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A NOTE ON A NEW APPROACH TO BOTH PRICE AND VOLATILITY JUMPS: AN APPLICATION TO THE PORTFOLIO MODEL

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Abstract

A new approach to jump diffusion is introduced, where the jump is treated as a vertical shift of the price (or volatility) function. This method is simpler than the previous methods and it is applied to the portfolio model with a stochastic volatility. Moreover, closed-form solutions for the optimal portfolio are obtained. The optimal closed-form solutions are derived when the value function is not smooth, without relying on the method of viscosity solutions.

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1. Introduction

Previous literature on jump-diffusion models adopted a diffusion-jump process (typically a Brownian motion and a Poisson process) in modelling a sudden increase (decrease) in the asset price (see, for example, [1, 5, 7, 8]). Moreover, previous models assumed a specific form of the utility function in order to derive closed-form solutions for the portfolio weights. For example, Ait-Sahalia et al. [1] and Liu et al. [7] assumed a power utility. In addition, obtaining closed-form solutions for the portfolio weights using previous models (especially, the model of Liu et al. [7]) is cumbersome; the limitations of their work is discussed by Ait-Sahalia et al. [1].

This paper presents a simpler model of price jumps. In doing so, it introduces a (stochastic) shift parameter (in the sense of Alghalith [3]) to represent the jump (shift) in a risky asset price. It also obtains a closed-form solution for the optimal portfolio without an orthogonal decomposition and the assumption of power utility (in contrast with the findings of Ait-Sahalia et al. [1]). Moreover, it derives optimal

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closed-form solutions when the value function is not smooth, without relying on the viscosity solutions adopted by the previous literature. This approach is applied to the portfolio model with a stochastic volatility.

2. The model

We use a stochastic portfolio model (see, for example, [1, 2, 4, 6, 7]) with a standard Brownian motion $\{W_{1s}, \mathcal{F}_s\}_{t\leq s\leq T}$ defined on the probability space $(\Omega, \mathcal{F}, \mathcal{F}_s, P)$, where $\{\mathcal{F}_s\}_{t\leq s\leq T}$ is the augmentation of filtration. The price of a risk-free asset is given by

$$S_0 = e^{\int_t^T r_s\, ds},$$

where $r_s \in C_b^2(R)$ (continuously differentiable and bounded functions) is the rate of return.

The risky-asset pricing process is given by

$$S_s = S_t e^{(\mu_s - \sigma_s^2/2)s + \sigma_s W_{1s}} + \sum_{i=0}^{n_s} \alpha_{1i}; \quad \alpha_{1s} \gtreqless 0; \quad \alpha_{1t} = 0; \quad S_s = \hat{S}_s + \alpha_{1s}$$

for $t \leq s$. Here $\hat{S}_s$ is the regular component of the current price, $\mu_s \in C_b^2(R)$ and $\sigma_s \in C_b^2(R)$ are the rate of return and the stochastic volatility, respectively, while α_{1s} is a stochastic (vertical) shift parameter occurring at time s (in the sense of Alghalith [3]). A change in the shift parameter represents a jump (vertical shift) in the price, where α_{1s} is the size of the jump at time s. Clearly, if $\alpha_{1s} > 0$, there is a positive jump (shift) at time s; if $\alpha_{1s} < 0$, there is a negative jump (drop); and, if $\alpha_{1s} = 0$, there is neither jump nor drop. Also, n_s is a random process representing the number of jumps up to time s (in the interval $(0, s]$). The fact that the random process represents n_s is irrelevant to our purpose and, therefore, no further description is needed. Also, it is irrelevant in forecasting the price, since the expected value of the shift parameter is zero. Note that it is a martingale and, hence, the price is a martingale. Also observe that if there is a jump at time s, then the differential $d\alpha_{1s} = \alpha_{1s} - \alpha_{1u-} = \alpha_{1s} - 0 = \alpha_{1s}$ (since α is equal to zero just before the jump); similarly, if there is no jump at time s, $d\alpha_{1s} = 0$. Therefore, the dynamics of α_{1s} are

$$d\alpha_{1s} = \gamma_{1s}\, dB_{1s},$$

where $\gamma \in C_b^2(R)$ is the deterministic volatility of α_{1s} and B_{1s} is a standard Brownian motion independent of W_{1s} (another Brownian motion). The dynamics of the risky-asset price and the stochastic volatility are given by

$$dS_s = S_{s-}\{\mu_s\, ds + \sigma_s\, dW_{1s}\} + d\alpha_{1s}$$

and

$$d\sigma_s^2 = (\varphi - k\sigma_s^2)\, ds + \lambda\sigma_s\, dW_{2s} + d\alpha_{2s}, \quad \sigma_t^2 \equiv \bar{\sigma}, \quad \sigma_s^2 = \hat{\sigma}_s + \alpha_{2s},$$

respectively, where φ, k and λ are constants, α_{2s} is a stochastic shift parameter (similar to α_{1s}) and $\hat{\sigma}_s$ is the volatility in the absence of the jump. As in the paper by Liu et al. [7], B_s and W_{2s} are independent Brownian motions and

$$d\alpha_{2s} = \gamma_{2s}\, dB_{2s},$$

where γ_{2s} is the deterministic volatility.

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The wealth process is given by

$$X_T^\pi = x + \int_t^T \{r_s X_s^\pi + \pi_s(\mu_s - r_s)\}\,ds + \int_t^T \pi_s \sigma_s\, dW_{1s} + \int_t^T \pi_s \frac{\gamma_s}{S_s}\, dB_s, \tag{2.1}$$

where x is the initial wealth and $\{\pi_s, \mathcal{F}_s\}_{t\le s\le T}$ is the portfolio process (as a dollar value, not as a *weight*), with $E[\int_t^T \pi_s^2\, ds] < \infty$. The trading strategy $\pi_s \in \mathcal{A}(x, \bar\sigma)$ is admissible. The objective of the investor is to maximize the expected utility of the terminal wealth

$$V(t, x, \bar\sigma) = \sup_\pi E[U(X_T^\pi) \mid \mathcal{F}_t].$$

Theorem 2.1. *The optimal portfolio is*

$$\pi_t^* = -\frac{(\mu_t - r_t)V_x}{(\bar\sigma^2 + \gamma_t^2/(\bar S_t + \alpha_{1t})^2)V_{xx}} - \frac{[\lambda\rho_{12}(\hat\sigma + \alpha_{2t}) + \rho\gamma_{1s}\gamma_{2s}]V_{x\bar\sigma}}{V_{xx}},$$

where the subscripts are partial derivatives of V.

Proof. With regular assumptions, the value function satisfies the Hamilton–Jacobi–Bellman partial differential equation (PDE) (however, in the next section, we relax these assumptions)

$$V_t + rxV_x + (\varphi - k\bar\sigma)V_{\bar\sigma} + \frac{1}{2}\lambda^2\bar\sigma V_{\bar\sigma\bar\sigma} + \sup_{\pi_t}\Big\{\pi_t(\mu - r)V_x + \frac{1}{2}\pi_t^2\Big(\bar\sigma^2 + \frac{\gamma_t^2}{S_t^2}\Big)V_{xx} + \pi_t(\rho_{12}\lambda\bar\sigma + \rho\gamma_{1s}\gamma_{2s})V_{x\bar\sigma}\Big\} = 0,$$

where $|\rho_{12}| < 1$ is the correlation factor between W_{1s} and W_{2s}. Thus, the solution yields

$$(\mu_t - r_t)V_x + \pi_t^*\Big(\bar\sigma^2 + \frac{\gamma_t^2}{S_t^2}\Big)V_{xx} + (\lambda\rho_{12}\bar\sigma + \rho\gamma_{1s}\gamma_{2s})V_{x\bar\sigma} = 0,$$

where $|\rho| < 1$ is the correlation factor between B_{1s} and B_{2s}; these two Brownian motions can also be assumed independent, that is, $\rho = 0$ (see Liu et al. [7]). Therefore, the optimal portfolio is

$$\pi_t^* = -\frac{(\mu_t - r_t)V_x}{(\bar\sigma^2 + \gamma_t^2/(\bar S_t + \alpha_{1t})^2)V_{xx}} - \frac{[\lambda\rho_{12}(\hat\sigma + \alpha_{2t}) + \rho\gamma_{1s}\gamma_{2s}]V_{x\bar\sigma}}{V_{xx}}.$$

This completes the proof of the theorem. □

This result is totally intuitive. The volatility of the shift (jump), γ_t, is negatively related to the optimal portfolio, while the size of the jump, α_t, is positively related to the optimal portfolio. That is, a jump increases the optimal portfolio, while a drop reduces the optimal portfolio. It is also guaranteed that $\bar S_t$ is positively related to the optimal portfolio. The impact of the size or volatility of the jump (α_{2t} or γ_{2s}) is generally ambiguous (depending on the signs of ρ_{12}, ρ and $V_{x\bar\sigma}$).

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3. Nonsmooth value functions

Following Alghalith [3] and suppressing the notation, we can also rewrite equation (2.1) as

$$X_T^\pi = x + b + r_t x + a\pi_t^*\Big[\mu_t - r_t + \sigma_t W_{1t} + \frac{\gamma_t}{S_t} B_t\Big] + \int_{\hat{t}>t}^T \{r_s X_s^\pi + (\mu_s - r_s)\pi_s\}\, ds + \int_{\hat{t}>t}^T \pi_s \sigma_s \, dW_{1s} + \int_{\hat{t}>t}^T \pi_s \frac{\gamma_s}{S_s}\, dB_s, \quad t < \hat{t} \le T,$$

where $\hat{t}$ is any time excluding the current time and a and b are nonstochastic shift parameters with initial values equal to one and zero, respectively. Therefore, we rewrite the objective function as

$$\begin{aligned} V(t,x,a,b) &= \sup_\pi E[U(X_T^\pi) \mid \mathcal{F}_t] \\ &= E\left[U\left(\begin{matrix} x + b + r_t x + a\pi_t^*\Big[\mu_t - r_t + \sigma_t W_{1t} + \frac{\gamma_t}{S_t} B_t\Big] \\ + \int_{\hat{t}>t}^T \{r_s X_s^\pi + (\mu_s - r_s)\pi_s^*\}\, ds + \int_{\hat{t}>t}^T \pi_s^* \sigma_s \, dW_{1s} + \int_{\hat{t}>t}^T \pi_s^* \frac{\gamma_s}{S_s}\, dB_s \end{matrix}\right) \middle| \mathcal{F}_t\right]. \end{aligned} \tag{3.1}$$

Differentiating both sides of (3.1) with respect to a and b, respectively, since V is smooth in the shift parameters by construction,

$$V_a(\cdot) = \pi_t^*\Big(\mu_t - r_t + \sigma_t W_{1t} + \frac{\gamma_t}{S_t} B_t\Big) E[U'(X_T^\pi) \mid \mathcal{F}_t], \tag{3.2}$$

$$V_b(\cdot) = E[U'(X_T^\pi) \mid \mathcal{F}_t], \tag{3.3}$$

where the subscripts denote partial derivatives; dividing equation (3.2) by equation (3.3) yields

$$\frac{V_a(\cdot)}{V_b(\cdot)} = \pi_t^*\Big(\mu_t - r_t + \sigma_t W_{1t} + \frac{\gamma_t}{S_t} B_t\Big). \tag{3.4}$$

Now we define

$$c \equiv -\pi_t^*\Big(\mu_t - r_t + \sigma_t W_{1t} + \frac{\gamma_t}{S_t} B_t\Big)$$

and rewrite equation (3.4) as

$$V_a(\cdot) + cV_b(\cdot) = 0.$$

This PDE has a strong solution of the form $V = F(b - ca)$, even if the value function is not smooth in x or t.

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4. Conclusion

A simpler and more general model of jump diffusion for both the price and volatility is introduced and this method is applied to the portfolio model. The results are totally intuitive. In future, this approach can be applied to numerous areas of mathematical finance, such as options and futures.

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