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SETS HOMOTHETIC TO THEIR INTERSECTION WITH A TRANSLATE

BY
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1. **Introduction.** Inspired by a question of J. B. Miller, L. Fejes Tóth asked for a catalogue of those subsets of Euclidean n-space which are homothetic (similar and similarly situated) to their intersection with a suitably translated copy of themselves. For example, a triangle is homothetic to its intersection with an arbitrarily translated replica provided only that the intersection has non-void interior. In 3-space a cube is homothetic to its intersection with a replica translated part way along any body diagonal. With these two preliminary examples for motivation, let us make a definition.

DEFINITION. Let S be a subset of Euclidean n-space. Let T be translation through $\vec{t}$ and K dilatation with centre O_1 and scale factor k. Then $S \in (T, K)$ if and only if $S \cap S^T = S^K$.

There are two natural problems.

PROBLEM 1. Given T and K find all subsets S such that $S \in (T, K)$.

PROBLEM 2. Given S find all T and K such that $S \in (T, K)$.

If E denotes the identity transformation on Euclidean n-space then any subset S belongs to (E, E). For the rest of the paper, let us insist that T is a proper translation $[\vec{t} \neq \vec{0}]$ and K is a proper dilatation $[k \neq 1]$. In the spirit of the original question we will concentrate on Problem 1 although certain aspects of Problem 2 will arise naturally.

2. **More examples.** Let Euclidean n-space be given Cartesian coordinates so that it may be identified with $\mathbb{R}^n$. Then $\mathbb{R}^n$ and the empty set $\varnothing$ belong to (T, K) for any T and any K.

Let $\mathbb{Q}$ denote the rational numbers so that $\mathbb{Q}^n$ is a subset of $\mathbb{R}^n$. Then if $\vec{t}$ and the coordinate vector of O_1 belong to $\mathbb{Q}^n$ and if $k \in \mathbb{Q}$, it follows that $\mathbb{Q}^n \in (T, K)$.

In 1 dimension, let S_1 be the set of all x such that $0 \leq x \leq 1.111\ldots$ and such that the decimal expansion of x uses only the digits 0 and 1. Then if T is translation through -1 and K is contraction by $\frac{1}{10}$ from the origin we have $S_1 \in (T, K)$.

In 2 dimensions we may take S_2 to be the union of the countably many lines $x=0$ and $x=\pm 2^m$ $[m=0, \pm 1, \pm 2, \ldots]$. Then if T_0 is translation through $(0, -1)$ and K_0 is contraction by $\frac{1}{2}$ from the origin we have $S_2 \in (T_0, K_0)$.

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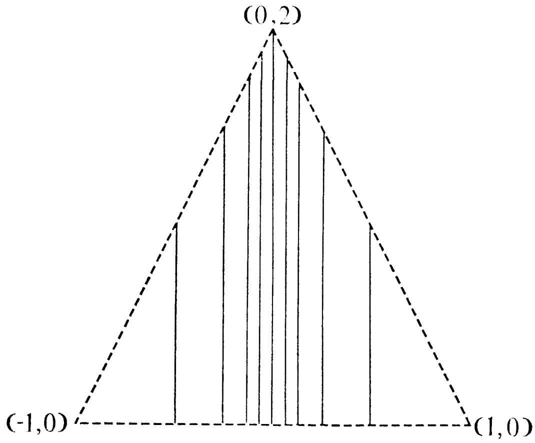

Figure 1.

An intriguing modification of S_2 is the "candelabra" S_3 drawn in Fig. 1. It consists of the part of S_2 lying in the interior of the triangle with vertices $(-1, 0)$, $(0, 2)$ and $(1, 0)$. Notice that, while the candelabra belongs to (T_0, K_0), its closure does not! To obtain a closed candelabra we should take the part of S_2 lying in the closure of the above triangle. This adds to S_3 not only the endpoints of its candlesticks but also the two isolated points $(\pm 1, 0)$.

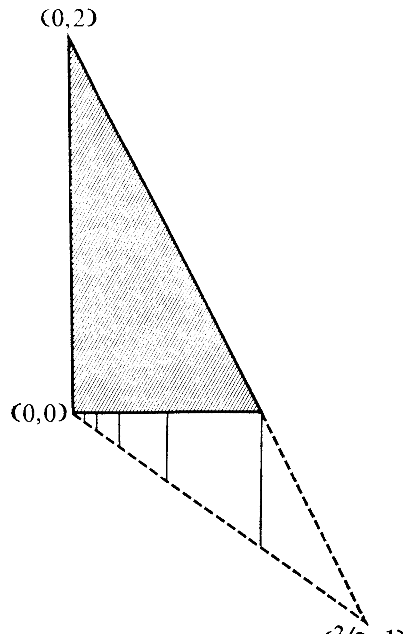

Figure 2.

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A second interesting modification of S_2 is the "feathered wing" S_4 drawn in Fig. 2. To obtain this example add the halfplane $y \geq 0$ to S_2 to make it connected. Then take the portion of this example which lies in the closed triangle with vertices (0, 0), (0, 2) and $(\frac{3}{2}, -1)$. The feathered wing belongs to (T_0, K_0) and is closed, bounded and connected, but not convex.

3. **A generalized problem.** Let us temporarily replace Euclidean n-space by an arbitrary space X and the group of homotheties [translations and dilatations] by an arbitrary group G acting on X. Suppose S is a subset of X and f, g and h are elements of G. Then let us write $S \in (f \cap g=h)$ if $S^f \cap S^g=S^h$ and $S \in (f \cup g=h)$ if $S^f \cup S^g=S^h$. In each case we may pose a problem by considering either the subset S or the group elements f, g and h to be unknown.

Two major simplifications in these problems are possible because f, g and h are bijections. First, De Morgan's laws imply that $S \in (f \cap g=h)$ if and only if its complement satisfies $S' \in (f \cup g=h)$. Second, $S \in (f \cap g=h)$ if and only if $S \in (e \cap gf^{-1}=hf^{-1})$ where e denotes the identity in G. Setting $gf^{-1}=T$ and $hf^{-1}=K$ we see that this last relation may be written $S \in (T, K)$ in the notation of the introduction. Thus the general problems suggested in the preceding paragraph reduce to the analogues of the introductory Problems 1 and 2.

A number of properties of the relation $S \in (T, K)$ can be developed in the general context in which S is a subset of a space X acted on by a group G containing T and K. Property A is somewhat specialized but it is included here to contrast with the behavior exhibited by the closure of the candelabra.

[A]: *If X is a topological space, T and K are homeomorphisms and $S \in (T, K)$ then the interior of S satisfies $S^i \in (T, K)$.*

Proof. $S^{iK}=S^{Ki}=(S \cap S^T)^i=S^i \cap S^{Ti}=S^i \cap S^{iT}$.

[B]: *If S_i $[i \in I]$ is a family of sets belonging to (T, K) then $\bigcap S_i \in (T, K)$.*

Proof. $(\bigcap S_i)^K=\bigcap S_i^K=\bigcap(S_i \cap S_i^T)=(\bigcap S_i) \cap (\bigcap S_i^T)=(\bigcap S_i) \cap (\bigcap S_i)^T$.

[C]: *If S_i $[i \in I]$ is a family of sets belonging to (T, K) and satisfying $\bigcup(S_i \cap S_j^T) \subset \bigcup(S_i \cap S_i^T)$ then $\bigcup S_i \in (T, K)$.*

Proof. $(\bigcup S_i) \cap (\bigcup S_i)^T=(\bigcup S_i) \cap (\bigcup S_j^T)=\bigcup(S_i \cap S_j^T)=\bigcup(S_i \cap S_i^T)=\bigcup S_i^K=(\bigcup S_i)^K$. The third equality depends on the special hypothesis.

[D]: *If $S \in (T, K)$ then $S \in (T^{-1}, KT^{-1})$.*

Proof. $S^{KT^{-1}}=(S \cap S^T)^{T^{-1}}=S \cap S^{T^{-1}}$.

[E]: *If $L \in G$ and $S \in (T, K)$ then $S^L \in (T^L, K^L)$.*

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Proof. $S^L \cap S^{L(L^{-1}TL)} = S^L \cap S^{TL} = (S \cap S^T)^L = S^{KL} = S^{L(L^{-1}KL)}$.

4. **The (T, K)-hull.** If R is any subset of X, let us define the (T, K)-hull of R to be $\langle R\rangle = \bigcap\{S : R \subset S \subset X \text{ and } S \in (T, K)\}$. Since $X \in (T, K)$ this set contains R and because of Property B it belongs to (T, K). It is therefore the smallest subset of X containing R and belonging to (T, K).

There is a construction for the (T, K)-hull of R. Let $K_1 = K$ and $K_2 = KT^{-1}$. Then let $R_0 = R$ and consider the increasing sequence of sets defined by

$$R_{n+1} = R_n \cup R_n^{K_1} \cup R_n^{K_2} \cup (R_n^{K_1^{-1}} \cap R_n^{K_2^{-1}}).$$

THEOREM 1. $\langle R\rangle = \bigcup_{n=1}^{\infty} R_n$.

Proof. The condition $S \cap S^T = S^K$ implies that $S \cap S^T \supset S^K$ and therefore that $S^K \subset S$ and $S^{KT^{-1}} \subset S$. It also implies that $S \cap S^T \subset S^K$ and therefore that $S^{K^{-1}} \cap S^{TK^{-1}} \subset S$. These remarks show that no superfluous points have been included in $\tilde{R} = \bigcup_{n=1}^{\infty} R_n$ and hence $\langle R\rangle \supset \tilde{R}$. To prove that $\langle R\rangle = \tilde{R}$ it remains to show that $\tilde{R} \in (T, K)$.

For $i = 1, 2$, $\tilde{R}^{K_i} = (\bigcup R_n)^{K_i} = \bigcup R_n^{K_i} \subset \bigcup R_{n+1}$. Thus $\tilde{R}^K \subset \tilde{R}$ and $\tilde{R}^{KT^{-1}} \subset \tilde{R}$ or $\tilde{R}^K \subset \tilde{R}^T$. It follows that $\tilde{R}^K \subset \tilde{R} \cap \tilde{R}^T$. On the other hand,

$$\begin{aligned}\tilde{R}^{K_1^{-1}} \cap \tilde{R}^{K_2^{-1}} &= (\bigcup R_n)^{K_1^{-1}} \cap (\bigcup R_n)^{K_2^{-1}} = (\bigcup R_n^{K_1^{-1}}) \cap (\bigcup R_n^{K_2^{-1}}) \\ &= \bigcup (R_n^{K_1^{-1}} \cap R_m^{K_2^{-1}}) = \bigcup (R_n^{K_1^{-1}} \cap R_n^{K_2^{-1}}) \subset \bigcup R_{n+1} \subset \tilde{R}\end{aligned}$$

and therefore $\tilde{R}^{K^{-1}} \cap \tilde{R}^{TK^{-1}} \subset \tilde{R}$ or $\tilde{R} \cap \tilde{R}^T \subset \tilde{R}^K$.

The candelabra provides two interesting illustrations of this theorem. If n candlesticks $x = 2^{-m}$ $[m = 1, 2, \ldots, n]$ are removed from the right side of S_3 to leave a set R then $R \notin (T_0, K_0)$. However the construction of Theorem 1 replaces these missing candlesticks one by one from the left and we obtain $\langle R\rangle = S_3 = R_m[m \geq n]$. As a second illustration let R be the closure of S_3. Then $R \notin (T_0, K_0)$ but the construction of Theorem 1 finds the points $(\pm 1, 0)$ in the first step and we obtain $\langle R\rangle = R_m[m \geq 1]$.

5. **Fundamental solutions.** A subset Σ of X is called a *fundamental solution* of the (T, K)-problem if $\Sigma^T = \Sigma$ and $\Sigma^K = \Sigma$. This obviously guarantees that $\Sigma \in (T, K)$.

A partition of X is called a *(T, K)-fundamental partition* if the sets involved are fundamental solutions of the (T, K)-problem. For such a partition we have $X = \bigcup \Sigma_i$ where $\Sigma_i \cap \Sigma_j = \varnothing$ and $\Sigma_i = \Sigma_i^T = \Sigma_i^K$.

Let $\Gamma = \langle T, K\rangle$ be the subgroup of G generated by T and K. Let $X = \bigcup X_i$ be the orbit decomposition of X relative to Γ. This is the finest fundamental partition of X and any other fundamental partition is obtained by letting each $\Sigma_i = \bigcup X_j$ be a collection of orbits.

Now let $X = \bigcup \Sigma_i$ be a fixed (T, K)-fundamental partition and let S be a subset

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of X. We may write $S=\bigcup S_i=\bigcup(S \cap \Sigma_i)$ and speak of the S_i as the parts of S relative to the partition.

THEOREM 2. *A set S belongs to* (T, K) *if and only if each of its parts* S_i *belongs to* (T, K).

Proof. If $S \in (T, K)$ then since each $\Sigma_i \in (T, K)$ it follows from Property B that $S_i=S \cap \Sigma_i$ belongs to (T, K).

On the other hand if each $S_i \in (T, K)$ then since $S_i \cap S_j^T \subset \Sigma_i \cap \Sigma_j^T= \Sigma_i \cap \Sigma_j=\varnothing$ it follows from Property C that $S=\bigcup S_i$ beongs to (T, K).

When Theorem 2 is used with the orbit decomposition $X=\bigcup X_i$ we see that the problem really belongs to the theory of permutation groups. We have a two-generator group $\Gamma=\langle T, K\rangle$ acting transitivity on a countable set X_i and we are looking for subsets $S_i \subset X_i$ which satisfy $S_i \cap S_i^T=S_i^K$.

In the original problem with $X=R^n$ there are a continuum of orbits and for each we have at least the two choices $S_i=\varnothing$ or $S_i=X_i$. The cardinality of the set of all solutions $S=\bigcup S_i$ is therefore equal to the cardinality of the set of all subsets of R^n.

In general the solutions described in the last paragraph do not have pleasant geometrical properties. They are far more pathological than the examples of §§1 and 2 and this runs contrary to the intent of the original question. In the remaining sections we shall concentrate on geometrically appealing solutions to the original problem.

6. **The original problem.** Now we return to the case when $X = R^n$, T is a proper translation and K is a proper dilatation. According to Property D, if $S \in (T, K)$ then $S \in (T^{-1}, KT^{-1})$ as well. If T is translation through $\vec{t}$ then T^{-1} is translation through $-\vec{t}$. If $K=K_1$ is a dilatation with scale factor k and centre O_1 then $KT^{-1}=K_2$ is a dilatation with the same scale factor k but a different centre ${}_2O\neq O_1$.

LEMMA 1. *The vector from* O_1 *to* O_2 *is* $\overrightarrow{O_1O_2}=(k-1)^{-1}\vec{t}$.

Proof. If we choose coordinates with O_1 as origin then KT^{-1} is given by $\vec{x}\to k\vec{x}-\vec{t}$. Its unique fixed point is O_2.

LEMMA 2. *Let* $S \in (T, K)$. *If* $k>1$ *then either S in not bounded or* $S=\varnothing$. *If* $k<1$ *and* $S\neq\varnothing$ *then* O_1 *and* O_2 *belong to* $\bar{S}$, *the closure of S.*

Proof. If S is bounded it has a diameter d and S^K has a diameter kd. Since $S^K \subset S$ we must have $kd\leq d$. If $k>1$ this implies $d=0$ and, as singleton solutions are impossible, $S=\varnothing$.

If $L=K_1$ or K_2, $\bar{S}^L=\overline{S^L} \subset \bar{S}$. $\bar{S}$ is a complete metric space. If $k<1$, L is a contraction mapping of $\bar{S}$ and its fixed point must belong to $\bar{S}$.

On the basis of our discussion to this point it is easy to construct a large class of geometrically appealing sets belonging to a given (T, K). If C_1 is a cone with

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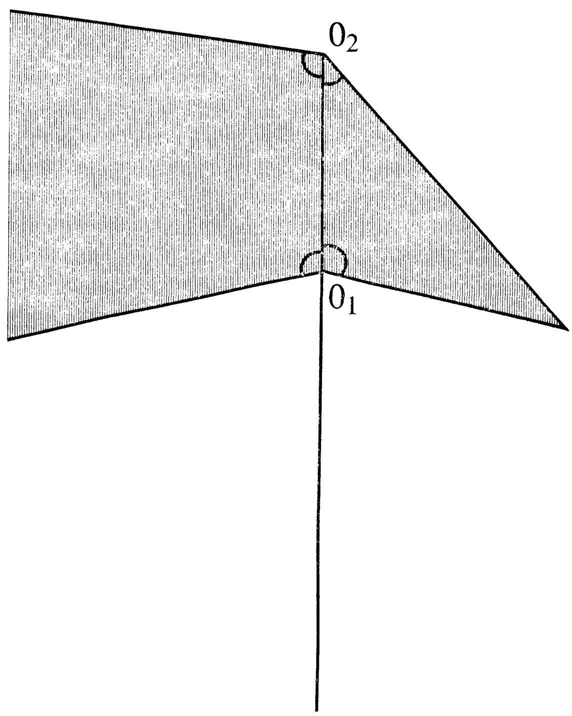
vertex at O_1 then $C_1^K = C_1$. In order to get $C_1 \in (T, K)$ it remains to insure that $C_1 \cap C_1^T = C_1$. This is guaranteed if C_1 meets a sphere about O_1 in a set which is spherically star-like with respect to the pole in the direction $-\vec{t}$ from O_1. An analogous construction about O_2 gives a cone C_2 which belongs to (T^{-1}, KT^{-1}) and hence to (T, K). It follows that $S = C_1 \cap C_2$ belongs to (T, K). Let us call this general method of building examples the *cone construction*. The triangle and cube of §1 may be considered as instances of this construction.

With the cone construction for motivation, the n-dimensional problem may be reduced to convenient one and two-dimensional problems. Let Λ be the line joining O_1 and O_2 and let π be any 2-dimensional halfplane bounded by Λ but not including Λ. Then Λ and the π's give a (T, K)-fundamental partition of R^n and according to Theorem 2 the most general (T, K) set in R^n is a union of its one and two-dimensional parts. This special partition allows us to give a part by part characterization of certain solution sets obtained from the cone construction.

THEOREM 3. *Let $S \neq \varnothing$ be a solution of the (T, K)-problem on the line $O_1 O_2$ or in a 2-dimensional halfplane bounded by this line. Then if S is closed and convex it may be obtained from closed convex cones by the cone construction.*



$\vec{t} = (0, -1)$; $k < 1$
Figure 3.

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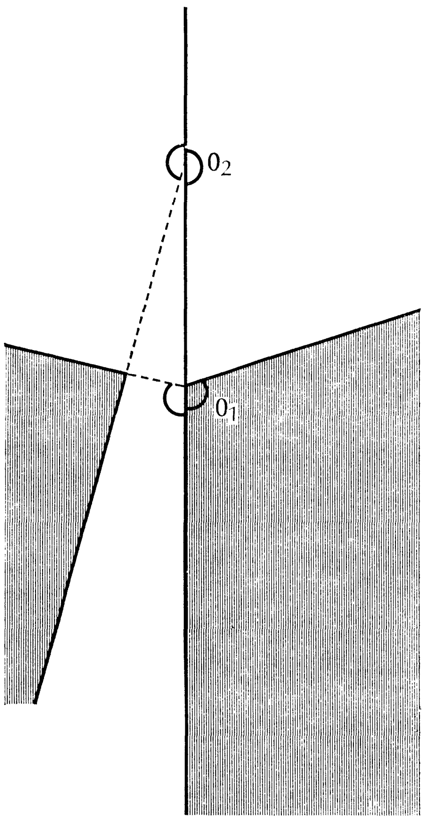
The feathered wing S_4 of §2 shows that the hypothesis of Theorem 3 cannot be weakened from convex to connected even when we are dealing with bounded parts.

The proof of the halfplane case is deferred to §7. The proof of the 1-dimensional case is almost immediate since the only closed, convex subsets of a line are intervals, rays and the line itself. When $k<1$, Lemma 2 implies that both O_1 and O_2 must belong to S and it is easy to check that the interval $[O_1, O_2]$, the ray from O_1 towards O_2, the ray from O_2 towards O_1 and the full line are the only possible solutions. When $k>1$ Lemma 2 eliminates finite intervals and it is easy to check that the only possibilities are the ray from O_1 away from O_2, the ray from O_2 away from O_1 and the full line.

As a corollary to Lemma 2 and Theorem 3 it is possible to list the compact, convex solutions of the (T, K)-problem in 2-dimensions. If $k>1$, the only solution is $\varnothing$. If $k<1$, we also have the quadrangles with diagonal $O_1 O_2$. The term quadrangle must be understood to include the line segment $[O_1, O_2]$ and any triangle with one vertex at one centre and the opposite edge containing the other centre.

The "wild tomahawk" of Fig. 3 and the "ship over shoal" of Fig. 4 illustrate the



$\vec{t}=(0, 1)$; $k>1$
Figure 4.

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general 2-dimensional solution obtained from Theorem 3 when $k<1$ and when $k>1$ respectively. The situation in n dimensions is analogous. It is worth emphasising that the parts of S are quite independent and while each is assumed to be closed and convex there is no reason why these properties must hold for S itself.

7. **Proof of Theorem 3.** Without loss of generality we may assume that $O_1=(0,0)$, $O_2=(0,|k-1|^{-1})$ and the closed convex set S lies in the halfplane $x>0$. With this assumption, $\vec{t}=(0,-1)$ if $k<1$ and $\vec{t}=(0,1)$ if $k>1$. Let $\vec{t}_1=-\vec{t}$ and $\vec{t}_2=\vec{t}$ and let θ_i denote the ray from O_i into the halfplane $x>0$ that makes the angle $\theta_i[0<\theta_i<\pi]$ with $\vec{t}_i[i=1,2]$. Let θ_i^* denote the supremum of the angles θ_i determined by rays θ_i which meet S. Let C_i be the cone from O_i consisting of all rays $0<\theta_i\leq\theta_i^*$.

The cones C_1 and C_2 are closed convex candidates for the cone construction and it is clear that $S\subset C_1\cap C_2$. To complete the proof of Theorem 3 it remains to show that $S=C_1\cap C_2$. Since S is closed it suffices to prove that $S\supset(C_1\cap C_2)^i$.

Lemma 3. *A ray θ_i from O_i $[i=1,2]$ belongs to one of three mutually exclusive classes:* (a) $\theta_i\cap S=\varnothing$ (b) $\theta_i\cap S=\theta_i$ *and* (c) $\theta_i\cap S=\tilde{\theta}_i$ *where if $k<1$ $\tilde{\theta}_i$ is a closed initial segment of θ_i and if $k>1$ $\tilde{\theta}_i$ is a closed terminal ray of θ_i.*

Proof. If θ_i does not belong to class (a) or (b) then $\tilde{\theta}_i$ is a non-void proper subset of θ_i. Since S is closed and convex, $\tilde{\theta}_i$ is closed and convex. Since $S^{K_i}\subset S$ while $\theta_i^{K_i}=\theta_i$, $\tilde{\theta}_i^{K_i}\subset\tilde{\theta}_i$. It follows that $\tilde{\theta}_i$ must be of the form described in class (c).

Lemma 4. *Two rays θ_1 and θ_2 of class* (c) *cannot meet in the complement of S.*

Proof. Suppose to the contrary that θ_1 meets θ_2 in a point Q of the complement of S. Let P_i be the boundary point of $\tilde{\theta}_i$ $[i=1,2]$ and note that the convex hull of $\tilde{\theta}_1\cup\tilde{\theta}_2$ including the closed line segment $[P_1,P_2]$ lies in S. We may therefore find a point P on the open line segment (Q,P_1) such that $P\notin S$ but $P^{K_1}=P^K\in S$ and $P^{K_2}=P^{KT^{-1}}\in S$. Since $P\notin S$, $P^K\notin S^K$. But $S^K=S\cap S^T$ and $P^K\in S$. It follows that $P^K\notin S^T$ and hence that $P^{KT^{-1}}\notin S$. This contradiction proves the lemma.

Lemma 5. *A ray θ_i from O_i* $[i=1,2]$ *which passes into the interior of $C_1\cap C_2$ must be of the class* (b) *or* (c).

Proof. To simplify notation let us argue the case $i=1$. The proof of the case $i=2$ is identical except for an interchange of subscripts.

Since the ray θ_1 satisfies $0<\theta_1<\theta_1^*$ we can find a ray θ_1' of class (b) or (c) such that $\theta_1<\theta_1'<\theta_1^*$. Since θ_1 enters $C_1\cap C_2$ so does θ_1' and consequently the rays of C_2 sufficiently close to θ_2^* cut them both. Let θ_2 be a common transversal chosen to be of class (b) or (c). Then Lemma 4 implies that $\theta_2\cap\theta_1'$ is a point of S and it follows from Lemma 3 that $\theta_2\cap\theta_1$ must also be a point of S. This completes the proof of Lemma 5.

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To complete the proof of Theorem 3, let P be a point of $(C_1 \cap C_2)^i$. Lemma 5 ensures that $\theta_1=O_1P$ and $\theta_2=O_2P$ are both of class (b) or (c). Lemma 4 then implies that $P=\theta_1 \cap \theta_2$ is a point of S.

8. **Related literature.** When I had completed the work of the preceding sections, L. Fejes Tóth kindly forwarded to me the references [1]–[4] which he had just received from G. Fejes Tóth and A. Florian. To simplify discussion of the relationship between these papers and the present one let us make another definition.

DEFINITION. A subset S of Euclidean n-space has Property H with respect to a translation T if there exists a dilatation $K=K(S, T)$ such that $S \cap S^T=S^K$ i.e. $S \in (T, K)$.

The theme of these papers is to prove that if S is "sufficiently nice" and if S has Property H with respect to "enough" translations then S is a simplex. For the most part the methods of the other authors depend on Property H holding simultaneously for independent translations. Moreover they deal exclusively with bounded sets S and therefore with dilatations K whose scale factor k satisfies $k<1$.

In [4] C. A. Rogers and G. C. Shephard investigate the volume of the difference body DS of an n-dimensional convex body S [convex body=compact convex set with non-void interior]. They find that the difference body attains a maximum volume of $\binom{2n}{n} V(S)$ exactly when S has Property H with respect to every translation T such that $S \cap S^T$ is n-dimensional. They characterize the instances of equality by proving that a convex body is a simplex if it enjoys Property H with respect to every translation such that $S \cap S^T$ is n-dimensional.

In [2] P. Grüber strengthens this characterization of the simplex in three different ways. First, if S is assumed to be a convex body, then Property H need only hold for all translations in an arbitrarily small ball about the origin. Second, if Property H holds for all translations such that $S \cap S^T$ contains more than one point, then one can drop the convexity condition and assume only that S is a compact set with non-void interior. Finally, in dimension $n=1$, the second characterization can be further strengthened by assuming only that S is a bounded measurable set with positive Lebesgue measure. This forces S to be an open interval together with a subset of its endpoints.

In [1] H. G. Eggleston verifies the conjecture of P. Grüber that the preceding 1-dimensional characterization actually holds in n dimensions. Specifically he shows that if S is a bounded measurable set with positive Lebesgue measure which enjoys Property H with respect to every translation T such that $S \cap S^T$ contains more than one point then S is the interior of a simplex together with a subcollection of its r-dimensional faces $[0 \le r \le n-1]$.

In [3] P. Grüber returns to the strong hypothesis that S is an n-dimensional convex body. He proves that if S contains a ball of radius α and is contained in a

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ball of radius $1/\alpha$ then there is a set of $N=N(n, \alpha)$ universal testing translations $\tau_\alpha=\{T_1, T_2, \ldots, T_N\}$ such that if S enjoys Property H with respect to every $T \in \tau_\alpha$ then S is a simplex. In the special case $n=2$, τ_α may be taken to consist of the three translations $(\alpha, 0)$, $(0, \alpha)$ and (α, α) and it is only necessary to assume that S is a convex body containing a disk of radius α. Hilfssatz 5 of [3] provides an alternative argument for Theorem 3 in the special case with $k<1$ and S bounded.

Addendum. G. C. Shephard has drawn my attention to Theorem 5 in the recent paper [1′] where four other characterizations of the n-simplex are deduced from the assumption that it is a polytope P satisfying our property H with respect to every translation T such that $P \cap P^T$ is n-dimensional. Related theorems in [1′] give alternative characterizations of the broader classes of polytopes defined by the following conditions. For every translations T, the set $P \cap P^T$ is

(i) less than n-dimensional or affinely equivalent to P (Theorem 4), or
(ii) empty or a summand of P (Theorem 2), or
(iii) empty or homothetic to a summand of P (Theorem 1).

In their concluding remarks the authors of [1′] ask what sets would arise in these cases if the assumption that P is a polytope were dropped. They cite interesting partial answers in their bibliography.

(1′) P. McMullen, R. Schneider and G. C. Shephard, *Monotypic polytopes and their intersection properties*. Geometriae Dedicata **3** (1974), 99–129.

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