim_{\mathbb{C}} M \geqslant 2$. Let $D \subset\subset M$ be an open subset whose boundary $bD$ is a connected $C^k$ submanifold of $M$ ($1 \leqslant k \leqslant$



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$\infty$). Consider a covering $r : N' \to N$ and set $M' = r^{-1}(M)$, $D' = r^{-1}(D)$ and $bD' = r^{-1}(bD)$. Let $\psi : N' \to \mathbb{R}_+$ be such that $\log \psi$ is uniformly continuous with respect to the path metric induced by a Riemannian metric pulled back from $N$. Let $f \in C^s(bD')$, $0 \leqslant s \leqslant k$, be a CR-function satisfying the hypotheses of Theorem 2.7.

(A) First, we will prove the theorem for $s = 0$ under the additional assumption

$$f \in C_{p,\psi}(bD') \quad \text{for } p \in [1,\infty) \quad \text{and} \quad f \in C_{1,1}(bD') \quad \text{for } p = \infty. \tag{6.1}$$

(We use here that $C_{1,1}(bD') \subset C_{\infty,1}(bD') = C_{\infty,\psi}(bD')$, see Definition 3.3.)

For a CR-function $f$ satisfying (6.1) we define a continuous $\mathcal{H}_{p,\psi}(M')$-valued function $F$ on $bD'$ by the formula

$$F(v) := L_v(f|_{r^{-1}(v)}), \quad v \in bD,$$

where $L_v$ are operators from Theorem 3.4, see § 3.3.

Lemma 6.1. *$F$ is a $\mathcal{H}_{p,\psi}(M')$-valued continuous CR-function.*

*Proof.* Let $U \subset M$ be a simply connected coordinate neighbourhood of a point $x \in bD$. It suffices to check that $F|_{U\cap bD}$ satisfies the required property. Note that by Proposition 3.5

$$[F(v)](z) = \sum_{y\in r^{-1}(x)} \widetilde{f}_y(v) H_y(v,z), \quad v \times z \in (U \cap bD) \times M', \tag{6.2}$$

where $\widetilde{f}_y(v) = f(s_y(v))$, $v \in U \cap bD$, and $s_y : U \to V_y$ is a biholomorphic map onto the connected component $V_y$ of $r^{-1}(U)$ containing $y$. By the definition of operators $L_v$ functions $H_y$ are restrictions to $bD \times M'$ of some holomorphic functions on $U \times M'$. Moreover, Proposition 3.5 implies that the series in (6.2) converges uniformly to $F$ on every compact subset of $(U \cap bD) \times M'$. Next, since $f|_{V_y\cap bD'}$ is a continuous CR-function, $\widetilde{f}_y$ is a continuous CR-function on $U \cap bD$. Also, by the definition of $H_y$, for a fixed $z \in M'$ every $H_y(\cdot, z)$ is a continuous CR-function on $U \cap bD$. Hence, $\widetilde{f}_y \cdot H_y(\cdot, z)$ is a continuous CR-function on $U \cap bD$, as well. Indeed, for each $(n, n-2)$-form $\omega$ with a compact support in $U$ we have

$$\int_{U\cap bD} \widetilde{f}_y(v) \cdot H_y(v,z)\overline{\partial}\omega(v) = \int_{U\cap bD} \widetilde{f}_y(v)\overline{\partial}(H_y(v,z)\cdot\omega(v)) = 0$$

because $\widetilde{f}_y$ is CR. Since the series in (6.2) converges uniformly to $[F(\cdot)](z)$ on every compact subset of $(U \cap bD) \times z$, every $[F(\cdot)](z)$, $z \in M'$, is a continuous CR-function on $U \cap bD$. This implies the required statement. □

Further, since $[F(v)](z)$ from Lemma 6.1 is holomorphic in $z \in M'$, we can expand it in the Taylor series in a complex coordinate neighbourhood $U_z$,

$$[F(v)](w) = \sum_{0\leqslant|\alpha|<\infty} F_\alpha(v)w^\alpha, \quad v \in bD. \tag{6.3}$$

Here $\alpha = (\alpha_1,\dots,\alpha_s) \in (\mathbb{Z}_+)^s$, $|\alpha| = \sum_{i=1}^s \alpha_i$, $w^\alpha = w_1^{\alpha_1}\dots w_n^{\alpha_n}$ and $w = (w_1,\dots,w_n)$ are coordinates on $U_z$ such that $w(z) = 0$. Now, from Lemma 6.1 it follows that each $F_\alpha$ in (6.3) is a continuous CR-function on $bD$. Then by Theorem 3.14 of Harvey [Har77], for every $F_\alpha$ there exists a function $\widetilde{F}_\alpha \in \mathcal{O}(D)\cap C(\overline{D})$ such that $\widetilde{F}_\alpha|_{bD} = F_\alpha$. Also, for a sufficiently small $U_z$ using estimates of the Cauchy integrals for derivatives of a holomorphic function and compactness of $bD$ we get, from (6.3),

$$M := \sup_{\alpha, v\in bD} |F_\alpha(v)| < \infty.$$

Thus, by the maximum modulus principle

$$\sup_{\alpha, v\in\overline{D}} |\widetilde{F}_\alpha(v)| = M < \infty.$$

The latter implies that, for a sufficiently small $U_z$, the series

$$\widetilde{F}_z(v,w) = \sum_{0\leqslant|\alpha|<\infty} \widetilde{F}_\alpha(v)w^\alpha, \quad v\times w\in\overline{D}\times U_z,$$

converges absolutely and uniformly. Hence, $\widetilde{F}_z\in\mathcal{O}(D\times U_z)\cap C(\overline{D}\times U_z)$. Further, assume that for $y,z\in M'$ we have $U_y\cap U_z\neq\emptyset$. Then for every $w\in U_y\cap U_z$ and $v\in bD$

$$\widetilde{F}_y(v,w)-\widetilde{F}_z(v,w) = [F(v)](w)-[F(v)](w) = 0.$$

This leads to the identity

$$\widetilde{F}_y(\cdot,w) = \widetilde{F}_z(\cdot,w), \quad w\in U_y\cap U_z.$$

Thus, we can define a function $F\in\mathcal{O}(D\times M')\cap C(\overline{D}\times M')$ by the formula

$$\widetilde{F}(v,w) := \widetilde{F}_z(v,w), \quad v\times w\in\overline{D}\times U_z. \tag{6.4}$$

Lemma 6.2. *We have $\widetilde{F}(v,\cdot)\in\mathcal{H}_{p,\psi}(M')$ for any $v\in D$.*

*Proof.* Observe that the evaluation at $v\in D$ is a linear continuous functional on the Banach space $\mathcal{O}(D)\cap C(\overline{D})$ equipped with supremum norm. Identifying $\mathcal{O}(D)\cap C(\overline{D})$ with its trace space on $bD$ and using the Hahn–Banach and Riesz theorems we have

$$h(v) = \int_{bD} h(\xi)\,d\mu(\xi), \quad h\in\mathcal{O}(D)\cap C(\overline{D}),$$

where $\mu$ is a complex regular Borel measure on $bD$ with the total variation $\operatorname{Var}\mu = 1$. Thus for every fixed $w\in M'$ we have

$$\widetilde{F}(v,w) = \int_{bD}[F(\xi)](w)\,d\mu(\xi).$$

Now, by the definition of the norm on $\mathcal{H}_{p,\psi}(M')$ using the triangle inequality, the identity $\widetilde{F}(v,\cdot) = F(v)$, $v\in bD$, and the fact that $F$ is a continuous $\mathcal{H}_{p,\psi}(M')$-valued function on $bD$ we obtain

$$\begin{aligned}
|\widetilde{F}(v,\cdot)|^M_{p,\psi} &:= \sup_{z\in M}\biggl(\sum_{y\in r^{-1}(z)}|\widetilde{F}(v,y)|^p\psi(y)\biggr)^{1/p}\\
&\leqslant \sup_{z\in M}\biggl(\sum_{y\in r^{-1}(z)}\biggl(\int_{bD}|\widetilde{F}(\xi,y)||\,d\mu(\xi)|\biggr)^p\psi(y)\biggr)^{1/p}\\
&\leqslant \sup_{z\in M}\biggl(\int_{bD}\biggl(\sum_{y\in r^{-1}(z)}|[F(\xi)](y)|^p\psi(y)\biggr)^{1/p}|\,d\mu(\xi)|\biggr) \leqslant \sup_{\xi\in bD}|F(\xi)|^M_{p,\psi}<\infty. \qquad\square
\end{aligned}$$

Further, set

$$f'(z) := \widetilde{F}(r(z),z), \quad z\in\overline{D'}. \tag{6.5}$$

Then using the inequalities of Lemma 6.2 we get

$$f'\in\mathcal{O}(D')\cap C(\overline{D'}), \quad f'|_{bD'} = F|_{bD'} = f, \quad \text{and}$$

$$|f'|_{p,\psi,z} := \biggl(\sum_{y\in r^{-1}(z)}|f'(y)|^p\psi(y)\biggr)^{1/p} \leqslant \sup_{\xi\in bD}|F(\xi)|^M_{p,\psi}\leqslant c|f|^{bD}_{p,\psi}, \quad z\in D,$$



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see (3.5) for the definition of $|\cdot|^{bD}_{p,\psi}$. (Here the last inequality follows directly from Theorem 3.4.) The latter implies that $f'|_{r^{-1}(z)} \in l_{p,\psi,z}(M')$, $z \in D$, see § 2.3. Thus, $f' \in \mathcal{H}_{p,\psi}(D') \cap C(\overline{D'})$ and

$$|f'|^{D}_{p,\psi} := \sup_{z\in D} |f'|_{p,\psi,z} \leqslant c \sup_{z\in bD} |f|_{p,\psi,z} \; (:= c|f|^{bD}_{p,\psi}). \tag{6.6}$$

This completes the proof of the theorem for $s = 0$ under assumption (6.1).

(B) Let us consider the general case of a continuous CR-function $f$ on $bD'$ satisfying

$$f|_{r^{-1}(x)} \in l_{p,\psi,x}(M') \quad \text{for any } x \in D \quad \text{and} \quad m := \sup_{x\in bD} |f|_{p,\psi,x} < \infty. \tag{6.7}$$

According to Remark 4.2(2) and Example 4.3 there is a constant $c > 0$ such that for any $0 < c_1 < c$ and $c_2 > 0$ there exists a function $F_{c_1,c_2} \in \mathcal{O}(M') \cap C(\overline{M'})$ satisfying

$$e^{-c_3 e^{c_1 d_o(z)}} \leqslant |F_{c_1,c_2}(z)| \leqslant e^{-c_2 e^{c_1 d_o(z)}} \quad \text{for all } z \in \overline{M'} \tag{6.8}$$

with $c_3$ depending on $c_2$, $c_1$, $c$, $M$, $M'$ such that $c_3 \to 0$ as $c_2 \to 0$. (Recall that $d_o$, $o \in M'$, is the distance on $N'$ defined as in Corollary 2.2.) Define a continuous CR-function $f_{c_1,c_2}$ by the formula

$$f_{c_1,c_2}(z) := f(z)F_{c_1,c_2}(z), \quad z \in bD'.$$

Lemma 6.3. *The continuous CR-function $f_{c_1,c_2}$ satisfies assumption (6.1).*

*Proof.* Note that for any $l \in \mathbb{N}$ there is a nonnegative $r$ such that

$$e^{-c_2 e^{c_1 d_o(z)}} < e^{-l d_o(z)} \quad \text{for } d_o(z) > r,\; z \in \overline{M'}.$$

Let $U \subset\subset M$ be a neighbourhood of $\overline{D}$ and $U' = r^{-1}(U) \subset M'$. From the above inequality arguing as in the proof of Corollary 2.2 we obtain that $F_{c_1,c_2} \in \mathcal{H}_{p,1}(U')$ for any $p \in [1,\infty]$. Then from [Bru06a, Proposition 2.4] follows that $F_{c_1,c_2}|_{bD'}$ belongs to $C_{p,1}(bD')$ for all $p$.

Next, take a point $x \in bD$ and prove that $f_{c_1,c_1}$ is $C_{p,\psi}$-continuous over $x$. Let $U_x \subset M$ be a complex (simply connected) coordinate neighbourhood of $x$. We will identify $r^{-1}(U_x)$ with $U_x \times X_H$ where $X_H$ is the fibre of $r : M' \to M$. Consider a sequence $\{x_n\} \subset U_x \cap bD$ converging to $x$. For $g \in X_H$ put $a_n(g) := f(x_n, g)$, $a(g) := f(x, g)$, $b_n(g) := F_{c_1,c_2}(x_n, g)$, $b(g) := F_{c_1,c_2}(x, g)$ and $c_n := a_n b_n$, $c := ab$. Then we must check that

$$\lim_{n\to\infty} |c - c_n|_{p,\psi,x} := \lim_{n\to\infty} \left( \sum_{g\in X_H} |c(g) - c_n(g)|^p \psi(x, g) \right)^{1/p} = 0.$$

Using the triangle inequality we have

$$|c - c_n|_{p,\psi,x} \leqslant |(a - a_n)b|_{p,\psi,x} + |a_n(b - b_n)|_{p,\psi,x} := I + II.$$

According to (6.8) for any $\epsilon > 0$ we can decompose $b$ in the sum $b' + b''$ where $b' = 0$ outside a finite subset $S_\epsilon \subset X_H$ and $b'' = 0$ on $S_\epsilon$ such that $|b''(g)| < \epsilon$ for all $g$. Note also that $|b'(g)| \leqslant 1$ for all $g$. Also, (6.7) and uniform continuity of $\log \psi$ on the compact set $bD$ imply that $|a - a_n|_{p,\psi,x} \leqslant km$ for some $k > 0$. Finally, by continuity of $f$ on $bD'$ we can find a number $N$ such that for any $n \geqslant N$ we have $|(a - a_n)\chi_\epsilon|_{p,\psi,x} < \epsilon$, where $\chi_\epsilon$ is the characteristic function of $S_\epsilon$. Using all of these facts we get, for $n \geqslant N$,

$$I \leqslant |(a - a_n)\chi_\epsilon b'|_{p,\psi,x} + |(a - a_n)b''|_{p,\psi,x} \leqslant \epsilon + km\epsilon = (1 + km)\epsilon.$$

To estimate $II$ observe that from (6.7) and uniform continuity of $\log \psi$ follow that for each $g \in X_H$

$$(|a_n(g)|^p \psi(x, g))^{1/p} \leqslant |a_n|_{p,\psi,x} \leqslant k'|a_n|_{p,\psi,x_n} \leqslant k'm$$



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(for some $k'$). Moreover, since $F_{c_1,c_2}|_{bD'} \in C_{p,1}(bD')$, there is an integer $N'$ such that for any $n \geqslant N'$ we have $|b - b_n|_{p,1,x} < \epsilon$. These two inequalities yield for $n \geqslant N'$

$$II \leqslant \sup_{g \in X_H} (|a_n(g)|^p \psi(x,g))^{1/p} |b - b_n|_{p,1,x} \leqslant k' m\epsilon.$$

Combining the estimates for $I$ and $II$ we obtain

$$\lim_{n \to \infty} |c - c_n|_{p,\psi,x} = 0.$$

This is equivalent to $C_{p,\psi}$-continuity of $f_{c_1,c_2}$ over $x$.

Similarly one can check that if $p = \infty$, then $f_{c_1,c_2}$ belongs to $C_{1,1}(bD')$. We leave it as an exercise to the readers. □

Let us finish the proof of the theorem. According to Lemma 6.3 and case (A) there is a function $f'_{c_1,c_2} \in \mathcal{O}(D') \cap C(\overline{D'})$ such that $f'_{c_1,c_2}|_{bD'} = f_{c_1,c_2}$. Note also that since $|F_{c_1,c_2}| \leqslant 1$ inequality (6.6) yields

$$|f'_{c_1,c_2}|^D_{p,\psi} \leqslant c \sup_{z \in bD} |f|_{p,\psi,z} := cm.$$

Consider $f' := f_{c_1,c_2}/F_{c_1,c_2}$. Then $f' \in \mathcal{O}(D') \cap C(\overline{D'})$ and $f'|_{bD'} = f$. The uniqueness property for holomorphic functions implies that $f'$ does not depend on $c_1$ and $c_2$. Since $F_{c_1,c_2}$ converges uniformly on compact subsets of $M'$ to 1 as $c_2 \to 0$ from the last inequality we get

$$|f'|^D_{p,\psi} \leqslant c \sup_{z \in bD} |f|_{p,\psi,z}.$$

Therefore, $f' \in \mathcal{H}_{p,\psi}(D') \cap C(\overline{D'})$.

The proof of the theorem for $s = 0$ is complete. If, in addition, $f \in C^s(bD)$ for $1 \leqslant s \leqslant k$, then, in fact, the extended function $f' \in C^s(\overline{D'})$ (see, e.g., Theorem 3.14 in [Har77], and the discussion that follows it). □

*Proof of Corollary 2.9.* Let us consider the function $F_{c_1,c_2}$ from (6.8). Suppose that $f \in \mathcal{O}(U')$ satisfies the hypotheses of Corollary 2.9 with $c_1$, $c_2$ and $c$ as in the definition of $F_{c_1,c_2}$. Then $f_{c_1,c_2} := fF_{c_1,c_2} \in H^\infty(U')$ with the norm bounded by 1. Further, by the hypotheses we can find a connected $C^\infty$ compact submanifold $bS \subset U$ that bounds a domain $S$ containing $K$. By Theorem 2.7 (applied to $S' := r^{-1}(S)$ and $bS' := r^{-1}(bS)$) the function $f_{c_1,c_2}$ admits an extension $f'_{c_1,c_2} \in H^\infty(D')$. Since $f'_{c_1,c_2} = f_{c_1,c_2}$ on $U'$ and $h(z) := \sup_{y \in r^{-1}(z)} |f'_{c_1,c_2}(y)|$ is a continuous plurisubharmonic function on $D$,

$$|f'_{c_1,c_2}|^D_\infty = \sup_{z \in U'} |f(z)| \leqslant 1.$$

Then the function $f' := f'_{c_1,c_2}/F_{c_1,c_2} \in \mathcal{O}(D')$ extends $f$ and satisfies

$$|f'(z)| \leqslant e^{c_3 e^{c_1 d_o(z)}}, \quad z \in D',$$

with $c_3$ as in (6.8). □

## 7. Proof of Theorem 2.10

Let $r : M_G \to M$ be the regular covering of $M$ satisfying condition (1.1) (for some $\widetilde{M}$ and $N$) with transformation group $G$. Let $G_1 \subset G$ be a subgroup of a finite index and let $r_1 : M_1 \to M$ be the covering with fibre $G/G_1$. Then there are coverings $N_1$ and $\widetilde{M}_1$ of $N$ and $\widetilde{M}$ with fibre $G/G_1$ such that $M_1 \subset\subset \widetilde{M}_1 \subset N_1$. Clearly this triple also satisfies condition (1.1). Thus, without loss of generality we may assume that $M := M_1$, $G := G_1$, $\widetilde{M} := \widetilde{M}_1$ and $N := N_1$, and so $G \in \mathcal{T}(M)$. The latter means that $G$ admits a linear representation $\rho$ into $GL_k(\mathbb{C})$ and that the flat vector bundle



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$E_\rho$ on $N$ associated with $\rho$ is topologically trivial. Then the restriction $E_\rho|_{\widetilde{M}}$ is topologically trivial. Since $\widetilde{M}$ is Stein, according to the Oka-Grauert principle (see [Gra58b]), $E_\rho|_{\widetilde{M}}$ is holomorphically trivial. In particular, $\rho$ can be obtained as the monodromy of an equation $dF = \omega F$ on $\widetilde{M}$ where $\omega$ is a matrix-valued holomorphic 1-form on $\widetilde{M}$ satisfying $d\omega - \omega \wedge \omega = 0$. Let $\widetilde{\omega} = r^*\omega$ be the pullback of $\omega$ on $\widetilde{M}_G = r^{-1}(\widetilde{M})$. (Here $r : N_G \to N$ is the regular covering of $N$ with the transformation group $G$ so that $M_G$ and $\widetilde{M}_G$ are domains in $N_G$.) Then there exists a function $\widetilde{F} \in \mathcal{O}(\widetilde{M}_G, GL_k(\mathbb{C}))$ such that $d\widetilde{F} = \widetilde{\omega}\widetilde{F}$. This follows from the fact that the monodromy of the last equation is the restriction of $\rho$ to $\pi_1(M_G)$ and so it is trivial (since $\pi_1(M_G) \subset \operatorname{Ker}\rho$). Note that $\widetilde{F}$ can be obtained by Picard iteration applied to $\widetilde{\omega}$. Since $\overline{M}$ is a compact subset of $\widetilde{M}$ (and so $\omega|_M$ is bounded), the Picard iteration produces, for some positive $c = c(M, \omega)$, the estimate

$$\|\widetilde{F}(z)\|_2 \leqslant e^{cd_o(z)}, \quad z \in M_G, \tag{7.1}$$

where $\|\cdot\|_2$ is the $l_2$-norm on $GL_k(\mathbb{C})$ and $o \in M_G$. (Here as before $d_o$ is the distance from $o$ in the path metric induced by a Riemannian metric pulled back from $N$.) Moreover, for every $z \in \widetilde{M}_G$ there exists a matrix $C_z \in GL_k(\mathbb{C})$ such that $\widetilde{F}(gz) = C_z^{-1}\rho(g)C_z$ for any $g \in G$. These are standard facts of the theory of flat connections. In particular, from the last identity we derive easily that $\widetilde{F}$ separates points in every orbit of the action of $G$ on $\overline{M}_G$.

Next, let $f$ be the function from Corollary 2.2. Then by (7.1) we get (for some $c_1 = c_1(f, \alpha)$)

$$\|e^{-\alpha f(z)}\widetilde{F}(z)\|_2 \leqslant c_1 e^{(c-\alpha)d_o(z)}, \quad z \in M_G.$$

From here, arguing as in the proof of Corollary 2.2, we deduce that for a sufficiently large $\alpha$ all entries of the matrix $e^{-\alpha f} \cdot \widetilde{F}$ belong to $\mathcal{H}_{2,1}(M_G) \cap C(\overline{M}_G)$. Now the family consisting of these entries and the function $e^{-\alpha f}$ separate all points in any orbit of the action of $G$ on $\overline{M}_G$; for otherwise, there are $x, y \in \overline{M}_G$, $y = gx$, $g \neq 1$, $g \in G$, such that $e^{-\alpha f(x)}\widetilde{F}(x) = e^{-\alpha f(y)}\widetilde{F}(y)$ and $e^{-\alpha f(x)} = e^{-\alpha f(y)}$. However, this implies that $\widetilde{F}(x) = \widetilde{F}(y)$, a contradiction. Finally, since $M \subset\subset \widetilde{M}$ and $\widetilde{M}$ is Stein, by the Remmert embedding theorem there are holomorphic functions $h_1, \dots, h_l$ from $H^\infty(M) \cap C(\overline{M})$ that separate all points in $\overline{M}$. We set $\widetilde{h}_i = e^{-\alpha f} r^* h_i$, $1 \leqslant i \leqslant l$. Then, by the definition, $\widetilde{h}_i \in \mathcal{H}_{2,1}(M_G) \cap C(\overline{M}_G)$ and so the family consisting of all $\widetilde{h}_i$, entries of $e^{-\alpha f}\widetilde{F}$ and $e^{-\alpha f}$ separates all points in $\overline{M}_G$.

The proof of Theorem 2.10 is complete.

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