1. Introduction. A vector lattice $W$ is boundedly complete when each subset $\{a_j; j \in J\}$ of $W$ which is bounded above by an element of $W$ has a least upper bound in $W$. The least upper bound of $\{a_j; j \in J\}$ is denoted by $\bigvee_{j \in J} a_j$ and the greatest lower bound by $\bigwedge_{j \in J} a_j$ whenever these exist.

Let $C(S)$ be the algebra of real valued continuous functions on a compact Hausdorff space $S$. Stone [4] shows that the vector lattice $C(S)$ is boundedly complete if and only if the closure of each open subset of $S$ is open; in this event we call $C(S)$ a Stone algebra. For example, if $(X, \mathcal{B}, \mu)$ is a probability space, then $L^\infty(X, \mathcal{B}, \mu)$ is a Stone algebra satisfying the countable chain condition.

Let $\{a_n\} (n = 1, 2, \ldots)$ be a bounded sequence in a Stone algebra $S$; then

$$
\bigvee_{n=1}^\infty \bigwedge_{r=n} a_r \leq \bigwedge_{n=1}^\infty \bigvee_{r=n} a_r.
$$

When these two terms are equal we define $\operatorname{LIM} a_n$ to be their common value and say the sequence is order convergent with order limit $\operatorname{LIM} a_n$. In the special case where $S$ is of the form $L^\infty(X, \mathcal{B}, \mu)$ and $\mu$ is a probability measure, if a sequence $\{b_n\} (n = 1, 2, \ldots)$ has order limit $b$, then the sequence $\{b_n\} (n = 1, 2, \ldots)$ converges to $b$ in the $L^1$-topology ($L^\infty$ is the dual of $L^1$). But Floyd [3] gives an example of a Stone algebra $S$ such that there is no Hausdorff vector topology for $S$ in which each bounded monotone increasing sequence converges to its least upper bound.

We shall postpone all further definitions till §2. In [7] we investigated Moy averaging operators on Stone algebras satisfying the countable chain condition. In this paper we consider a monotone increasing sequence $\{S_n\} (n = 1, 2, \ldots)$ of Stone subalgebras of a Stone algebra $S_\infty$ such that the smallest Stone subalgebra containing $\bigcup_{n=1}^\infty S_n$ is $S_\infty$. Let $S_\infty$ satisfy the countable chain condition and let $T_0: S_\infty \rightarrow S_1$ be a Moy operator satisfying certain conditions. Then we show that there exists a sequence $\{T_n\} (n = 1, 2, \ldots)$ of Moy operators on $S_\infty$ such that:

(i) $T_n$ is a projection of $S_\infty$ onto $S_n$ for $n \geq 1$.
(ii) $T_r T_n = T_r$ for $0 \leq r < n$.
(iii) If $b$ is a positive element of $S_\infty$ and $T_n b = 0$ then $b = 0$.
(iv) For each $z \in S_\infty$, the order limit $\operatorname{LIM} T_n z$ exists and $\operatorname{LIM} T_n z = z$.

This result is a Corollary of Theorem 2.

Theorem 1 is a convergence theorem for a sequence of generalized conditional expectations with respect to a modular Stone algebra valued measure. For conditional expectations with respect to real valued measures such results are known in probability theory as martingale convergence theorems for sequences of Stone algebras.
theorems; see Doob [2]. Theorem 1 was suggested by the classical work of Sparre Andersen and Jessen in [1]. The key step in generalizing their result to Stone algebra valued measures is Lemma 1.

In a later publication I intend to discuss applications of the results of this paper to Boolean algebras.

The work of this paper depends essentially on that of [6]. This is because in [7] we used the results of [6] to establish the existence, under certain conditions, of generalized conditional expectations.

2. Convergence theorems. Throughout this paper $(X, \mathcal{B})$ is a measurable space and $C(S)$ is a Stone algebra. Stone algebra valued measures were defined in [5]. We require $\rho$ to be a finite $C(S)$-valued measure on $(X, \mathcal{B})$; that is, $\rho$ is to be a map of $\mathcal{B}$ into $C(S)$ such that

(i) $\rho E \geq 0$ for each $E \in \mathcal{B};$
(ii) if $\{E_j\}$ ($j = 1, 2, \ldots$) is a pairwise disjoint family of sets in $\mathcal{B}$ then

$$\rho \bigvee_{j=1}^{\infty} E_j = \bigvee_{n=1}^{\infty} \bigwedge_{j=1}^{n} \rho E_j.$$  

A Stone algebra $\mathcal{S}$ satisfies the countable chain condition when each bounded subset of $\mathcal{S}$ contains a countable subset such that the two sets have the same least upper bound. This condition on $\mathcal{S}$ is equivalent (see Proposition 3.2 of [6]) to the Boolean algebra of idempotent elements of $\mathcal{S}$ satisfying the countable chain condition. From now onward we require $C(S)$ to satisfy the countable chain condition.

We define $L^p$-spaces with respect to Stone algebras in [6], and it follows from Proposition 3.3 that $L^\infty(X, \mathcal{B}, \rho)$ is a Stone algebra satisfying the countable chain condition because $C(S)$ satisfies this condition.

We require the existence of an algebra homomorphism $\pi: C(S) \to L^\infty(X, \mathcal{B}, \rho)$ such that

$$\int_X \pi(a) f d\rho = a \int_X f d\rho$$

for each $f \in L^1(X, \mathcal{B}, \rho).$

Then $\rho$ is a modular measure with respect to $\pi$, as defined in [6]. Close connections between modular measures and averaging operators were exhibited in [7].

Let $\mathcal{S}$ be a Stone algebra and $\mathcal{U}$ a subalgebra. $\mathcal{U}$ is a Stone subalgebra of $\mathcal{S}$, if the least upper bound, in $\mathcal{S}$, of each upper bounded subset of $\mathcal{U}$ is in $\mathcal{U}$; i.e. $\mathcal{U}$ is a Stone algebra and a bounded subset of $\mathcal{U}$ has the same least upper bound in $\mathcal{S}$ and $\mathcal{U}$.

Let $T$ be a linear operator on a Stone algebra $\mathcal{S}$. $T$ is an averaging operator if $T(fTg) = (Tf)(Tg)$ for each $f$ and $g$ in $\mathcal{S}$. $T$ is a Moy averaging operator when $T$ is a positive averaging operator and, if $\{f_n\}$ ($n = 1, 2, \ldots$) is a monotone increasing sequence in $\mathcal{S}$ which is bounded above, then $T \bigvee_{n=1}^{\infty} f_n = \bigvee_{n=1}^{\infty} T f_n$. For any operator $T$ on $\mathcal{S}$ let

$$\mathcal{E}(T) = \{a \in \mathcal{S}: aTb = Tab \text{ for all } b \in \mathcal{S}\}.$$  

When $T$ is an averaging operator the range of $T$ is a subset of $\mathcal{E}(T)$. It is shown in [7] that when $T$ is a Moy operator and $\mathcal{S}$ satisfies the countable chain condition then $\mathcal{E}(T)$ is a Stone subalgebra of $\mathcal{S}$.
When \(\mathcal{B}_1\) is a Boolean \(\sigma\)-subalgebra of \(\mathcal{B}\) and \(\rho_1\) is the restriction of \(\rho\) to \(\mathcal{B}_1\) then \(L^\infty(X, \mathcal{B}_1, \rho_1)\) can be identified with a Stone subalgebra of \(L^\infty(X, \mathcal{B}, \rho)\). If \(\pi[C(S)]\) is a subalgebra of \(L^\infty(X, \mathcal{B}_1, \rho_1)\), then for each \(f \in L^1(X, \mathcal{B}, \rho)\) we can find a \(\mathcal{B}_1\)-measureable function \(f_1 \in L^1(X, \mathcal{B}_1, \rho_1)\) such that

\[
\int_E f \, d\rho = \int_E f_1 \, d\rho_1 \quad \text{for each } E \in \mathcal{B}_1.
\]

This is Lemma 2.1 of [7].

**Definition 1.** Let \(\mathcal{B}_1\) be a \(\sigma\)-subalgebra of \(\mathcal{B}\) such that \(\pi[C(S)] \subset L^\infty(X, \mathcal{B}_1, \rho_1)\). The conditional expectation of \(\mathcal{B}_1\) with respect to \(\rho\) is the map \(C: L^1(X, \mathcal{B}, \rho) \to L^1(X, \mathcal{B}_1, \rho_1)\) such that for each \([f]_\rho \in L^1(X, \mathcal{B}, \rho)\) we have

\[
C[f]_\rho = [f]_\rho_1,
\]

where

\[
\int_E f \, d\rho = \int_E f_1 \, d\rho_1 \quad \text{for each } E \in \mathcal{B}_1.
\]

The generalized conditional expectation operator \(C\), defined above, is a positive linear map of \(L^1(X, \mathcal{B}, \rho)\) onto \(L^1(X, \mathcal{B}_1, \rho_1)\) such that \(C^2 = C\). The restriction of \(C\) to \(L^\infty(X, \mathcal{B}, \rho)\) is a Moy averaging operator whose range is \(L^\infty(X, \mathcal{B}_1, \rho_1)\).

**Lemma 1.** Let \(\mathcal{W}\) be a Boolean subalgebra of \(\mathcal{B}\) such that \(\mathcal{B}\) is the smallest \(\sigma\)-algebra of subsets of \(X\) containing \(\mathcal{W}\). Let \(f \in L^1(X, \mathcal{B}, \rho)\) be such that \(\int_E f \, d\rho \geq 0\) for each \(E \in \mathcal{W}\). Then

\[
\int_E f \, d\rho \geq 0 \quad \text{for each } E \in \mathcal{B}.
\]

**Proof.** Let \(\mathcal{U} = \left\{E \in \mathcal{B}: \int_E f \, d\rho \geq 0\right\}\); then by hypothesis \(\mathcal{W} \subset \mathcal{U}\). An argument using Zorn's lemma shows that there is a maximal Boolean algebra \(\mathcal{M}\) such that \(\mathcal{W} \subset \mathcal{M} \subset \mathcal{U}\).

Let \(\mathcal{M}^* = \{E \subset X: \chi_E = \lim \chi_{E_n}\}\), where each \(E_n \in \mathcal{M}\), so that \(\mathcal{M} \subset \mathcal{M}^*\). If \(A \in \mathcal{M}^*\) and \(B \in \mathcal{M}^*\) then \(A \cap B\) and \(X - A\) are in \(\mathcal{M}^*\). Hence \(\mathcal{M}^*\) is a Boolean algebra containing \(\mathcal{M}\).

Let \(E \in \mathcal{M}^*\); then \(\chi_E = \lim \chi_{E_n}\), where \(E_n \in \mathcal{M}\) for each \(n\). Then, by the analogue for Stone algebra valued measures of the Dominated Convergence Theorem established in [5], we have

\[
\int_E f \, d\rho = \lim_X \int_X f \, \chi_E \, d\rho = \lim_X \int_X f \, \chi_{E_n} \, d\rho = \lim_X \int_{E_n} f \, d\rho.
\]

Thus \(\int_E f \, d\rho \geq 0\) and so \(\mathcal{M}^* \subset \mathcal{U}\). It now follows from the maximality of \(\mathcal{M}\) that \(\mathcal{M} = \mathcal{M}^*\). Thus \(\mathcal{M}\) is a Boolean \(\sigma\)-algebra containing \(\mathcal{W}\) and thus \(\mathcal{M} = \mathcal{U} = \mathcal{B}\).

**Theorem 1.** Suppose that \(\rho\) is a finite \(C(S)\)-valued measure on the measurable space \((X, \mathcal{B})\) and suppose that \(\rho\) is modular with respect to \(\pi\). Let \(\{\mathcal{B}_n\} (n = 1, 2, \ldots)\) be a monotone increasing sequence of \(\sigma\)-subalgebras of \(\mathcal{B}\) such that \(\mathcal{B}\) is the smallest \(\sigma\)-subalgebra of \(\mathcal{B}\) containing \(\bigcup_{n=1}^\infty \mathcal{B}_n\). Further, let \(\pi[C(S)]\) be a subalgebra of \(L^\infty(X, \mathcal{B}_1, \rho)\). For each \(n\) let \(T_n\) be the
generalized conditional expectation of $\mathcal{B}_n$ with respect to $\rho$. Let $f \in L^1(X, \mathcal{B}, \rho)$ and let $f_n \in L^1(X, \mathcal{B}_n, \rho)$ be such that $[f_n]_\rho = T_n[f]$ for each $n$. Then $\lim f_n(x) = f(x)$ almost everywhere with respect to $\rho$.

**Proof.** The set $F = \{x \in X : \lim f_n(x) < f(x)\}$ is the countable union of all sets of the form

$$F_{\alpha, \beta} = \{x \in X : \lim f_n(x) \leq \alpha < \beta \leq f(x)\},$$

where $\alpha$ and $\beta$ are rational and $\alpha < \beta$. Assume that $\rho F \neq 0$; then $\rho F_{\lambda, \mu} \neq 0$ for some rational numbers $\lambda$ and $\mu$, $\lambda < \mu$.

Let $L_\lambda = \{x \in X : \lim f_n(x) \leq \lambda\}$ and, for each natural number $n$, let

$$H_n = \left\{x \in X : \inf_{r \geq n} f_r(x) < \lambda + \frac{1}{n}\right\}.$$

Let

$$H_{n,1} = \left\{x \in X : f_{n+1}(x) < \lambda + \frac{1}{n}\right\}$$

and, for $q \geq 2$,

$$H_{n,q} = \left\{x \in X : \min\left\{f_r(x) : n < r < n+q\right\} \geq \lambda + \frac{1}{n} \text{ and } f_{n+q}(x) < \lambda + \frac{1}{n}\right\}.$$

Since $f_{n+q}$ is $\mathcal{B}_{n+q}$-measurable, $H_{n,q} \in \mathcal{B}_{n+q}$. Also $\{H_{n,q}\} (q = 1, 2, \ldots)$ is a pairwise disjoint family such that $H_n = \bigcup_{q=1}^{\infty} H_{n,q}$. We also have $L_\lambda = \bigcap_{n=1}^{\infty} H_n$.

Choose $A \in \bigcup_{n=1}^{\infty} \mathcal{B}_n$, so that $A \in \mathcal{B}_n$ for some $N$. Then $H_{n,q} \cap A \in \mathcal{B}_{n+q}$ for $n \geq N$ and $q \geq 1$.

By Proposition 3.3 of [6]

$$\int_A f \chi_{H_n} \, d\rho = \lim \sum_{q=1}^{r} f \chi_{H_{n,q}} \, d\rho.$$

From the definition of $T_{n+q}$ and $f_{n+q}$ we have, for $n \geq N$,

$$\int_A f \chi_{H_{n,q}} \, d\rho = \int_{A \cap H_{n,q}} f_{n+q} \, d\rho \leq \left(\lambda + \frac{1}{n}\right) \int_A \chi_{H_{n,q}} \, d\rho.$$

So

$$\int_A f \chi_{H_n} \, d\rho \leq \left(\lambda + \frac{1}{n}\right) \int_A \chi_{H_n} \, d\rho \text{ for } n \geq N.$$

Thus

$$\int_A \left(\lambda + \frac{1}{n}\right) \chi_{H_n} - f \chi_{H_n} \, d\rho \geq 0 \text{ for } n \geq N.$$

But $\lim \chi_{H_n} = \chi_{L_\lambda}$ and so
So, by Proposition 3.5 of [6],
\[
\int_A (\lambda - f) \chi_{L^*_A} \, d\rho \geq 0 \quad \text{for each} \quad A \in \bigcup_{n=1}^{\infty} \mathcal{B}_n.
\]

We observe that \( \bigcup_{n=1}^{\infty} \mathcal{B}_n \) is a Boolean subalgebra of \( \mathcal{B} \) and, by hypothesis, \( \mathcal{B} \) is the \( \sigma \)-algebra generated by \( \bigcup_{n=1}^{\infty} \mathcal{B}_n \). It now follows from Lemma 1 that
\[
\int_A (\lambda - f) \chi_{L^*_A} \, d\rho \geq 0 \quad \text{for each} \quad A \in \mathcal{B}.
\]

We replace \( A \) by \( F_{\lambda,\mu} \) in the above inequality and since \( F_{\lambda,\mu} \subseteq L_\lambda \), obtain
\[
\lambda \rho F_{\lambda,\mu} \geq \int_{F_{\lambda,\mu}} f \, d\rho \geq \mu \rho F_{\lambda,\mu}.
\]

Since \( \mu > \lambda \) this implies that \( F_{\lambda,\mu} = 0 \). This is a contradiction; so the assumption \( \rho F \neq 0 \) must be false. Thus \( f(x) \leq \lim f_n(x) \) for almost all \( x \).

Applying this result to \( -f \) we obtain \( f(x) \leq \lim f_n(x) \) for almost all \( x \).

So \( \lim f_n \) exists and equals \( f \) almost everywhere with respect to \( \rho \).

We now strip away the measure theory of Theorem 1 and obtain the following abstract martingale theorem.

**Theorem 2.** Let \( \{ \mathcal{A}_n \} (n = 1, 2, \ldots) \) be an increasing sequence of Stone subalgebras of a Stone algebra \( \mathcal{A}_\infty \) such that the smallest Stone subalgebra containing \( \bigcup_{n=1}^{\infty} \mathcal{A}_n \) is the whole of \( \mathcal{A}_\infty \). Let \( \mathcal{A}_0 \) be a Stone algebra satisfying the countable chain condition and \( \pi : \mathcal{A}_0 \to \mathcal{A}_1 \) an algebra homomorphism. Let \( T_0 : \mathcal{A}_\infty \to \mathcal{A}_0 \) be a positive linear map such that:

(i) If \( b \geq 0 \) and \( T_0 b = 0 \) then \( b = 0 \).

(ii) \( T_0 (\pi(a)z) = a T_0 z \) for each \( z \in \mathcal{A}_\infty \) and each \( a \in \mathcal{A}_0 \).

(iii) If \( \{ z_n \} (n = 1, 2, \ldots) \) is a bounded monotone increasing sequence of positive elements of \( \mathcal{A}_\infty \) then
\[
T_0 \left( \bigvee_{n=1}^{\infty} z_n \right) = \bigvee_{n=1}^{\infty} T_0 z_n.
\]

Then there exists a sequence of Moyal operators \( \{ T_n \} (n = 1, 2, \ldots) \) such that:

(i) \( T_n \) is a projection of \( \mathcal{A}_\infty \) onto \( \mathcal{A}_n \) for each \( n \geq 1 \).

(ii) If \( b \geq 0 \) and \( T_n b = 0 \) then \( b = 0 \).

(iii) \( T_r T_n = T_r \) for \( 0 \leq r < n \).

(iv) For each \( z \in \mathcal{A}_\infty \) the order limit \( \LIM T_n z \) exists and \( \LIM T_n z = z \).

**Proof.** Let \( \mathcal{A}_\infty \cong C(E) \), the ring of continuous functions on an extremally disconnected compact Hausdorff space \( E \). For each Borel set \( A \) in \( E \) there is a unique idempotent \( k(A) \)
in \( C(E) \) which differs from \( \chi_A \) only on a meagre Borel set. We recall from [5] that \( k \) is a \( C(E) \)-valued measure, the map \( f \to \int_E f dk \) is an algebra homomorphism of \( B^\sigma(E) \) (the bounded Borel functions on \( E \)) onto \( C(E) \) and the kernel of this homomorphism is the set of Borel functions vanishing outside a meagre Borel set.

Let \( m \) be defined on the Borel sets of \( E \) by \( mB = T_0(kB) \). Then \( m \) is a (finite) \( \mathcal{A}_0 \)-valued measure on the Borel sets of \( E \) and for each \( f \in B^\sigma(E) \) we have

\[
\int_E f dm = T_0 \left( \int_E f dk \right).
\]

Let \( B \) be any Borel set of \( E \); then \( mB = 0 \) if and only if \( kB = 0 \), that is, if and only if \( B \) is meagre. Thus

\[
L^\sigma(E, m) \cong C(E) \cong \mathcal{A}_\infty.
\]

For each \( a \in \mathcal{A}_0 \) and \( f \in B^\sigma(E) \) we have

\[
\int_E \pi(a) f dm = T_0 \left( \int_E \pi(a) f dk \right) = T_0 \left( \pi(a) \int_E f dk \right).
\]

But, by hypothesis,

\[
T_0 \left( \pi(a) \int_E f dk \right) = a T_0 \int_E f dk = a \int_E f dm.
\]

Thus \( m \) is a modular \( \mathcal{A}_0 \)-valued measure with respect to \( \pi \).

Let \( \mathcal{B}_n \) be the collection of all Borel sets \( B \) of \( E \) such that \( kB \in \mathcal{A}_n \). Then \( L^\sigma(E, \mathcal{B}_n, m) \cong \mathcal{A}_n \) for each \( n \geq 1 \). Let \( \mathcal{B}_\infty \) be the smallest \( \sigma \)-subalgebra of the Borel sets of \( E \) which contains \( \bigcup_{n=1}^\infty \mathcal{B}_n \). Thus \( L^\sigma(E, \mathcal{B}_\infty, m) \) is a Stone subalgebra of \( L^\sigma(E, m) \cong \mathcal{A}_\infty \) and contains each of the algebras \( \mathcal{A}_n \) \((n = 1, 2, \ldots)\). Thus \( L^\sigma(E, \mathcal{B}_\infty, m) \cong \mathcal{A}_\infty \cong L^\sigma(E, m) \), although \( \mathcal{B}_\infty \) may not contain all the Borel sets of \( E \).

Since \( \pi[\mathcal{A}_0] \subset \mathcal{A}_\infty \) for \( n \geq 1 \) and \( m \) is an \( \mathcal{A}_0 \)-valued measure, which is modular with respect to \( \pi \), there exists a generalized conditional expectation operator \( T_n \) mapping \( \mathcal{A}_\infty \) onto \( \mathcal{A}_n \). Thus \( T_n \) is a projection of \( \mathcal{A}_\infty \) onto \( \mathcal{A}_n \); if \( b \) is a positive element of \( \mathcal{A}_\infty \) and \( T_n b = 0 \), then \( b = 0 \); \( T_n \) is the unique linear operator from \( \mathcal{A}_\infty \) into \( \mathcal{A}_n \) such that for each idempotent \( e \in \mathcal{A}_\infty \) and each \( z \in \mathcal{A}_\infty \) we have \( T_0(eT_n z) = T_0(ez) \). Let \( 1 \leq r < n \) and let \( e \) be an idempotent of \( \mathcal{A}_\infty \) and \( z \in \mathcal{A}_\infty \); then \( T_0(eT_r T_n z) = T_0(eT_n z) = T_0(ez) \), and so \( T_r T_n = T_r \).

It remains to show that, if \( z \in \mathcal{A}_\infty \), then the order limit \( \text{LIM} T_n z \) exists and equals \( z \). Let us identify \( \mathcal{A}_\infty \) with \( C(E) \) so that \( z \) and each \( T_n z \) \((n \geq 1)\) are continuous functions in \( C(E) \). We have from Theorem 1 that there exists a Borel set \( B \) such that \( mB = 0 \) and \( \text{lim}(T_n z)(t) \) exists and equals \( z(t) \) for each \( t \in E - B \). The sequence \( \{T_n z\} \) \((n = 1, 2, \ldots)\) is uniformly bounded because each \( T_n \) is a positive operator and \( T_n 1 = 1 \). Since \( mB = 0 \) only if \( kB = 0 \), we have, by the analogue of the Dominated Convergence Theorem proved in [5], that

\[
\text{LIM} \int_E T_n z dk \text{ exists and equals } \int_E z dk.
\]

Thus \( \text{LIM} T_n z \) exists and equals \( z \).
**Corollary.** Let \( \{A_n\} (n = 1, 2, \ldots) \) be an increasing sequence of Stone subalgebras of a Stone algebra \( A_\infty \) such that the smallest Stone subalgebra containing \( \bigcup_{n=1}^{\infty} A_n \) is the whole of \( A_\infty \). Let \( A_\infty \) satisfy the countable chain condition. Let \( T_0 \) be a Moy operator on \( A_\infty \) whose range is a subset of \( A_1 \), and is such that if \( b \) is a positive element of \( A_\infty \) and \( T_0 b = 0 \) then \( b = 0 \). Then there exists a sequence of Moy operators \( \{T_n\}, n = 1, 2, \ldots \), such that:

(i) \( T_n \) is a projection of \( A_\infty \) onto \( A_n \) for \( n \geq 1 \).

(ii) If \( b \) is a positive element of \( A_\infty \) and \( T_n b = 0 \) then \( b = 0 \).

(iii) \( T_r = T_r T_n \) for \( 0 \leq r < n \).

(iv) For each \( z \in A_\infty \) the order limit \( \lim T_n z \) exists and \( \lim T_n z = z \).

**Proof.** Since \( A_\infty \) satisfies the countable chain condition, we have that \( \sigma(T_0) = \{a \in A_\infty : a T_0 b = T_0 a b \text{ for all } b \in A_\infty\} \) is a Stone subalgebra of \( A_\infty \). Let \( A_0 \) be the smallest Stone subalgebra of \( A_\infty \) containing the range of \( T_0 \). Thus \( A_0 \subseteq A_1 \) and \( A_0 \subseteq \sigma(T_0) \). Let \( \pi : A_0 \rightarrow A_\infty \) be the natural embedding. Then \( T_0 \pi(a) z = a T_0 z \) for each \( a \in A_0 \) and each \( z \in A_\infty \). The corollary now follows from Theorem 2.

These methods can be adapted to prove analogous convergence theorems, where instead of \( \{A_n\} (n = 1, 2, \ldots) \) being monotone increasing it is monotone decreasing and

\[
\pi[A_0] \subseteq \bigcap_{n=1}^{\infty} A_n = A_\infty.
\]

In Theorem 1 we required the measure \( \rho \) to be modular so as to ensure the existence of the generalized conditional expectations \( T_n \). We observe that we can dispense with the hypothesis that \( \rho \) is modular if we know that the conditional expectation \( T_1 \) of \( \mathcal{B}_1 \) with respect to \( \rho \) exists. This is because \( T_1 \) may be regarded as an \( L^\infty(X, \mathcal{B}_1, \rho) \)-valued modular measure and so there is a conditional expectation \( T_n \) of \( \mathcal{B}_n \) with respect to \( T_1 \) for each \( n \). A straightforward computation shows that \( T_n \) is the conditional expectation of \( \mathcal{B}_n \) with respect to \( \rho \). The proof, in Theorem 1, that \( \{T_nf\} (n = 1, 2, \ldots) \) converges almost everywhere to \( f \) depends only on the existence of the conditional expectations \( T_n \) and not on the modularity of \( \rho \).

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St. Catherine’s College
Oxford